Unit 13: Sides and Angles of Triangles

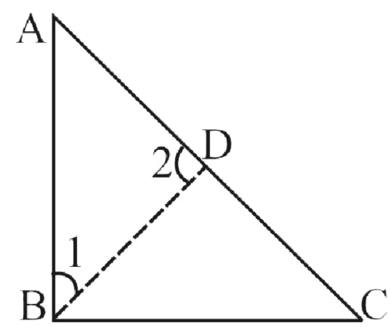
Overview

Theorem 13.1.1

If two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.

Given:

In ΔABC, $m\overline{AC} > m\overline{AB}$



To prove

$m\angle ABC > m\angle ACB$

Construction

On $\overline{AC}$ take a point D such that

$\overline{AD} \cong \overline{AB}$. Join B to D so that ΔADB is an isosceles triangle. Label $\angle 1$ and $\angle 2$ as shown in the given figure.

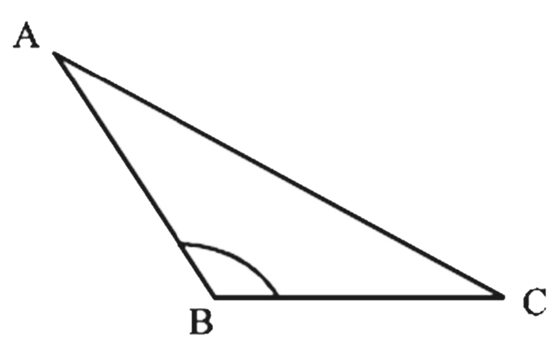
Proof

Statements	Reasons
In ΔABD	
$m\angle 1 = m\angle 2$ … (i)	Angles opposite to congruent sides (construction)
In ΔBCD, $m\angle ACB < m\angle 2$	
i.e. $m\angle 2 > m\angle ACB$ _____ (ii)	(An anterior angle of a triangle is greater than a non adjacent interior angle.)
$\therefore$ $m\angle 1 > m\angle ACB$ _____ (iii)	By (i) and (ii)
But $m\angle ABC = m\angle 1 + m\angle DBC$	Postulate of addition of angles
$\therefore$ $m\angle ABC > m\angle 1$ _____ (iv)	
$\therefore$ $m\angle ABC > m\angle 1 > m\angle ACB$	By (iii) and (iv)
Hence $m\angle ABC > m\angle ACB$	(Transitive property of inequality of real number)

Example 1

Prove that in a scalene triangle, the angle opposite to the largest side is of measure greater than $60°$.

(i.e., two-third of a right-angle)



Given

In $\Delta ABC, m\overline{AC} > m\overline{AB}, m\overline{AC} > m\overline{BC}$.

To prove

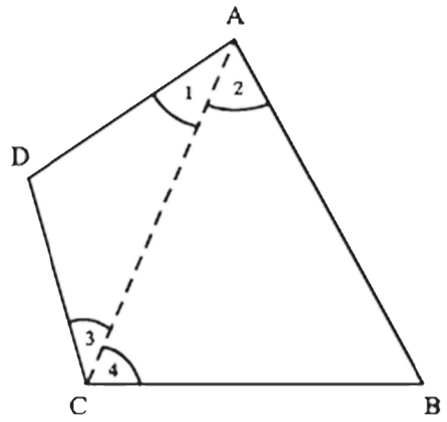
$m\angle B > 60°$

Proof

Statements	Reasons
In ΔABC	
$m\angle B > m\angle C$	$m\overline{AC} > m\overline{AB}$ (given)
$m\angle B > m\angle A$	$m\overline{AC} > m\overline{BC}$ (given)
But $m\angle A + m\angle B + m\angle C = 180^\circ$	$\angle A, \angle B, \angle C$ are the angles of ΔABC
$\therefore m\angle B + m\angle B + m\angle B > 180^\circ$	$m\angle B > m\angle C, m\angle B > m\angle A$ (proved)
Hence $m\angle B > 60^\circ$	$\frac{180^\circ}{3} = 60^\circ$

Example 2

In a quadrilateral ABCD, $\overline{AB}$ is the longest side and $\overline{CD}$ is the shortest side. Prove that ΔBCD>mBAD



Given

In quad. ABCD, $\overline{AB}$ is the longest side and $\overline{CD}$ is the shortest side.

To prove

$m\angle BCD > m\angle BAD$

Construction

Joint A to C.

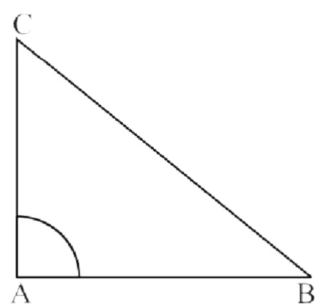
Name the angles $\angle 1, \angle 2, \angle 3$ and $\angle 4$ as shown in the figure.

Proof

Statements	Reasons
In $\Delta ABC, m\angle 4 > \angle 2 ...(i)$	$m\overline{AB} > m\overline{BC}$ (given)
In $\Delta ACD, m\angle 3 > \angle 1 ...(ii)$	$m\overline{AD} > m\overline{CD}$ (given)
$\therefore m\angle 4 + m\angle 3 > m\angle 2 + m\angle 1$	From (i) and (ii)
Hence $m\angle BCD > m\angle BAD$	$\therefore \begin{cases} m\angle 4 + m\angle 3 = m\angle BCD \\ m\angle 2 + m\angle 1 = m\angle BAD \end{cases}$

Theorem 13.1.2 (Converse of theorem 13.1.1)

If two angles of a triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.



Given:

In ΔABC, $m\angle A > m\angle B$

To prove

$m\overline{BC} > m\overline{AC}$

Proof

Statements	Reasons
If $m\overline{BC} \not> m\overline{AC}$, then Either (i) $m\overline{BC} = m\overline{AC}$ Or (ii) $m\overline{BC} < m\overline{AC}$	(Trichotomy property of real numbers)
From (i) if $m\overline{BC} = m\overline{AC}$, then	
$m\angle A = m\angle B$	(Angles opposite to congruent sides are congruent)
Which in not possible From (ii) if $m\overline{BC} < m\overline{AC}$, then	
$m\angle A < m\angle B$	(The angle opposite to longer side is greater than angle opposite to smaller side?
This is also not possible	Contrary to the given
$\therefore\ m\overline{BC} \neq m\overline{AC}$	
And $m\overline{BC} \not< m\overline{AC}$	Trichotomy property of real numbers.
Thus $m\overline{BC} > m\overline{AC}$	

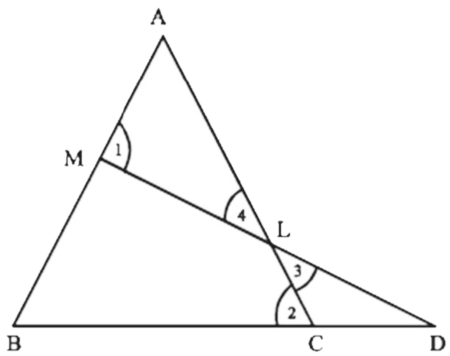
Corollaries

(i) The hypotenuse of a right triangle is longer than each of the other two sides.

(ii) In an obtuse angled triangle, the side opposite to the obtuse angle is longer than each of the other two sides.

Example

ABC is an isosceles triangle with base $\overline{BC}$. On $\overrightarrow{BC}$ a point D is taken away from C. A line segment through D cuts $\overline{AC}$ at L and $\overline{AB}$ at M. prove that $m\overline{AL} > m\overline{AM}$.



Given

In $\Delta ABC, \overline{AB} \cong \overline{AC}$

D is a point on $\overrightarrow{BC}$ away from C

A line segment through D cuts $\overline{AC}$ at L and $\overline{AB}$ at M.

To Prove

$m\overline{AL} > m\overline{AM}$

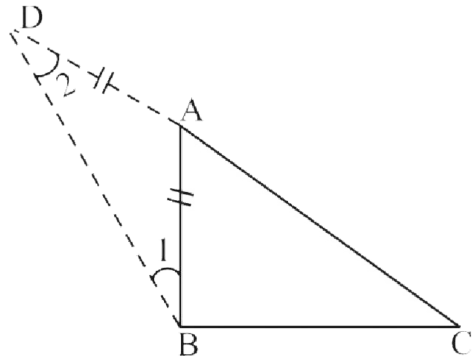
Proof

Statements	Reasons
In ΔABC	
$\angle B \cong \angle 2$..(i)	$\overline{AB} \cong \overline{AC}$ (given)
In ΔMBD	
$m\angle 1 > m\angle B$...(ii)	($\angle 2$ is an ext. $\angle$ and $\angle 3$ is its internal opposite $\angle$)
$\therefore m\angle 1 > m\angle 2$...(iii)	From (i) and (ii)
In ΔLCD	
$m\angle 2 > m\angle 3$	($\angle 1$ is an ext. $\angle$ and $\angle 3$ is its internal opposite $\angle$)
$\therefore m\angle 1 > m\angle 3$...(v)	From (iii) and (iv)

But $m\angle 3 \cong m\angle 4 ...(vi)$ $\therefore m\angle 1 > m\angle 4$ Hence $m\overline{AL} > m\overline{AM}$	Vertical angles From (v) and (vi) In $\Delta ALM, m\angle 1 > m\angle 4$ (proved)

Theorem 13.1.3

The sum of the lengths of any two sides of a triangle is greater than the length of third side.



Given ΔABC

To prove

(i) $m\overline{AB} + m\overline{AC} > m\overline{BC}$

(ii) $m\overline{AB} + m\overline{BC} > m\overline{AC}$

(iii) $m\overline{BC} + m\overline{AC} > m\overline{AB}$

Construction

Take a point D on $\overrightarrow{CA}$ such that $\overline{AD} \cong \overline{AB}$ join B to D and name the angles $\angle 1$, $\angle 2$ as shown in the given figure.

Proof

Statements	Reasons
In ΔABD,	
$\angle 1 \cong \angle 2$ _____ (i)	$\overline{AD} \cong \overline{AB}$ (construction)
$m\angle DBC > m\angle 1$ _____ (ii)	$m\angle DBC = m\angle 1 + m\angle ABC$
$\because m\angle DBC > m\angle 2$ _____ (iii)	From (i) and (ii)
In ΔDBC	
$m\overline{CD} > m\overline{BC}$	By (iii)
i.e. $m\overline{AD} + m\overline{AC} > m\overline{BC}$	$m\overline{CD} = m\overline{AD} > m\overline{AC}$
Hence $m\overline{AB} + m\overline{AC} > m\overline{BC}$	$m\overline{AD} = m\overline{AB}$ (Construction)
Similarly	
$m\overline{AB} + m\overline{BC} > m\overline{AC}$	
And $m\overline{BC} + m\overline{CA} > m\overline{AB}$	

Example 1

Which of the following sets of lengths can be the lengths of the sides of a triangle?

(a) 2cm, 3cm, 5cm (b) 3cm, 4cm, 5cm, (c) 2cm, 4cm, 7cm,

(a) $\because 2 + 3 = 5$

$\therefore$ This set of lengths cannot be those of the sides of a triangle.

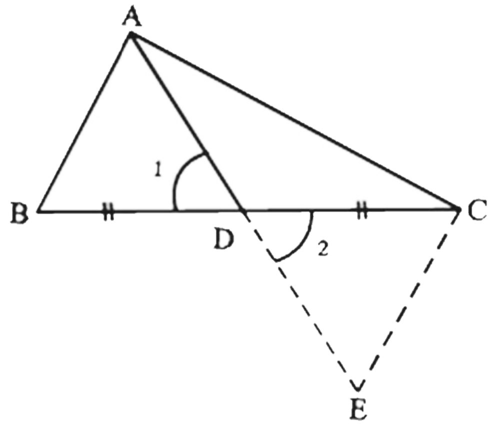
(b) $\because 3 + 4 > 5, 3 + 5 > 4, 4 + 5 > 3$

$\therefore$ This set can form a triangle

(c) $\because 2 + 4 < 7$

$\therefore$ This set of lengths cannot be the sides of a triangle.

Example 2



Prove that the sum of the measures of two sides of a triangle is greater than twice the measure of the median which bisects the third side.

Given

In ΔABC, median AD bisects side $\overline{BC}$ at D.

To prove

$m\overline{BC} + \overline{AC} > 2m\overline{AD}$.

Construction

On $\overrightarrow{AD}$, Take a point E, such that $\overline{DE} \cong \overline{AD}$.

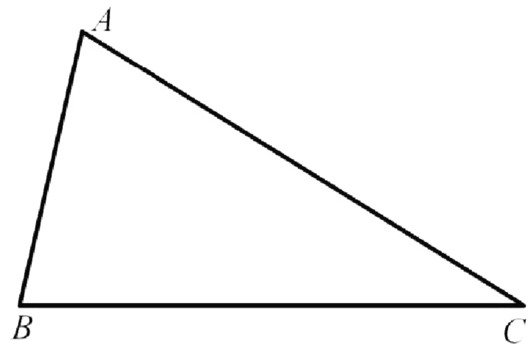
Join C to E. Name the angles $\angle 1, \angle 2$ as shown in the ________ figure.

Proof

Statements	Reasons
In $\Delta ABD \leftrightarrow \Delta ECD$	
$\overline{BD} \cong \overline{CD}$	Given
$\angle 1 \cong \angle 2$	Vertical angles
$\overline{AD} \cong \overline{ED}$	Construction
$\Delta ABD \cong \Delta ECD$	S.A.S. Postulate
$\overline{AB} \cong \overline{EC}$...(i)	Corresponding sides of $\cong \Delta$s
$m\overline{AC} + m\overline{EC} > m\overline{AE}$...(ii)	ACE is a triangle
$m\overline{AC} + m\overline{AB} > m\overline{AE}$...(ii)	From (i) and (ii)
Hence $m\overline{AC} + m\overline{AB} > 2m\overline{AD}$	$m\overline{AE} = 2m\overline{AD}$ (Construction)

Example 3

Prove that the difference of measures of two sides of a triangle is less than the measure of the third side.



Given

ΔABC

To Prove

$m\overline{AC} - m\overline{AB} < m\overline{BC}$

$m\overline{BC} - m\overline{AB} < m\overline{AC}$

$m\overline{BC} - m\overline{AC} < m\overline{AB}$

Proof

Statements	Reasons
$m\overline{AB} + m\overline{BC} > m\overline{AC}$	ABC is a triangle
$\left(\cancel{m\overline{AB}} + m\overline{BC} - \cancel{m\overline{AB}}\right) > \left(m\overline{AC} - m\overline{AB}\right)$	Subtracting $m\overline{AB}$ from both sides
$\therefore m\overline{BC} > \left(m\overline{AC} - m\overline{AB}\right)$	
or $m\overline{AC} - m\overline{AB} < m\overline{BC}$...(i)	$a > b \Rightarrow b < a$
Similarly	
$\left.\begin{array}{l} m\overline{BC} - m\overline{AB} < m\overline{AC} \\ m\overline{BC} - m\overline{AC} < m\overline{AB} \end{array}\right\}$	Reason similar to (i)

Exercise 13.1

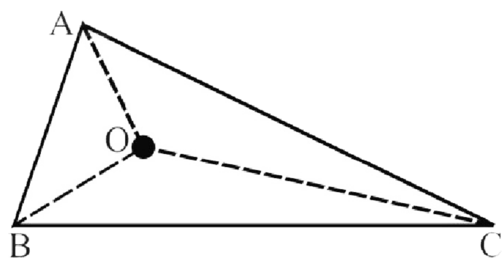
Q.1 Two sides of a triangle measure 10cm and 15 cm which of the following measure is possible for the third side?

(a) 5*cm*
(b) 20 *cm*
(c) 25 *cm*
(d) 30 *cm*

Solution
Lengths of two sides are 15 and 10 cm.
So, sum of two lengths of triangle = 10 + 15 = 25 m
$10 + 15 > 20$
$10 + 20 > 15$
$15 + 20 > 10$
$\therefore$ 20 cm is possible for third side
Or
Sum of length of two sides is always greater than the third sides of a triangle.
Given

Q.2 Point O is interior of ΔABC
Show that

$$m\overline{OA}+m\overline{OB}+m\overline{OC}>\frac{1}{2}\left(m\overline{AB}+m\overline{BC}+m\overline{CA}\right)$$



Given
Point O is interior of ΔABC
To prove:

$$m\overrightarrow{OA}+m\overline{OB}+m\overline{OC}<\frac{1}{2}\left(m\overline{AB}+m\overline{BC}+m\overline{AC}\right)$$

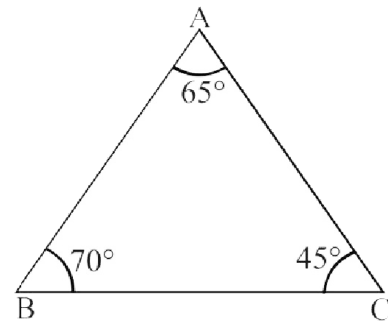
Construction
Join O with A, B and C.
So that we get three triangle ΔOAB, ΔOBC and ΔOAC
Proof

Statements	Reasons
In ΔOAB $m\overline{OA}+OB > m\overline{AB}$ _____(i)	In any triangle the sum of length of two sides is greater then the third sides.
In ΔOAC $m\overline{OC}+m\overline{OA} > m\overline{AC}$ _____(ii)	As in (i)
In ΔOBC $mOB+\overline{OC} > m\overline{BC}$ _____(iii)	As in (i)
Adding equation i, ii and iii $\overline{OA}+\overline{OC}+\overline{OA}+\overline{OB}+\overline{OB}+\overline{OC}>\overline{AC}+\overline{AB}+\overline{BC}$ $2\overline{OA}+2\overline{OC}+2\overline{OB}>\overline{AB}+\overline{BC}+\overline{CA}$ $2\left(OA+OC+OB\right)>\overline{AB}+\overline{BC}+\overline{CA}$	

$\frac{\cancel{2}(OA+OC+OB)}{\cancel{2}} > \frac{\overline{AB}+\overline{BC}+\overline{CA}}{2}$ $(OA+OC+OB) > \frac{1}{2}(\overline{AB}+\overline{BC}+\overline{CA})$	Dividing both sides by 2

Q.3 In the ΔABC m∠B = 70° and m∠C = 45° which of the sides of the triangle is longest and which is the shortest.

Solution



Sum of three angle in a triangle is 180°

∠A + ∠B + ∠C = 180

∠A + 70 + 45 = 180

∠A + 115 = 180

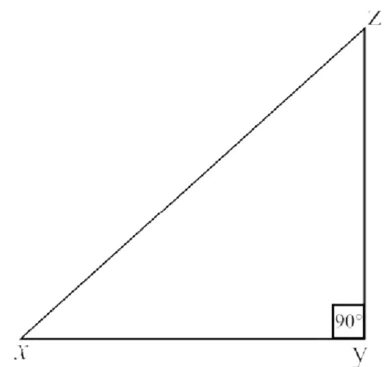
∠A = 180 – 115

∠A = 65°

Sides of the triangle depend upon the angles largest angle has largest opposite side and smallest angle has smallest opposite side here $\angle B$ is largest so, $\overline{AC}$ is largest $\angle C$ is smallest, so $\overline{AB}$ is smallest side.

Q.4 Prove that in a right-angled triangle, the hypotenuse is longer than each of the other two sides.

Solution



Sum of three angles in a triangle is equal to 180°. So in a triangle one angle will be equal to 90° and rest of two angles are acute angle (less than 90°)

∴ m∠y = 90

And m∠x + m∠z = 90

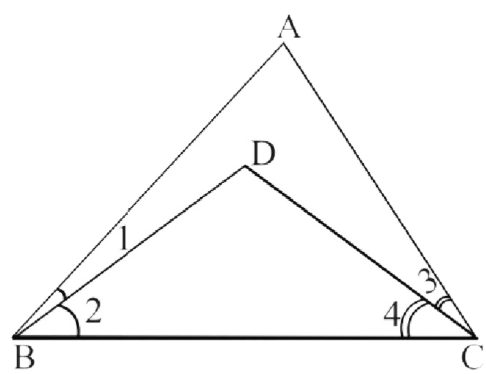
So m∠x and m∠z are acute angle

∴ Opposite to m∠y = 90° is hypotenuse

It is largest side.

Q.5 In the triangular figure $\overline{AB} > \overline{AC}$. $\overline{BD}$ and $\overline{CD}$ are the bisectors of ∠B and ∠C respectively prove that $\overline{BD} > \overline{DC}$

Given



InΔABC

$\overline{AB} > \overline{AC}$

$\overline{BD}$ and $\overline{CD}$ are the bisectors of $\angle B$ and $\angle C$

To prove

$\overline{BD} > \overline{CD}$

Construction

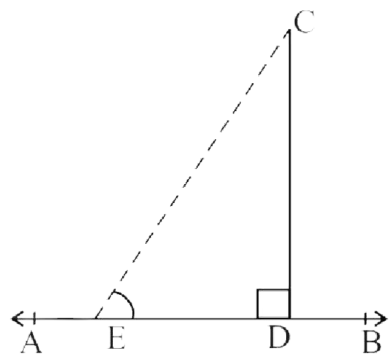
Label the angles $\angle 1, \angle 2, \angle 3$ and $\angle 4$

Proof

Statements	Reasons
In ΔABC	
$\overline{AB} > \overline{AC}$	Given
$\overline{BD}$ is the bisector of $\angle B$	
$\frac{1}{2} m\angle ACB > \frac{1}{2} m\angle ABC$	
$m\angle ABC$	
$m\angle 2 < m\angle 4$	
$\overline{CD}$ is the bisector of $\angle C$	Given
In ΔBCD	
$\overline{BD} > \overline{DC}$	Side opposite to greater angle is greater

Theorem 13.1.4

From a point, out side a line, the perpendicular is the shortest distance from the point to the line.



Given:

A line AB and a point C

(Not lying on $\overleftrightarrow{AB}$)and a point D on $\overleftrightarrow{AB}$ such that $\overline{CD} \perp \overleftrightarrow{AB}$

To prove

$m\overline{CD}$ is the shortest distance from the point C to $\overleftrightarrow{AB}$

Construction

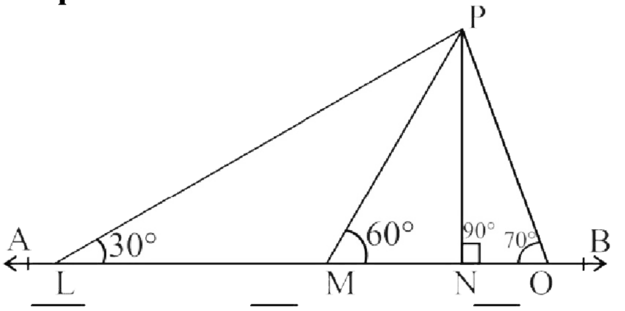
Take a point E on $\overleftrightarrow{AB}$ .Join C and E to form a ΔCDE

Proof

Statements	Reasons
In ΔCDE	
$m\angle CDB > m\angle CED$	(An exterior angle of a triangle is greater than non adjacent interior angle)
But $m\angle CDB = m\angle CDE$	Supplement of right angle
$\therefore$ $m\angle CDE > m\angle CED$	
Or $m\angle CED < m\angle CDE$	
Or $m\overline{CD} < m\overline{CE}$	Side opposite to greater angle is greater.
But E is any point on $\overleftrightarrow{AB}$	
Hence $m\overline{CD}$ is the shortest distance from C to $\overleftrightarrow{AB}$	

Exercise 13.2

Q.1 In the figure P is any point and AB is a line which of the following is the shortest distance between the point P and the line AB.

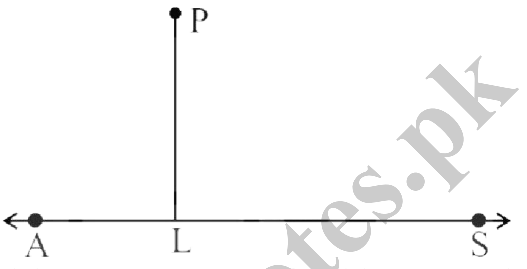


(a) $m\overline{PL}$ (b) $m\overline{PM}$ (c) $m\overline{PN}$ (d) $m\overline{PO}$

As we know that $\overline{PN} \perp \overleftrightarrow{AB}$

So $\overline{PN}$ is the shortest distance

Q.2 In the figure, P is any point lying away from the line $\overline{AB}$. Then $m\overline{PL}$ will be the shortest distance if



(a) $m\angle P \angle A = 80^o$ (b) $m\angle P\angle B = 100^o$ (c) $m\angle P\angle A = 90^o$

Solution:

$m\angle PLA = 90^o$

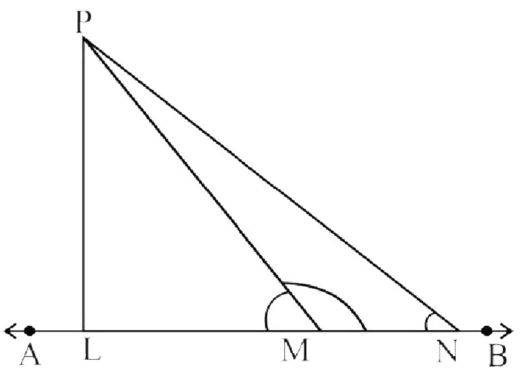
$\overline{PL} \perp \overleftrightarrow{AS}$

PL is the shortest distance

So $\angle PLA$ or PLS equal to 90^o

Q.3 In the figure, $\overline{PL}$ is perpendicular to the line AB and $\overline{LN} > m\overline{LM}$. Prove that $m\overline{PN} > m\overline{PM}$.

Given



$\overline{PL} \perp \overleftrightarrow{AB}$

$m\overline{LN} > m\overline{LM}$

To proved:

$m\overline{PN} > m\overline{PM}$

Proof

Statements	Reasons
ΔPLM	
$\angle PLM = 90^o$	
$\therefore \angle PMN > \Delta PLM$	Exterior angle
$\angle PMN > 90^o$	
InΔPLN	
$\angle PLN = 90^o$	
$m\angle PNL < 90^o$	Acute angle
ΔPMN	
$m\angle PMN > m\angle PNL$	
$\therefore \overline{PN} > \overline{PM}$	

Review Exercise 13

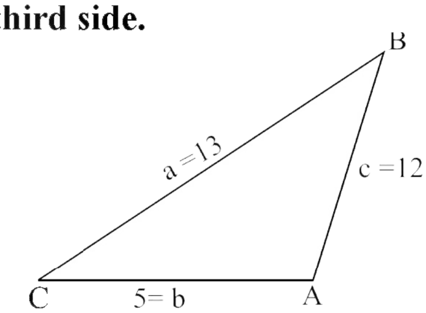
Q.1 Which of the following are true and which are false?

(i)	The angle opposite to the longer side is greater.	(True)
(ii)	In a right-angled triangle greater angle is of 60°.	(False)
(iii)	In an isosceles right-angled triangle, angles other than right angle are each of 45°.	(True)
(iv)	A triangle having two congruent sides is called equilateral triangle.	(False)
(v)	A perpendicular from a point to line is shortest distance.	(True)
(vi)	Perpendicular to line forms an angle of 90°.	(True)
(vii)	A point out side the line is collinear.	(False)
(viii)	Sum of two sides' of a triangle is greater than the third.	(True)
(ix)	The distance between a line and a point on it is zero.	(True)
(x)	Triangle can be formed of length 2cm, 3cm and 5cm.	(False)

Q.2 What will be angle for shortest distance from an outside point to the line?

The angle for shortest distance from an outside point to the line is 90° angle.

Q.3 If 13cm, 12cm and 5cm are the length of a triangle, then verify that difference of measures of any two sides of a triangle is less than the third side.



$a = 13$, $b = 5$, $c = 12$ cm

$a - b = 13 - 5 = 8$

$8 < c$

$c - b = 12 - 5 = 7$

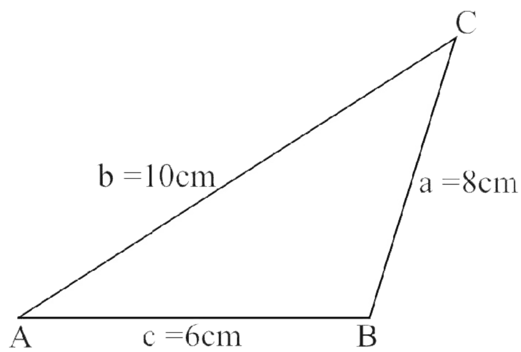
$7 < a$

$a - c = 13 - 12 = 1$

$1 < b$

This is the process which show the difference of any two sides of a triangle is less then the measure of the third.

Q.4 If 10cm, 6cm and 8cm are the length of a triangle, then verify that sum of measures of two sides of a triangle is greater than the third side.



$a = 8$cm, $b = 10$cm, $c = 6$cm

$8 + 10 = 18\text{cm} > 6\text{cm}$

$a + b > c$

$10 + 6 = 16\text{cm} > 8\text{cm}$

$b + c > a$

$6 + 8 = 14\text{cm} > 10\text{cm}$

$c + a > b$

$\therefore$ The sum of measures of two sides of a triangle is greater than the third side.

Q.5 3cm, 4cm and 7cm are not the length of the triangle. Give reasons.

a = 3cm b = 4cm c = 7cm

$3 + 4 = 7$

$a + b = c$

$b + c > a$

$4 + 7 > 3$

$c + a > b$

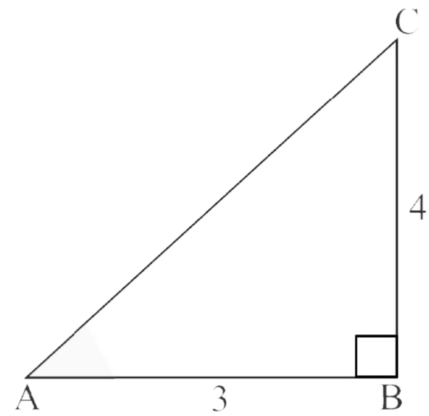
$7 + 3 > 4$

In a triangle sum of measures of two sides should be greater than the third sides.

Q.6 If 3cm and 4cm are the length of two sides of a right angle triangle than what should be the third length of the triangle.

If sum of the squares of two sides of a triangles is equal to the square of the third side then it is called right angled triangle.

So by Pythagoras theorem.



$(\overline{AC})^2 = (BC)^2 + (AB)^2$

$(\overline{AC})^2 = (4)^2 + (3)^2$

$(\overline{AC})^2 = 16 + 9$

$(\overline{AC})^2 = 25$

Taking square root on both sides

$\sqrt{(\overline{AC})^2} = \sqrt{25}$

$\overline{AC} = 5cm$

$\therefore$ Length of third side of right angled triangle is 5cm.