Unit 12: Line Bisectors and Angle Bisectors

Overview

Right Bisector of a line segment:

Right bisection of a line segment means to draw a perpendicular at the mid-point of line segment.

Bisector of an angle:

Bisection of an angle means to draw a ray to divide the given angle into two equal parts.

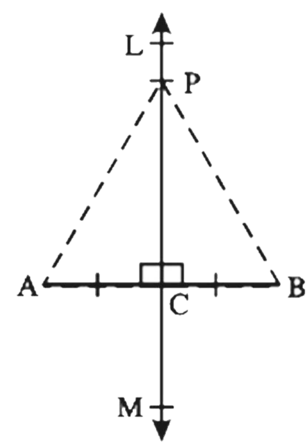
Theorem 12.1.1

Statement:

Any point on the right bisector of a line segment is equidistant from its end points.

Given

A line $\overleftrightarrow{LM}$ intersects the line segment AB at the point C Such that $\overleftrightarrow{LM} \perp \overline{AB}$ and $\overline{AC} \cong \overline{BC}$. P is a point on $\overleftrightarrow{LM}$



To prove

$\overline{PA} \cong \overline{PB}$

Construction

Join P to the point A and B

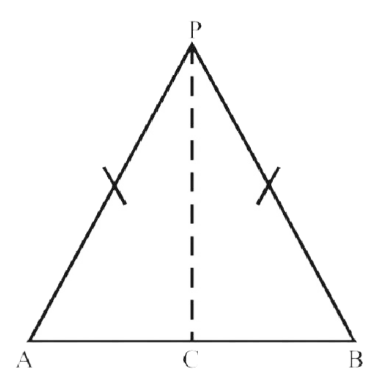
Proof

Statements	Reasons
In $\Delta ACP \leftrightarrow \Delta BCP$	
$\overline{AC} \cong \overline{BC}$	Given
$\angle ACP \cong \angle BCP$	Given $\overline{PC} \perp \overline{AB}$, so that each $\angle$ at $C = 90^o$
$\overline{PC} \cong \overline{PC}$	Common
$\therefore \Delta ACP \cong \Delta BCP$	S.A.S Postulate
Hence $\overline{PA} \cong \overline{PB}$	(Corresponding sides of congruent triangles)

Theorem 12.1.2

$\{$Converse of Theorem 12.1.1$\}$

Any point equidistant from the end points of a line segment is on the right bisector of it.



Given

$\overline{AB}$ is a line segment. Point P is such that $\overline{PA} \cong \overline{PB}$

To prove

The point P is the on the right bisector of $\overline{AB}$

Construction

Join P to C, the midpoint of $\overline{AB}$

Proof

Statements	Reasons
In $\Delta ACP \leftrightarrow \Delta BCP$	
$\overline{PA} \cong \overline{PB}$	Given
$\overline{PC} \cong \overline{PC}$	Common
$\overline{AC} \cong \overline{BC}$	Construction
$\therefore \Delta ACP \cong \Delta BCP$	$S.S.S \cong S.S.S$
$\angle ACP \cong \angle BCP$ ___________ (i)	Corresponding angles of congruent triangles
But $m\angle ACP + m\angle BCP = 180^o$ ___ (ii)	Supplementary angles
$\therefore m\angle ACP = m\angle BCP = 90^o$	From (i) and (ii)
i.e $\overline{PC} \perp \overline{AB}$ ___________ (iii)	$m\angle ACP = 90^o$ (Proved)
Also $\overline{CA} \cong \overline{CB}$ ___________ (iv)	**Construction**
$\therefore$ $\overline{PC}$ is a right bisector of $\overline{AB}$	from (iii) and (iv)
i.e. the point P is on the right bisector of $\overline{AB}$	

Exercise 12.1

Q.1 Prove that the centre of a circle is on the right bisectors of each of its chords.

Given

A, B, C are the three non-collinear points.

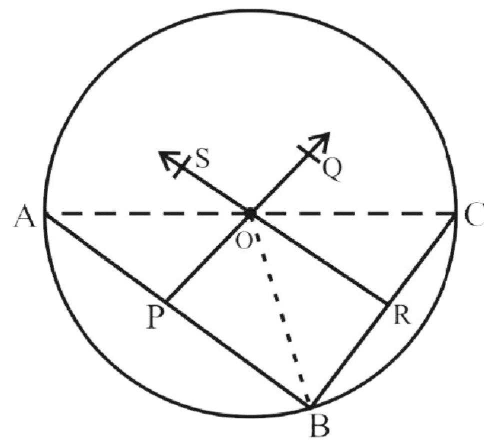
Required: To find the centre of the circle passing through A,B,C

Construction

Join B to C, A take $\overrightarrow{PQ}$ is right bisector of $\overline{AB}$ and $\overline{RS}$ right bisector of BC, they intersect at O.

Join O to A, O to B, O to C.

$\therefore$ O is the centre of circle.



Proof

Statements	Reasons
$\overline{OB} \cong \overline{OC}$ ___________ (i)	O is the right bisector of $\overline{BC}$
$\overline{OA} \cong \overline{OB}$ ___________ (ii)	O is the right bisector of $\overline{AB}$
$\overline{OA} = \overline{OB} = \overline{OC}$	From (i) and (ii)
Hence is equidistant from the A,B,C	
$\therefore O$ is center of circle which is required	

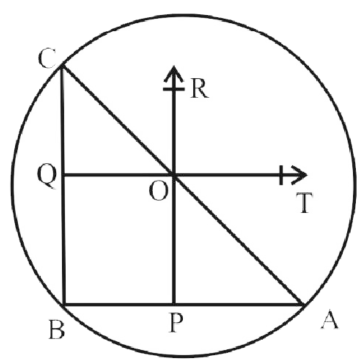
Q.2 Where will the center of a circle passing through three non-collinear points? And Why?

Given

A.B.C are three non collinear points and circle passing through these points.

To prove

Find the center of the circle passing through vertices A, B and C.



Construction

(i) Join B to A and C.

(ii) Take $\overrightarrow{QT}$ right bisector of $\overline{BC}$ and take also $\overrightarrow{PR}$ right bisector of $\overline{AB}$.

$\overrightarrow{PR}$ and $\overrightarrow{QT}$ intersect at point O. joint O to A,B and C. O is the center of the circle.

Proof

Statements	Reasons
$\overline{QO}$ is right bisector $\overline{BC}$	
$\overline{OB} \cong \overline{OC} \ldots$(i)	
$\overline{PO}$ is right bisector of $\overline{AB}$	
$\overline{OA} \cong \overline{OB} \ldots$(ii)	
So	
$\overline{OA} \cong \overline{OC} \cong \overline{OB}$	From (i) and (ii)
$\therefore$ It is proved that O is the center of the circle.	

Q.3 **Three village P,Q and R are not on the same line. The people of these villages want to make a children park at such a place which is equidistant from these three villages. After fixing the place of children park prove that the park is equidistant from the three villages.**

Given

P,Q,R are three villages not on the same straight line.

To prove

The point equidistant from P,R,Q.

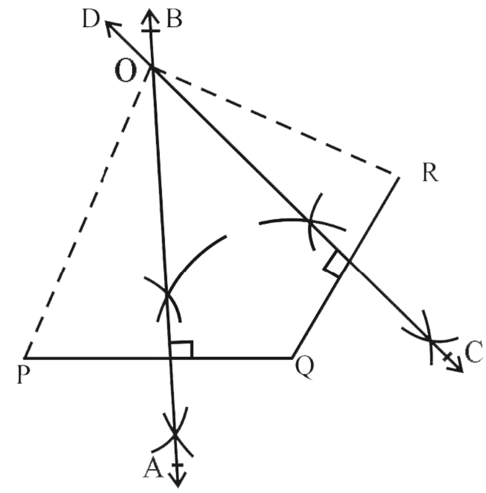
Construction

(i) Joint Q to P and R.

(ii) Take $\overleftrightarrow{AB}$ right bisector of $\overline{PQ}$ and $\overleftrightarrow{CD}$ right bisector of $\overline{QR}$. $\overline{AB}$ and $\overleftrightarrow{CD}$ intersect at O.

(iii) Join 0 to P, Q, R

The place of children part at point O.



Proof

Statements	Reasons
$\overline{OQ} \cong \overline{OR}$ __________ (i)	O is on the right bisector of $\overline{QR}$
$\overline{OP} \cong \overline{OQ}$ __________ (ii)	O is on the right bisector of $\overline{PQ}$
$\overline{OP} \cong \overline{OQ} \cong \overline{OR}$ __________ (iii)	From (i) and (ii)
$\therefore O$ is on the bisector of $\angle P$	
Hence $\overline{PO}$ is bisector of $\angle P$	

O is equidistant from P,Q and R

Theorem 12.1.3

The right bisectors of the sides of a triangle are concurrent.

Given

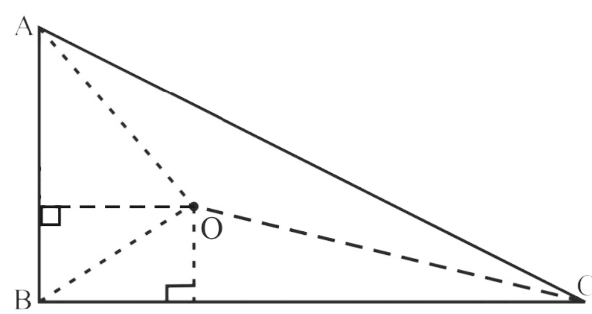
ΔABC

To prove

The right bisectors of $\overline{AB}, \overline{BC}$ and $\overline{CA}$ are concurrent.

Construction

Draw the right bisectors of $\overline{AB}$ and $\overline{BC}$ which meet each other at the point O. Join O to A, B and C.

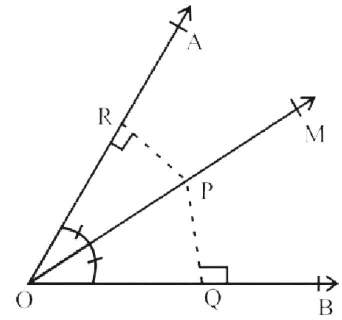


Proof

Statements	Reasons
$\overline{OA} \cong \overline{OB} \longrightarrow$ (i)	(Each point on right bisector of a segment is equidistant from its end points)
$\overline{OB} \cong \overline{OC} \longrightarrow$ (ii)	As in (i)
$\overline{OA} \cong \overline{OC}$	from (i) and (ii)
$\therefore$ Point O is on the right bisector of $\overline{CA}$ $\rightarrow$ (iv)	(O is equidistant from A and C)
But point O is on the right bisector of $\overline{AB}$ and of $\overline{BC}$ $\longrightarrow$ (v)	**Construction**
Hence the right bisectors of the three sides of triangle are concurrent at O	{from (iv) and (v)}

Theorem 12.1.4

Any point on the bisector of an angle is equidistant from its arms.



Given

A point P is on $\overrightarrow{OM}$, the bisector of $\angle AOB$

To Prove

$\overline{PQ} \cong \overline{PR}$ i.e P is equidistant from $\overrightarrow{OA}$ and $\overrightarrow{OB}$

Construction

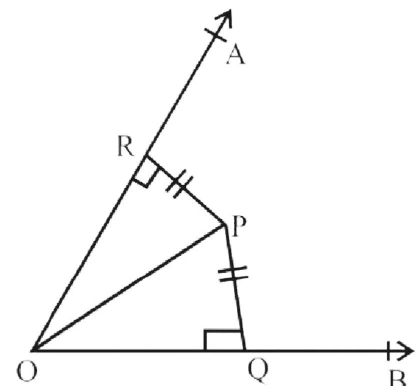
Draw $\overline{PR} \perp \overrightarrow{OA}$ and $\overline{PQ} \perp \overrightarrow{OB}$

Proof

Statements	Reasons
In $\Delta POQ \leftrightarrow \Delta POR$	
$\overline{OP} \cong \overline{OP}$	Common
$\angle PQO \cong \angle PRO$	Construction
$\angle POQ \cong \angle POR$	Given
$\therefore \Delta POQ \cong \Delta POR$	$S.A.A \cong S.A.A$
Hence $\overline{PQ} \cong \overline{PR}$	(Corresponding sides of congruent triangles)

Theorem 12.1.5 (Converse of Theorem 12.1.4)

Any point inside an angle, equidistant from its arms, is on the bisector of it.



Given

Any point P lies inside $\angle AOB$, such that

$\overline{PQ} \cong \overline{PR}$, where $\overline{PQ} \perp \overrightarrow{OB}$ and $\overline{PR} \perp \overrightarrow{OA}$

To prove

Point P is on the bisector of $\angle AOB$

Construction

Join P to O

Proof

Statements	Reasons
In $\Delta POQ \leftrightarrow \Delta POR$	
$\angle PQO \cong \angle PRO$	Given (Right angles)
$\overline{PO} \cong \overline{PO}$	Common
$\overline{PQ} \cong \overline{PR}$	Given
$\therefore \Delta POQ \cong \Delta POR$	H.S $\cong$ H.S
Hence $\angle POQ \cong \angle POR$	(Corresponding angles of congruent triangles)
i.e, P is on the bisector of $\angle AOB$	

Exercise 12.2

Q.1 In a quadrilateral ABCD $\overline{AB} \cong \overline{BC}$ and the right bisectors of $\overline{AD}, \overline{CD}$ meet each other at point N. Prove that $\overline{BN}$ is a bisector of $\angle ABC$

Given

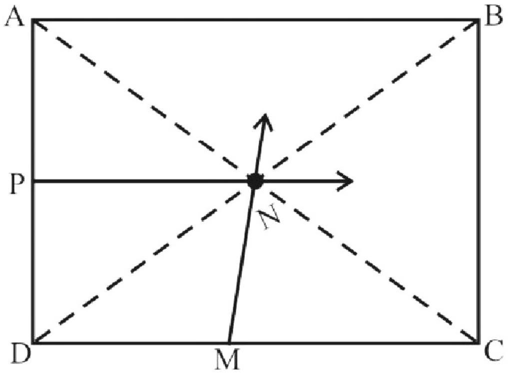
In the quadrilateral ABCD

$\overline{AB} \cong \overline{BC}$

$\overline{NM}$ is right bisector of $\overline{CD}$

$\overline{PN}$ is right bisector of $\overline{AD}$

They meet at N



To prove

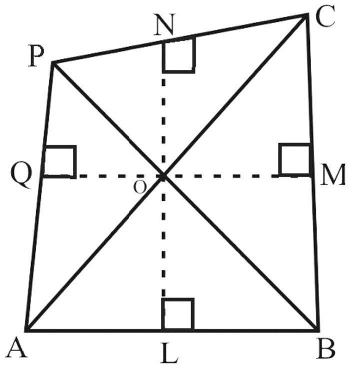
$\overline{BN}$ is the bisector of angle ABC

Construction join N to A,B,C,D

Proof

Statements	Reasons
$\overline{ND} \cong \overline{NA}$ ___________ (i)	N is an right bisector of $\overline{AD}$
$\overline{ND} \cong \overline{NC}$ ___________ (ii)	N is on right bisector of $\overline{DC}$
$\overline{NA} = \overline{NC}$ ___________ (iii)	from (i) and (ii)
$\Delta BNC \leftrightarrow \Delta ANB$	
$\overline{NC} = \overline{NA}$	Already proved (from iii)
$\overline{AB} \cong \overline{CB}$	Given
$\overline{BN} \cong \overline{BN}$	Common
$\therefore \Delta BNA \cong \Delta BNC$	$S.S.S \cong S.S.S$
Hence $\angle ABN \cong \angle NBC$	Corresponding angles of congruent triangles
Hence $\overline{BN}$ is the bisector of $\angle ABC$	

Q.2 The bisectors of $\angle A, \angle B$ and $\angle C$ of a quadrilateral ABCP meet each other at point O. Prove that the bisector of $\angle P$ will also pass through the point O.



Given

ABCP is quadrilateral. $\overline{AO}, \overline{BO}, \overline{CO}$ are bisectors of $\angle A, \angle B$ and $\angle C$ meet at point O.

To prove

$\overline{PO}$ is bisector of $\angle P$

Construction:

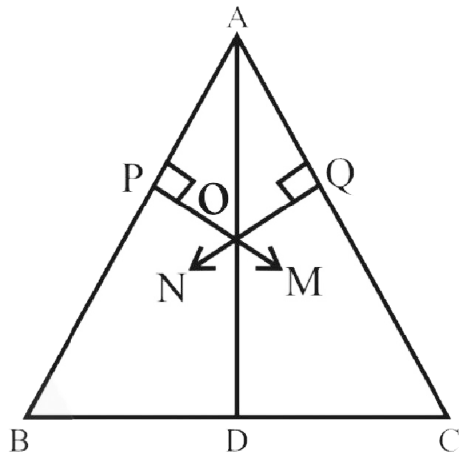
Join P to O.

Draw $\overline{OQ} \perp \overline{AP}$, $\overline{ON} \perp \overline{PC}$ and $\overline{OL} \perp \overline{AB}$, $\overline{OM} \perp \overline{BC}$

Proof:

Statements	Reasons
$\overline{OM} \cong \overline{ON}$ ________ (i)	O is on the bisector of $\angle C$
$\overline{OL} \cong \overline{OM}$ ________ (ii)	O is on the bisector of $\angle B$
$\overline{OL} \cong \overline{OQ}$ ________ (iii)	O is on the bisector of $\angle A$
$\overline{OQ} \cong \overline{ON}$	From i, ii, iii
Point O lines on the bisector of $\angle P$	
$\therefore \overline{OP}$ is the bisector of angle P	

Q.3 Prove that the right bisector of congruent sides of an isosceles triangle and its altitude are concurrent.



Given

ΔABC

$\overline{AB} \cong \overline{AC}$ due to isosceles triangle $\overline{PM}$ is right bisector of $\overline{AB}$

$\overline{QN}$ is right bisector of $\overline{AC}$

$\overrightarrow{PM}$ and $\overrightarrow{QN}$ intersect each other at point O

Required

The altitude of ΔABC lies at point O

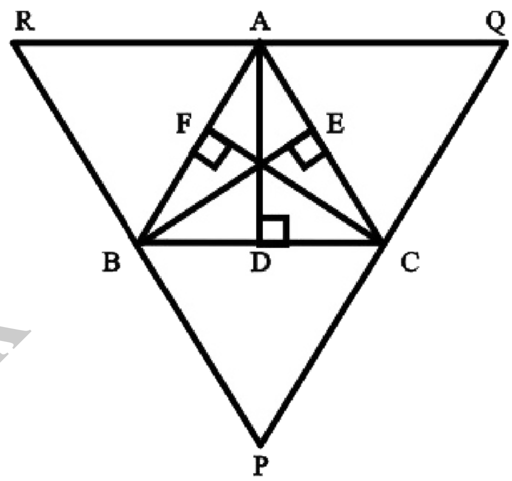
Join A to O and extend it to cut $\overline{BC}$ at D.

Proof

Statements	Reasons
$m\overline{AB} \cong m\overline{AC}$	Given
$\frac{1}{2}m\overline{AB} = \frac{1}{2}m\overline{AC}$	Dividing both side by 2
$\overline{AQ} \cong \overline{AP}$	
In $\Delta AQO \leftrightarrow \Delta APO$	
$\angle APO \cong \angle AQO$	Each 90^o (Given)
$\overline{AQ} \cong \overline{AP}$	Already Proved
$\overline{AO} \cong \overline{AO}$	Common
$\Delta APO \cong \Delta AQO$	$H.S \cong H.S$
$\angle PAO \cong \angle QAO$ (i)	Corresponding angles of congruent triangles
$\Delta BAD \leftrightarrow \Delta CAD$	
$\overline{AB} \cong \overline{AC}$	Given
$\overline{AD} \cong \overline{AD}$	Common

$\angle BAD \cong \angle CAD$	Proved from (i)
$\Delta BAD \cong \Delta CAD$	$S.A.S \cong S.A.S$
$\angle ODB \cong \angle ODC$	Each angle is 90^o (Given)
$m\angle ODM + m\angle ODC = 180^o$	Supplementary angle
$\therefore \overline{AD} \perp \overline{BC}$	
Point 0 lies on altitude $\overline{AD}$	

Q.4 Prove that the altitudes of a triangle are concurrent.



Given

In ΔABC

$\overline{AD}, \overline{BE}, \overline{CF}$ are its altitiudes

i.e $\overline{AD} \perp \overline{BC}, \overline{BE} \perp \overline{AC}, \overline{CF} \perp \overline{AB}$

Required $\overline{AD}, \overline{BE}$ and $\overline{CF}$ are concurrent

Construction:

Passing through A, B, C take

$\overline{RQ} \| \overline{BC}, \overline{RP} \| \overline{AC}$ and $\overline{QP} \| \overline{AB}$ respectively forming a ΔPQR

Proof

Statements	Reasons
$\overline{BC} \| \overline{AQ}$	Construction
$\overline{AB} \| \overline{QC}$	Construction
$\therefore$ ABCQ is a $\|^{gm}$	
Hence $\overline{AQ} \cong \overline{BC}$	
Similarly $\overline{AB} \cong \overline{QC}$	
Hence point A is midpoint RQ	
And $\overline{AD} \perp \overline{BC}$	Given
$\overline{BC} \| \overline{RQ}$	Opposite sides of parallelogram ABCQ
$\overline{AD} \| \overline{RQ}$	
Thus $\overline{AD} \perp$ is right bisector of $\overline{RQ}$	
similarly $\overline{BE}$ is a right bisector of $\overline{RP}$ and	
$\overline{CF}$ is right bisector of PQ	
$\therefore \perp^s$ $\overline{AD}, \overline{BE}, \overline{CF}$ are right bisector of sides of ΔPQR	
$\therefore \overline{AD}, \overline{BE}$ and $\overline{CF}$ are Concurrent	

Theorem12.1.6

The bisectors of the angles of a triangle are concurrent

Given

ΔABC

To Prove

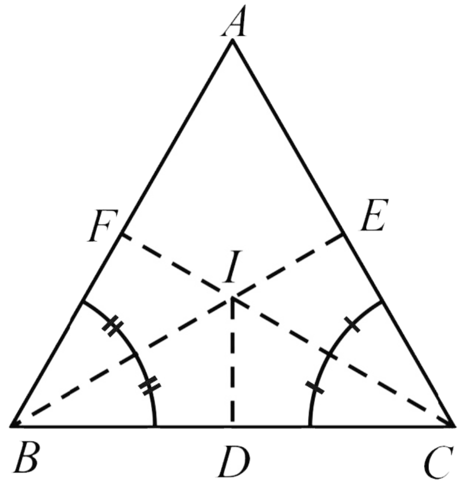
The bisector of ∠A, ∠B, and ∠C are concurrent

Construction:

Draw the bisectors of ∠B and ∠C which intersect at point I. From I, draw

$\overline{IF} \perp \overline{AB}$, $\overline{ID} \perp \overline{BC}$ and $\overline{IE} \perp \overline{CA}$

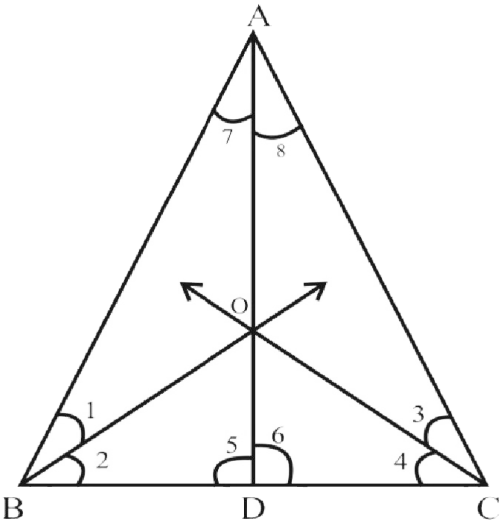
Proof



Statements	Reasons
$\overline{ID} \cong \overline{IF}$	(Any point on bisector of an angle is equidistance from its arms.
Similarly	
ID ≅ IE	
∴ $\overline{IE} \cong \overline{IF}$	Each ≅ ID
So the point I is on the bisector of ∠A … (i)	
Also the point I is on the bisectors of ∠ABC and ∠BCA … (ii)	Construction
Thus the bisector of ∠A, ∠B and ∠C are concurrent at I	{From (i) and (ii)}

Exercise 12.3

Q.1 Prove that the bisectors of the angles of base of an isosceles triangle intersect each other on its altitude.



Given

ΔABC

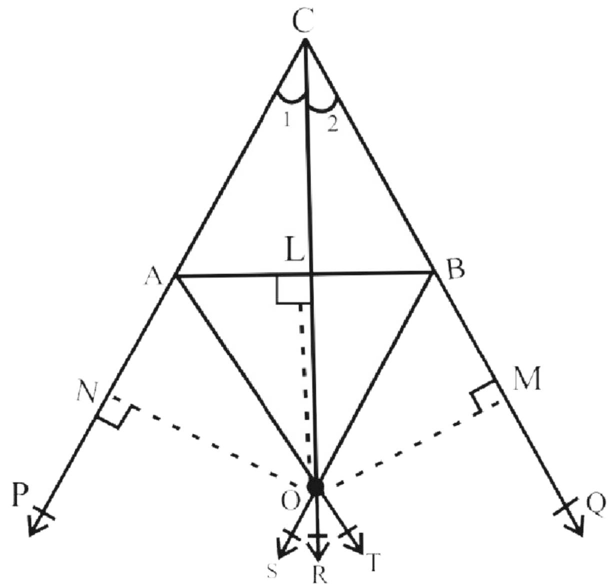
$\overline{AB} = \overline{AC}$ Due to isosceles triangle

Bisect $\angle B$ and $\angle C$ to intersect at point O Join A to D and extend to BC at D $\overline{AD}$ is the altitude of ΔABC $\overline{AD} \perp \overline{BC}$

Proof

Statements	Reasons
In ΔABC	
$\overline{AB} \cong \overline{AC}$	Given
$\angle B \cong \angle C$	Due to isosceles triangle opposite angle are congruent
$\frac{1}{2} m\angle B = \frac{1}{2} m\angle C$	Dividing both side by 2
$\angle 1 \cong \angle 3$	
$\Delta ABO \leftrightarrow \Delta ACO$	
$\overline{AO} = \overline{AO}$	
$\overline{AB} = \overline{AC}$	
$\overline{BO} \cong \overline{CO}$	Given
$\Delta ABO \cong \Delta ACO$	Due to isosceles triangle
$\Delta ABD \leftrightarrow \Delta ACD$	
$\overline{AD} \cong \overline{AD}$	
$\angle 7 \cong \angle 8$	
$\overline{AB} \cong \overline{AC}$	
$\Delta ABD \cong \Delta ACD$	
$\angle 5 + \angle 6 = 180$	
$\angle 5 = \angle 6 = 90^\circ$	
So $\overline{AD} \perp \overline{BC}$	Supplementary angles
$\overline{AD}$ Passes from point O	

Q.2 Prove that the bisectors of two exterior and third interior angle of a triangle are concurrent



Given

ΔABC

Exterior angles are $\angle ABQ$ and $\angle BAP$ $\overrightarrow{AT}$ and $\overrightarrow{BS}$ intersect each other at point O therefore join O to C

Draw the angle bisecter of C

$\angle 1 \cong \angle 2$

Construction

$\overline{OM} \perp \overrightarrow{CQ}, \overline{OL} \perp \overline{AB}$, $\overline{ON} \perp \overrightarrow{CP}$

Proof

Statements	Reasons
$\overline{ON} \cong \overline{OM}$...........(i)	
$\overline{OL} \cong \overline{OM}$...........(ii)	
$\overline{ON} \cong \overline{OL}$	
Hence Angle Bisector of C	Comparing equation (i) and (ii)
i,e $\angle 1 \cong \angle 2$	

Review Exercise 12

Q.1 Which of the following are true and which are false?

(i) Bisection means to divide into two equal parts (True)

(ii) Right bisection of line segment means to draw perpendicular which passes through the midpoint of line segment (True)

(iii) Any point on the right bisector of a line segment is not equidistant from its end points (False)

(iv) Any point equidistant from the end points of a line segment is on the right bisector of it (True)

(v) The right bisectors of the sides of a triangle are not concurrent (False)

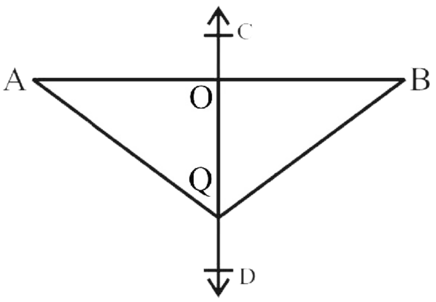
(vi) The bisectors of the angles of a triangle are concurrent (True)

(vii) Any point on the bisector of an angle is not equidistant from its arms (False)

(viii) Any point inside an angle equidistant from its arms, is on the bisector of it (True)

Q.2 If $\overleftrightarrow{CD}$ is right bisector of line segment $\overline{AB}$, then

(i) $m\overline{OA} =$ _______ (ii) $m\overline{AQ} =$ _______



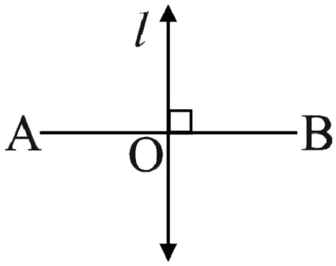
Solution

(i) $m\overline{OA} = m\overline{OB}$

(ii) $m\overline{AQ} = m\overline{BQ}$

Q.3 Define the following

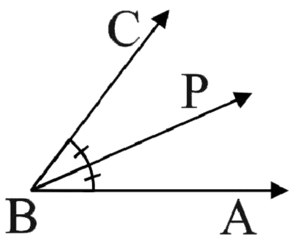
(i) Right Bisector of a Line Segment



A line l is called a right bisector of a line segment if l is perpendicular to the line segment and passes through its midpoint.

(ii) Bisector of an Angle

A ray BP is called the bisector of $m\angle ABC$, if P is a point in the interior of the angle and $m\angle ABP = m\angle PBC$.



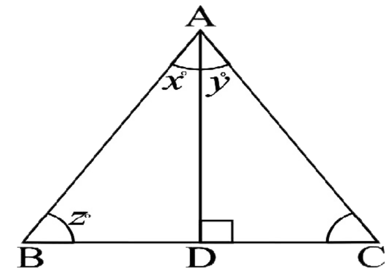
Q.4 The given triangle ABC is equilateral triangle and $\overline{AD}$ is bisector of angle A, then find, the values of unknown x^o, y^o and z^o.

Solution

In equilateral triangle all side are equal to each and there angle of the triangle equal to 60^o.

So

$\angle B = z^o = 60^o$



$\overline{AD}$ is the bisector of $\angle A$

$\angle A = 60^o$

$\therefore$ When angle A is bisected

$x^\circ = y^\circ$

$x^\circ = \frac{1}{2} m\angle A$

$= \frac{1}{2} \times 60^\circ$

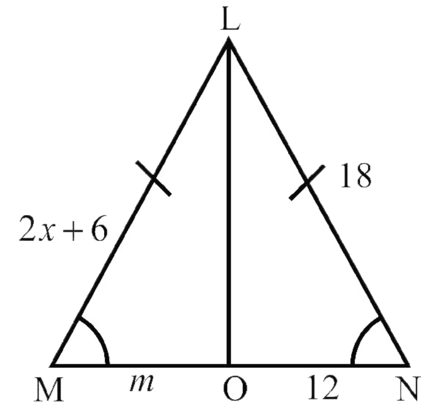
$x^\circ = 30^\circ$

$y^\circ = 30^\circ \qquad (\because x^\circ = y^\circ)$

So $x^o = y^o = 30^o$

Q.5 In the given congruent triangle LMO and LNO find the unknowns x and m given

$\Delta LMO \cong \Delta LNO$



$m\overline{LM} = m\overline{LN}$

$2x + 6 = 18$

$2x = 18 - 6$

$2x = 12$

$x = \frac{\cancel{12}^{6}}{\cancel{2}}$

$x = 6$ Unit

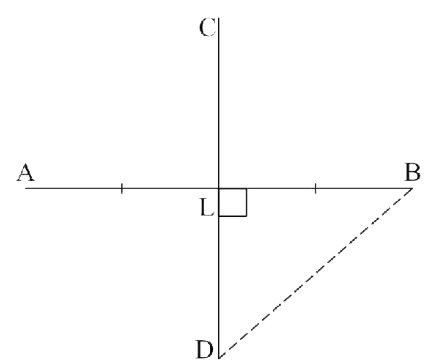
$m\overline{MO} = m\overline{ON}$

$\therefore m = 12$ unit

Q.6 CD is right bisector of the line segment AB

(i) If $m\overline{AB} = 6\text{cm}$ then find the $m\overline{AL}$ and $m\overline{LB}$

Solution



L is the midpoint of $\overline{AB}$

$\therefore m\overline{AL} = m\overline{LB}$

$m\overline{AL} = \frac{1}{2} mAB = \frac{1}{2} \times 6$

So $m\overline{AL} = 3\text{cm}$

$m\overline{LB} = 3cm \quad (\because m\overline{AL} = m\overline{LB})$

(ii) If $m\overline{BD} = 4cm$ **then find m** $\overline{AD}$

$m\overline{AD} = m\overline{BD}$ (Any point on the right bisector of a line segment is equidistant from its end points.)

$m\overline{AD} = 4$

$m\overline{AD} = 4cm$