CHAPTER NO. 2 (VECTORS AND EQUILIBRIUM)

Question 2.1:- Suppose, in rectangular coordinate system, a vector A has its tail at point P (-2, -3) and its tip at Q (3, 9). Determine the distance between these two points.

Solution:- Position vector of any point (a, b) is written as $\vec{r} = \text{a}\,\hat{\imath} + b\hat{\jmath}$.

Position vector of point P = $\overrightarrow{r_P}$ = $-2\,\hat{\imath} - 3\,\hat{\jmath}$

Position vector of point Q = $\overrightarrow{r_Q}$ = $3\,\hat{\imath} + 9\hat{\jmath}$

$\vec{A} = \overrightarrow{r_Q} - \overrightarrow{r_P} = (3\,\hat{\imath} + 9\hat{\jmath}) - (-2\,\hat{\imath} - 3\,\hat{\jmath})$

$\vec{A} = 5\,\hat{\imath} + 12\hat{\jmath}$

Magnitude of a vector $\vec{A} = \text{x}\,\hat{\imath} + y\hat{\jmath}$ is determined by $\text{A} = \sqrt{x^2 + y^2}$

$\text{A} = \sqrt{5^2 + 12^2} = \sqrt{169}$

A = 13 units

Question 2.2:- A certain corner of a room is selected as the origin of a rectangular coordinate system. If an insect is sitting on an adjacent wall at a point having coordinates (2, 1), where the units are in metres, what is the distance of the insect from this corner of the room?

Solution:- Position vector of any point (a, b) is written as $\vec{r} = \text{a}\,\hat{\imath} + b\hat{\jmath}$.

Position vector of point (2, 1) = $\vec{r} = (2\,\hat{\imath} + \hat{\jmath})$ m

Magnitude of position vector $\vec{r} = \text{q}\,\hat{\imath} + b\hat{\jmath}$ is determined by $\text{r} = \sqrt{a^2 + b^2}$

$\text{r} = \sqrt{2^2 + 1^2} = \sqrt{5}$

r = 2.2 m

Question 1.3:- What is the unit vector in the direction of the vector $\vec{A} = 4\,\hat{\imath} + 3\hat{\jmath}$?

Solution:- $\vec{A} = 4\,\hat{\imath} + 3\hat{\jmath}$

Unit vector in the direction of a vector $\vec{A}$ is determined as $\hat{A} = \frac{\vec{A}}{A}$

Magnitude of $\vec{A} = \text{A} = \sqrt{4^2 + 3^2} = \sqrt{25}$

A = 5

$\hat{A} = \frac{\vec{A}}{A} = \frac{4\,\hat{\imath} + 3\hat{\jmath}}{5}$

$\boldsymbol{\hat{A} = \frac{4\,\hat{\imath} + 3\hat{\jmath}}{5}}$

Question 2.4:- Two particles are located at $r_1 = 3\,\hat{\imath} + 7\hat{\jmath}$ and $r_2 = -2\,\hat{\imath} + 3\,\hat{\jmath}$ respectively. Find both the magnitude of the vector ($r_2 - r_1$) and its orientation with respect to x-axis.

Solution:- $\overrightarrow{r_1} = 3\,\hat{\imath} + 7\,\hat{\jmath}$

$\overrightarrow{r_2} = -2\,\hat{\imath} + 3\,\hat{\jmath}$

$\overrightarrow{r_2} - \overrightarrow{r_1} = (-2\,\hat{\imath} + 3\,\hat{\jmath}) - (3\,\hat{\imath} + 7\,\hat{\jmath}) = (-2\,\hat{\imath} - 3\,\hat{\imath}) + (3\,\hat{\jmath} - 7\,\hat{\jmath})$

$\vec{r_2} - \vec{r_1} = -5\,\hat{\imath} - 4\,\hat{\jmath}$

Magnitude of $\vec{r_2} - \vec{r_1} = |\vec{r_2} - \vec{r_1}| = \sqrt{(-5)^2 + (-4)^2} = \sqrt{41}$

$|\vec{r_2} - \vec{r_1}| = 6.4$

Since both x and y-components of vector $\vec{r_2} - \vec{r_1}$ are negative, so vector lies in third quadrant.

First of all we find $\phi = \tan^{-1}(\frac{y}{x})$ by neglecting negative signs of y and x.

$\phi = \tan^{-1}(\frac{4}{5}) = \tan^{-1}(0.8) = 38.65^o = 39^o$

Orientation of $\vec{r_2} - \vec{r_1}$ with respect to x-axis in third quadrant is $\theta = 180^o + \phi$

$\theta = 180^o + 39^o$

$\underline{\theta = 219^o}$

Question 2.5:- If a vector B is added to vector A, the result is $6\,\hat{\imath} + \hat{\jmath}$. If B is subtracted from A, the result is $-4\,\hat{\imath} + 7\,\hat{\jmath}$. What is the magnitude of vector A?

Solution:- $\vec{A} + \vec{B} = 6\,\hat{\imath} + \hat{\jmath}$

$\vec{A} - \vec{B} = -4\,\hat{\imath} + 7\hat{\jmath}$

Adding both sides of these two equations gives

$(\vec{A} + \vec{B}) + (\vec{A} - \vec{B}) = (6\,\hat{\imath} + \hat{\jmath}) + (-4\,\hat{\imath} + 7\hat{\jmath})$

$2\,\vec{A} = 2\,\hat{\imath} + 8\,\hat{\jmath}$

$\vec{A} = \hat{\imath} + 4\,\hat{\jmath}$ $\therefore$ Dividing both sides by 2

$A = \sqrt{(1)^2 + (4)^2} = \sqrt{17}$

$\underline{A = 4.1}$

Question 2.6:- Given that $A = 2\,\hat{\imath} + 3\,\hat{\jmath}$ and $B = 3\,\hat{\imath} - 4\,\hat{\jmath}$, find the magnitude and angle of (a) C = A + B and (b) D = 3A – 2B.

Solution:- $\vec{A} = 2\,\hat{\imath} + 3\,\hat{\jmath}$

$\vec{B} = 3\,\hat{\imath} - 4\,\hat{\jmath}$

$\vec{C} = \vec{A} + \vec{B}$ $\vec{C} = (2\,\hat{\imath} + 3\,\hat{\jmath}) + (3\,\hat{\imath} - 4\,\hat{\jmath})$ $\vec{C} = (2\,\hat{\imath} + 3\,\hat{\imath}) + (3\,\hat{\jmath} - 4\,\hat{\jmath})$ $\vec{C} = 5\,\hat{\imath} - \hat{\jmath}$ $C = \sqrt{(5)^2 + (-1)^2} = \sqrt{26}$ **$\underline{C = 5.1}$** x-component of $\vec{C}$ is positive and y-component is negative, $\vec{C}$ lies in 4th quadrant. We can find $\phi = \tan^{-1}(1/5) = 11^o$	$\vec{D} = 3\,\vec{A} - 2\,\vec{B}$ $\vec{D} = 3\,(2\,\hat{\imath} + 3\,\hat{\jmath}) - 2\,(3\,\hat{\imath} - 4\,\hat{\jmath})$ $\vec{D} = (6\,\hat{\imath} + 9\,\hat{\jmath}) + (-6\,\hat{\imath} + 8\,\hat{\jmath})$ $\vec{D} = (6\,\hat{\imath} - 6\,\hat{\imath}) + (9\,\hat{\jmath} + 8\,\hat{\jmath})$ $\vec{D} = 17\,\hat{\jmath}$ $D = \sqrt{(0)^2 + (17)^2} = \sqrt{289}$ **$\underline{D = 17}$** x-component of $\vec{D}$ is zero and y-component is

$\theta = 360^o - \phi = 360^o - 11^o$ **$\theta = 349^o$**	positive, $\vec{D}$ lies on positive y-axis. $\theta = \tan^{-1}(17/0) = \tan^{-1}(\infty)$ **$\theta = 90^o$**

Question 2.7:- Find the angle between two vectors, A = 5 $\hat{\imath} + \hat{\jmath}$ and B = 2 $\hat{\imath}$ + 4 $\hat{\jmath}$.

Solution:- $\vec{A} = 5\,\hat{\imath} + \hat{\jmath}$

$\vec{B} = 2\,\hat{\imath} + 4\,\hat{\jmath}$

$\vec{A} \cdot \vec{B} = A\,B\cos\theta$

$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{A\,B}$

$\theta = \cos^{-1}\left(\frac{\vec{A}\cdot\vec{B}}{A\,B}\right)$

$\theta = \cos^{-1}\frac{(5\,\hat{\imath}+\hat{\jmath}).(2\,\hat{\imath}+4\,\hat{\jmath})}{\sqrt{(5)^2+(1)^2}\sqrt{(2)^2+(4)^2}} = \cos^{-1}\frac{(5)(2)+(1)(4)}{\sqrt{26}\sqrt{20}} = \cos^{-1}\left(\frac{14}{\sqrt{520}}\right)$

$\theta = \cos^{-1}(0.61)$

$\theta = 52^o$

Question 2.8:- Fid the work done when the point of application of the force 3 $\hat{\imath}$ + 2 $\hat{\jmath}$ moves in a straight line from the point (2, -1) to the point (6, 4).

Solution:- $\vec{F} = 3\,\hat{\imath} + 2\,\hat{\jmath}$

Position vector of initial point (2, -1) = $\vec{r_1} = 2\,\hat{\imath} - \hat{\jmath}$

Position vector of final point (6, 4) = $\vec{r_2} = 6\,\hat{\imath} + 4\,\hat{\jmath}$

Displacement = $\vec{d} = \vec{r_2} - \vec{r_1} = (6\,\hat{\imath} + 4\,\hat{\jmath}) - (2\,\hat{\imath} - \hat{\jmath})$

$\vec{d} = 4\,\hat{\imath} + 5\,\hat{\jmath}$

$W = \vec{F} \cdot \vec{d}$

$W = (3\,\hat{\imath} + 2\,\hat{\jmath}).(4\,\hat{\imath} + 5\,\hat{\jmath})$

W = (3)(4) + (2)(5)

W = 12 + 10

W = 22 units

Question 2.9:- Show that the three vectors $\hat{\imath} + \hat{\jmath} + \hat{k}$, $2\,\hat{\imath} - 3\,\hat{\jmath} + \hat{k}$ and $4\,\hat{\imath} + \hat{\jmath} - 5\,\hat{k}$ are mutually perpendicular.

Solution:- $\vec{A} = \hat{\imath} + \hat{\jmath} + \hat{k}$

$\vec{B} = 2\,\hat{\imath} - 3\,\hat{\jmath} + \hat{k}$

$\vec{C} = 4\,\hat{\imath} + \hat{\jmath} - 5\,\hat{k}$

Two vectors are mutually perpendicular if their scalar or dot product is zero.

$\vec{A} \cdot \vec{B} = (\hat{\imath} + \hat{\jmath} + \hat{k}).(2\,\hat{\imath} - 3\,\hat{\jmath} + \hat{k}) = (1)(2)+(1)(-3)+(1)(1) = 0$

$\vec{B} \cdot \vec{C} = (2\,\hat{\imath} - 3\,\hat{\jmath} + \hat{k}).(4\,\hat{\imath} + \hat{\jmath} - 5\,\hat{k}) = (2)(4)+(-3)(1)+(1)(-5) = 0$

$\vec{C} \cdot \vec{A} = (4\,\hat{\imath} + \hat{\jmath} - 5\,\hat{k}).(\hat{\imath} + \hat{\jmath} + \hat{k}) = (4)(1)+(1)(1)+(-5)(1) = 0$

$\vec{A} \cdot \vec{B}, \vec{B} \cdot \vec{C}$ and $\vec{C} \cdot \vec{A}$ all are zero, we can deduce that $\vec{A}$, $\vec{B}$ *and* $\vec{C}$ are mutually perpendicular.

Question 2.10:- Given that A = $\hat{\imath} - 2\,\hat{\jmath} + 3\,\hat{k}$ and B = $3\,\hat{\imath} - 4\,\hat{k}$, find the projection of A on B.

Solution:- $\vec{A} = \hat{\imath} - 2\,\hat{\jmath} + 3\,\hat{k}$

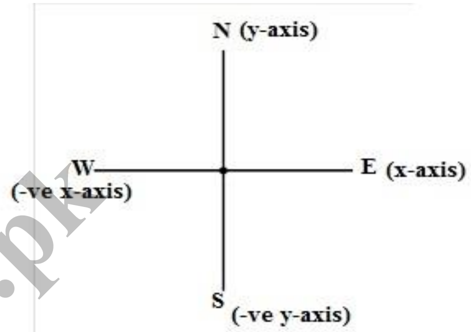
$\vec{B} = 3\,\hat{\imath} - 4\,\hat{k}$

Projection of $\vec{A}$ *on* $\vec{B}$ = A cos θ = $\vec{A} \cdot \hat{B} = \vec{A} \cdot \frac{\vec{B}}{B}$

A cos θ = $\frac{(\hat{\imath} - 2\,\hat{\jmath} + 3\,\hat{k}).(3\,\hat{\imath} - 4\,\hat{k})}{\sqrt{(3)^2 + (-4)^2}} = \frac{(1)(3)+(-2)(0)+(3)(-4)}{5}$

Projection of $\vec{A}$ *on* $\vec{B}$ = A cos θ = $\frac{-9}{5}$

Question 2.11:- Vectors A, B and C are 4 units north, 3 units west and 8 units east, respectively. Describe carefully (a) A x B (b) A x C (c) B x C.



Solution:- We can create a correspondence between directions of map and Cartesian coordinates.

$\vec{A} = 4\,\hat{\jmath}$

$\vec{B} = -3\,\hat{\imath}$

$\vec{C} = 8\,\hat{\imath}$

(a) $\vec{A}$ x $\vec{B}$ = $(4\,\hat{\jmath})$ x $(-3\,\hat{\imath})$ = -12 $(\hat{\jmath}$ x $\hat{\imath})$ = -12 $(-\hat{k})$ = 12 $\hat{k}$ (12 units vertically upwards)

(b) $\vec{A}$ x $\vec{C}$ = $(4\,\hat{\jmath})$ x $(8\,\hat{\imath})$ = 32 $(\hat{\jmath}$ x $\hat{\imath})$ = 32 $(-\hat{k})$ = -32 $\hat{k}$ (32 units vertically downwards)

(c) $\vec{B}$ x $\vec{C}$ = $(-3\,\hat{\imath})$ x $(8\,\hat{\imath})$ = -24 $(\hat{\imath}$ x $\hat{\imath})$ = $\vec{0}$ (Zero)

Question 2.12:- The torque or turning effect of force about a given point is given by r x F where r is the vector from the given point to the point of application of F. Consider a force F = -3 $\hat{\imath} + \hat{\jmath} + 5\,\hat{k}$ (newton) acting on the point 7 $\hat{\imath} + 3\,\hat{\jmath} + \hat{k}$ (m). What is the torque in N m about the origin?

Solution:- $\vec{F} = (-3\,\hat{\imath} + \hat{\jmath} + 5\,\hat{k})$ N

$\vec{r} = (7\,\hat{\imath} + 3\,\hat{\jmath} + \hat{k})$ m

We know that $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}}\ \boldsymbol{x}\ \vec{\boldsymbol{F}}$

Writing the cross product in determinant form as under:-

$$\vec{\tau} = \begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 7 & 3 & 1 \\ -3 & 1 & 5 \end{matrix}$$

$\vec{\tau} = \hat{\imath}\,[(3)(5)-(1)(1)] + \hat{\jmath}\,[(1)(-3)-(7)(5)] + \hat{k}\,[(7)(1)-(3)(-3)]$

$\underline{\vec{\tau} = (14\,\hat{\imath} - 38\,\hat{\jmath} + 16\,\hat{k})\text{ N m}}$

Question 2.13:- The line of action of force F = $\boldsymbol{\hat{\imath}} - 2\,\boldsymbol{\hat{\jmath}}$, passes through a point whose position vector is (-$\boldsymbol{\hat{\jmath}} + \boldsymbol{\hat{k}}$). Find (a) the moment of F about the origin, (b) the moment of F about the point of which the position vector is $\boldsymbol{\hat{\imath}} + \boldsymbol{\hat{k}}$.

Solution:- $\vec{F} = \hat{\imath} - 2\,\hat{\jmath}$

$\overrightarrow{r_P}$ = - $\hat{\jmath} + \hat{k}$

(a) We know that $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}}\ \boldsymbol{x}\ \vec{\boldsymbol{F}}$

In order to determine the torque of $\vec{F}$ about origin $\overrightarrow{r_P} = \vec{r}$

Writing the cross product in determinant form as under:-

$$\vec{\tau} = \begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{matrix}$$

$\vec{\tau} = \hat{\imath}$ [(-1)(0)-(1)(-2)] + $\hat{\jmath}$ [(1)(1)-(0)(0)] + $\hat{k}$ [(0)(-2)-(1)(1)]

<u>$\vec{\boldsymbol{\tau}} = \boldsymbol{2\,\hat{\imath} + \hat{\jmath} + \hat{k}}$</u>

(b) In order to find the toque of $\vec{F}$ about a point A whose position vector is $\overrightarrow{r_A} = \hat{\imath} + \hat{k}$ (other than origin), first of all we have to find position vector between point of application of force P and the pivot i.e. A as under:-

$\vec{r} = \overrightarrow{r_P}$ - $\overrightarrow{r_A}$ = (- $\hat{\jmath} + \hat{k}$) – ($\hat{\imath} + \hat{k}$)

$\vec{r} = -\,\hat{\imath} - \hat{\jmath}$

Now, we will find torque as $\vec{\tau} = \vec{r}\ x\ \vec{F}$.

Writing the cross product in determinant form as under:-

$$\vec{\tau} = \begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & -1 & 0 \\ 1 & -2 & 0 \end{matrix}$$

$\vec{\tau} = \hat{\imath}$ [(-1)(0)-(0)(-2)] + $\hat{\jmath}$ [(0)(1)-(0)(0)] + $\hat{k}$ [(-1)(-2)-(-1)(1)]

<u>$\vec{\boldsymbol{\tau}} = \boldsymbol{3\ \hat{k}}$</u>

Question 2.14:- The magnitude of dot and cross products of two vectors are $6\sqrt{3}$ and 6 respectively. Find the angle between the vectors.

Solution:- $|\vec{A}\,.\,\vec{B}|$ = A B cos θ = $6\sqrt{3}$ --- Eq. (1)

$|\vec{A}$ x $\vec{B}|$ = A B sin θ = 6 -- Eq. (2)

Divide Eq. (1) by Eq. (2)

$\frac{A\ B\ sin\ \theta}{A\ B\ cos\ \theta} = \frac{6}{6\sqrt{3}}$

$\frac{sin\ \theta}{cos\ \theta} = \frac{1}{\sqrt{3}}$

tan θ = $\frac{1}{\sqrt{3}}$

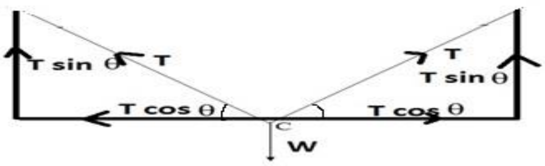
$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\theta = 30^o$

Question 2.15:- A load of 10.0 N is suspended from a clothes line. This distorts the line so that it makes an angle of 15° with the horizontal at each end. Find the tension in the clothes line.

Solution:- W = 10.0 N

$\theta = 15^o$



We can draw the free body diagram as shown in figure. As angle of distortion on both sides is same, so tension on both sides is same.

Apply first condition of equilibrium on x-axis as $\Sigma F_x = 0$.

$+ T\cos\theta + (-T\cos\theta) = 0$

Apply first condition of equilibrium on y-axis as $\Sigma F_y = 0$.

$+ T\sin\theta + T\sin\theta - W = 0$

$2T\sin\theta = W$

$T = \frac{W}{2\sin\theta} = \frac{10}{2\sin 15^o}$

T = 19.3 N

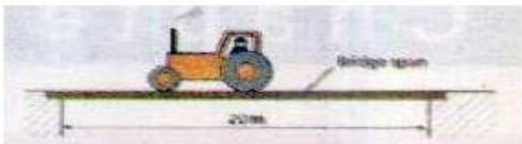
Question 2.16:- A tractor of weight 15,000 N crosses a single span bridge of weight 8000 N and the length 21.0 m. The bridge span is supported half a metre from either end. The tractor's front wheels take 1/3 of total weight of the tractor, and the rear wheels are 3 m behind the front wheel. Calculate the force on the bridge supports when the rear wheels are at the middle of the bridge span.

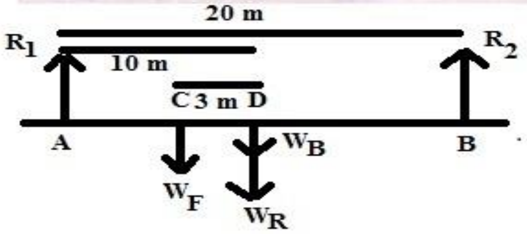
Solution:- Weight of the bridge = W_B = 8,000 N

Total weight of the tractor = W = 15,000 N

Weight of front wheels = $W_F = \frac{W}{3}$ = 15,000/3 = 5,000 N

Weight of rear wheels = $W_R = W - W_F$ = 15,000 - 5,000 = 10, 000 N





Consider the free body diagram. The upward directed reaction forces of the bridge supports are R_1 and R_2 are acting on point A and B, respectively. Weight of the bridge W_B and weight of the rear wheel W_R are acting downwards at point D exactly in the middle of the bridge. Weight of front wheels W_F is acting downward at point C.

Given AB = 20 m, AD = 10 m, AC = 7 m, CD = 3 m and BD = 10 m

All forces are in vertical direction, we can apply first condition of equilibrium $\Sigma F_y = 0$.

$R_1 + R_2 + (-W_B) + (-W_R) + (-W_F) = 0$

$R_1 + R_2 = W_B + W_R + W_F$

$R_1 + R_2 = 8,000 + 10,000 + 5,000$

$R_1 + R_2 = 23,000$ -- Eq. (1)

Now, we will apply second condition of equilibrium to the geometry as $\Sigma\tau = 0$.

We can choose A as pivot point. It is clear that R_2 will produce counter-clockwise torque about A. W_B, W_F and W_R will produce clockwise torque about A.

$(R_2 \times AB) - (W_F \times AC) - (W_B + W_R) \times AD = 0$

$20\ R_2 - (5,000 \times 7) - (10,000 + 8,000) \times 10 = 0$

$20\ R_2 = 35,000 + 180,000 = 215,000$

$R_2 = 215,000/20$

R_2 = 10, 750 N = 10.75 kN put value of R_2 in Eq. (1)

$R_1 + 10,750 = 23,000$

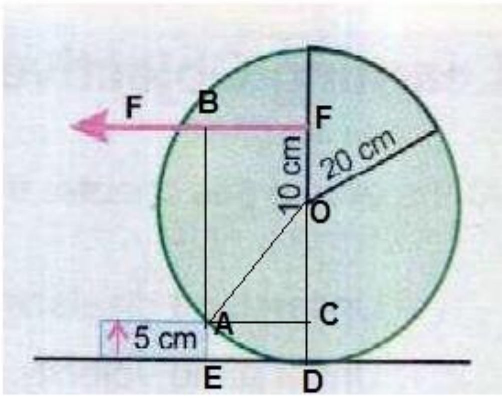
$R_1 = 23,000 - 10,750$

R_1 = 12,250 N = 12.25 kN

Question 2.17:- A spherical ball of weight 50 N is to be lifted over the step as shown in the figure. Calculate the minimum force needed just to lift it above the floor.

Solution:- W = 50 N

OA = OD = 20 cm, AE = CD = 5 cm and OC = 15 cm



In right angle triangle OAC, $OA^2 = OC^2 + AC^2$

$AC^2 = OA^2 - OC^2 = (20)^2 - (15)^2 = 175$

AC = 13.2 cm = 13 cm

AB = OC + OF = 15 + 10

AB = 25 cm

In order to lift the ball over the step at point A, the force F acting at point F is producing counter-clockwise torque while weight of the ball acting downwards at O is producing the clockwise torque at point A.

Apply second condition of equilibrium at point A as $\Sigma\tau = 0$.

F x AB – W x AC = 0

F x AB = W x AC

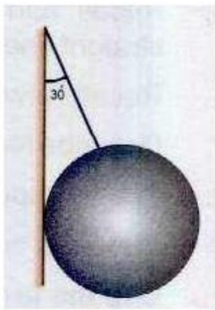
F x 25 = 50 x 13

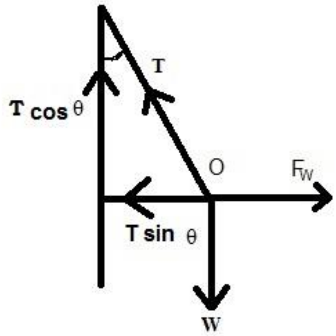
F = 26 N

Question 2.18:- A uniform sphere of weight 10.0 N is held by a string attached to a frictionless wall so that the string makes an angle of 30° with the wall as shown in the figure. Find the tension on the string and the force exerted on the sphere by the wall.

Solution:- W = 10.0 N

θ = 30°





The tension in the string can be divided into two parts as T cos θ towards the wall and T sin θ upwards. Weight of the ball is acting downwards at point O and force of the wall is acting on the ball rightwards.

Apply first condition of equilibrium along x-axis as $\Sigma F_x = 0$.

$+F_W + (-T \sin\theta) = 0$

$F_W = T \sin\theta$ --- Eq. (1)

Apply first condition of equilibrium along y-axis as $\Sigma F_y = 0$.

$-W + (T \cos\theta) = 0$

$W = T \cos\theta$

$T = W/\cos\theta = 10/\cos 30^\circ = 10/0.866$

<u>T = 11.547 N = 11.6 N</u> Put value of T in Eq. (1)

$F_W = (11.54) \sin 30^\circ = 11.54\ (0.5)$

<u>F_W = 5.77 N</u>