Scalar:
A quantity which has magnitude only is called scalar. $e.g$ time, volume, speed and work e.tc the scalars are denoted by letters.

Vectors:
A quantity which has both magnitude and direction is called vector. $e.g$ $Velocity$, displacement, force and torque $e.t.c.$ A vector $say\ V\ is\ denoted\ by\ \vec{V}\ or\ V$ e.t.c. or by bold face letter v.

Geometrical interpretation of Vector:
Geometrically, A vector is represented by a line segment AB with A its initial point and B its terminal point. It is often found convenient to denoted a vector by an arrow and written either as $\overrightarrow{AB}$ or as a bold face symbol like $v\ or\ in\ underlined\ form \underline{v}$.

Magnitude of a vector.
Let $v\ be\ a\ vector$, then its magnitude is denoted by $|\vec{v}|$
$or\ simly\ v. it\ is\ also\ called\ norm\ or\ length\ of\ vector$
$if\ the\ line\ \overrightarrow{AB}\ represent\ a\ vector, then\ distance\ from$
Point A to point B will be magnitude of $\overrightarrow{AB}$ $and\ is$
Denoted by $|\overrightarrow{AB}|$

Unit vector:
A vector whose magnitude is one (unity) is called unit vector. Unit vector of $\underline{v}\ is\ written\ as\ \hat{v}$ (read as v hat) and is defined as $\hat{v} = \frac{\underline{v}}{|\underline{v}|}$

Null vector:
A vector whose magnitude is zero but no specific direction is called null or zero vector. It is denoted by $\vec{o}$

Negative Vectors:
Two vectors are said to be negative of each other if they have the same magnitude but opposite direction.
If $\overline{AB} = \underline{v}$ then $\overline{BA} = -\overline{AB} = -\underline{v}\ and\ |\overline{BA}| = |-\overline{AB}|$
(
$\because\ the\ magnitude\ of\ a\ vector\ is\ a\ non\ negative\ member)$

Multiplication of vector by a scalar: let $k\ be\ a\ scalar$
Number ($k \in$
$R\ and\ \underline{v}\ be\ a\ vector. then\ k\vec{v}\ is\ a\ vector$
Which is k times to vector $\vec{v}$.

(i) $\vec{v}\ and\ k\vec{v}\ are\ in\ same\ direction\ if\ k > 0.$
(ii) $\vec{v}\ and\ k\vec{v}\ are\ in\ opposite\ direction\ if\ k < o$

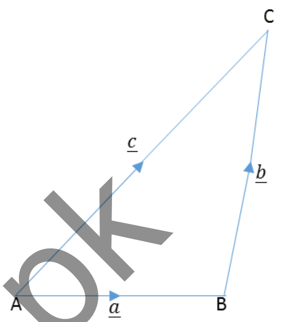
Equal Vectors:
Two vectors
$\overrightarrow{AB}\ and\ \overrightarrow{CD}\ are\ said\ to\ be\ equal, if\ they\ have\ same\ magnitude\ and\ same\ direction\ i.e\ |\overrightarrow{AB}| = |\overrightarrow{CD}|$

Parallel vectors:
Two vectors $\vec{u}, and\ \vec{v}\ are\ said\ to\ be\ parallel\ if\ \vec{u} \times k\ \vec{v}$
Or $\vec{v} \times k\ \vec{u}$ if $\vec{u} \times$
$k\ \vec{v}\ then\ \vec{u}\ and\ \vec{v}\ are\ same\ direction$
$if\ k > 0\ also\ \vec{u}\ and\ \vec{v}\ are\ in\ opposite\ direction\ if$
$k < o$

Addition of two vectors:
Addition of two vectors is explained by following two laws.

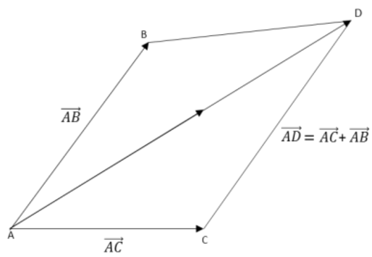
i. **Triangle law of addition:**
If two vectors
$\underline{a}\ and\ \underline{b}\ are\ represented\ by\ two$
Sides $AB\ and\ BC\ \ of\ \Delta ABC$ s. that the terminal point $\underline{a}\ coinside\ with\ the\ initial\ point\ of\ \underline{b}.$
Then $the\ third\ side\ AC\ of\ the\ triangle\ gives$
Vector sum AC of the triangle gives vector sum $\underline{a} + \underline{b}$

$$i.e\ \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \Rightarrow \underline{a} + \underline{b} = \overrightarrow{AC}$$



ii. **Parallelogram law of addition:**
If two vectors
$\underline{u}\ and\ \underline{v}\ are\ represented\ by\ two$
Sides
$AB\ and\ AC\ of\ ||gram\ as\ shown\ in\ figure. then$
Diagonal $AD\ give\ the\ sum\ of\ \overrightarrow{AB}\ and\ \overrightarrow{AC}\ i.e$

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC} = \underline{u} + \underline{v}$$



Subtraction of two vectors:
$let\ ABCD\ be\ a\ ||gram.$

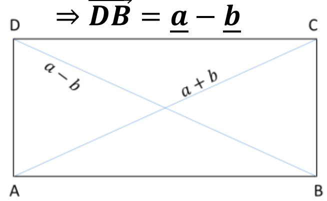
$$\overrightarrow{AB} = \overrightarrow{\boldsymbol{DC}} = \vec{\boldsymbol{a}}$$
$$\overrightarrow{BC} = \overrightarrow{\boldsymbol{AD}} = \vec{\boldsymbol{b}}$$

In $\Delta \boldsymbol{ABC}$

$$\overrightarrow{AC} = \overrightarrow{\boldsymbol{AB}} = \overrightarrow{\boldsymbol{BC}}$$
$$\overrightarrow{AC} = \vec{\boldsymbol{a}} = \vec{\boldsymbol{b}}$$

In $\Delta \boldsymbol{ABD}$, $\overrightarrow{AB} + \overrightarrow{\boldsymbol{BD}} = \overrightarrow{\boldsymbol{AD}}$

$$\Rightarrow \overrightarrow{BD} = \overrightarrow{\boldsymbol{AD}} - \overrightarrow{\boldsymbol{AB}}$$
$$\Rightarrow \overrightarrow{BD} = \vec{\boldsymbol{b}} - \vec{\boldsymbol{a}} \Rightarrow \overrightarrow{DB} = -(\boldsymbol{b} - \boldsymbol{a})$$
$$\Rightarrow \overrightarrow{\boldsymbol{DB}} = \underline{\boldsymbol{a}} - \underline{\boldsymbol{b}}$$



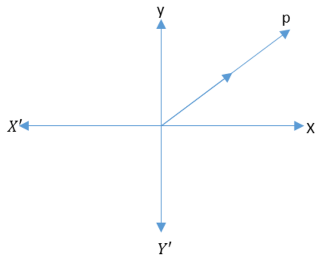
Position vector:

The vector whose initial point is at called position vector of point p.

Position vector of $p = \overrightarrow{OP}$

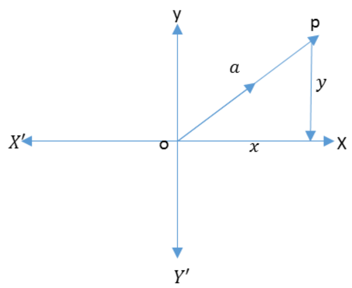
$$\overrightarrow{AB} = \overrightarrow{\boldsymbol{DC}} = \vec{\boldsymbol{a}}$$

$$\overrightarrow{AB} = \overrightarrow{\boldsymbol{DC}} = \vec{\boldsymbol{a}}$$



Vectors in plane:

$$let\ \overrightarrow{OP} = \underline{a}\ be\ a\ vector\ in\ xy$$
$$= plane.\ we\ resolve\ x$$
$$and\ y\ be\ the\ horizonatal\ and\ vertical\ components\ of\ a\ respectively. Then\ \underline{a}$$
$$= [x, y]\ \ is\ a\ vector\ in\ xy - plane$$



Addition:

$$let\ \vec{a} = [x_1, y_1],\ \ \vec{b}$$
$$= [x_2, y_2]\ be\ two\ vectors\ in\ plane$$
$$then\ \vec{a} + \vec{b} = [x_1, y_1] +\ [x_2, y_2], [x_1 + x_2, y_1 + y_2]$$

Subtraction:

$$let\ \vec{a} = [x_1, y_1],\ \ \vec{b}$$
$$= [x_2, y_2]\ be\ two\ vectors\ in\ plane$$
$$then\ \vec{a} - \vec{b} = [x_1, y_1] - [x_2, y_2], [x_1 - x_2, y_1 - y_2]$$

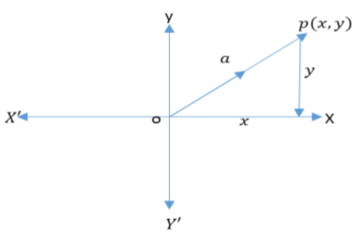
Magnitude of a vector in plane:

Let $\vec{\boldsymbol{a}} = [\boldsymbol{x}, \boldsymbol{y}]$ $\ be\ vector\ in\ plane\ as\ shown\ in\ fig.$

Then by pat agora's theorem

$$\left|\overrightarrow{OP}\right|^2 = \left|\overrightarrow{OA}\right|^2 + \left|\overrightarrow{AP}\right|^2$$
$$|\vec{a}|^2 = x^2 + y^2$$

⇨ $a = \sqrt{x^2 + y^2}\ \ \ which\ is\ magnitude\ of\ \ \vec{a}\ .$



Another notation for represent vectors in plane:

$$let\ \overline{OA}\ and\ \overline{OB}\ be\ two\ unit\ vectors\ along\ x - axis\ and\ y - axis\ respectively. then\ \left|\overline{OA}\right| = 1\ \ \left|\overline{OB}\right| = 1$$

as $\overrightarrow{OA} = [1,0]$ $\ ,\overrightarrow{OB} = [0,1]$

Then $\hat{\imath} = [1,0],\ j = [o, 1]$

The ratio formula

Statement:

$$let\ A\ and\ B\ be\ two\ points\ whose\ position\ vectors\ \underline{a}\ and\ \underline{b}\ respectively. if\ a\ point\ p\ divides\ AB\ in\ the\ ratio\ p{:}q\ then\ the\ position\ vector\ of\ p\ is\ given$$

By

$$\underline{r} = \frac{q\underline{a} + p\underline{b}}{p + q}$$

Proof:

In fig.

$$\overrightarrow{OA} = \vec{a},\qquad \overrightarrow{OB} = \vec{b}$$
$$\overrightarrow{OP} = \vec{r}$$

Given that

$$\overrightarrow{AP}{:}\overrightarrow{PB} = p{:}q\ \Rightarrow \frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{p}{q} \to (i)$$
$$\Delta OAP,\qquad \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$
$$\overrightarrow{AP} = \overrightarrow{OA} - \overrightarrow{OP} = \vec{r} - \vec{a}$$

In $\Delta\ OAB,\qquad \overrightarrow{OB} = \overrightarrow{OP} + \overrightarrow{PB}$

$$\overrightarrow{PB} = \overrightarrow{OB} + \overrightarrow{OP} = \vec{b} - \vec{r}$$

By (i)

$$\Rightarrow\ \frac{\vec{r} - \vec{a}}{\vec{b} - \vec{r}} = \frac{p}{q} \Rightarrow q\left(\vec{F} - \vec{a}\right) = p\left(\vec{b} - \vec{r}\right)$$

⇨ $q\vec{r} - q\vec{a} = p\vec{b} - p\vec{r} \Rightarrow\ q\vec{r} + p\vec{r} = p\vec{b} + q\vec{a}$

$$\Rightarrow (q + p)\vec{r} = p\vec{b} + q\vec{a}$$
$$\Rightarrow \vec{r} = \frac{p\vec{b} + q\vec{a}}{(q + p)}$$
$$Hence \Rightarrow \vec{r} = \frac{q\vec{b} + p\vec{a}}{(q + p)}$$

Corollary:

$$if\ p\ is\ the\ mid\ point\ of\ AB\ then\ p{:}q = 1{:}1$$

So, $\underline{\boldsymbol{r}} = \dfrac{\underline{\boldsymbol{a}}+\underline{\boldsymbol{b}}}{\boldsymbol{2}}$

Exercise NO 7.1

Question No.1

Write the vector $\overrightarrow{\mathbf{PQ}}$ in the form x $\underline{\boldsymbol{i}}$ + y $\underline{\boldsymbol{j}}$.

(i) $\boldsymbol{P}(\mathbf{2}, \mathbf{3})$, $\boldsymbol{Q}(\mathbf{6}, -\mathbf{2})$

$\overrightarrow{PQ}$= $(x_2 - x_1)\ \underline{\boldsymbol{i}} + (y_2 - y_1)\ \underline{\boldsymbol{j}}$

$\overrightarrow{PQ}$ = (6–2) $\underline{\boldsymbol{i}}$ + (–2–3) $\underline{\boldsymbol{j}}$

$\overrightarrow{PQ}$ = 4 $\underline{\boldsymbol{i}}$ – 5 $\underline{\boldsymbol{j}}$

(ii) $\boldsymbol{P}(\mathbf{0}, \mathbf{5})$, $\boldsymbol{Q}(-\mathbf{1}, -\mathbf{6})$

$\overrightarrow{PQ}$ = $(x_2 - x_1)\ \underline{\boldsymbol{i}} + (y_2 - y_1)\ \underline{\boldsymbol{j}}$

$\overrightarrow{PQ}$ = (–1–0) $\underline{\boldsymbol{i}}$ + (–6–5) $\underline{\boldsymbol{j}}$

$\overrightarrow{PQ}$ = –1 $\underline{\boldsymbol{i}}$– 11 $\underline{\boldsymbol{j}}$

Q.2: Find the magnitude of the vector u:

(i) $\underline{\mathbf{u}} = 2\,\underline{i} - 7\,\underline{j}$

$$|\underline{\mathbf{u}}| = \sqrt{(x)^2 + (y)^2}$$
$$|\underline{\mathbf{u}}| = \sqrt{(2)^2 + (-7)^2}$$
$$|\underline{\mathbf{u}}| = \sqrt{4 + 49}$$
$$|\underline{\mathbf{u}}| = \sqrt{53}$$

(ii) $\underline{\mathbf{u}} = \underline{i} + \underline{j}$

$$|\underline{\mathbf{u}}| = \sqrt{(x)^2 + (y)^2}$$
$$|\underline{\mathbf{u}}| = \sqrt{(1)^2 + (1)^2}$$
$$|\underline{\mathbf{u}}| = \sqrt{1 + 1}$$
$$|\underline{\mathbf{u}}| = \sqrt{2}$$

(iii) $\underline{\mathbf{u}} = [3, -4]$

$\underline{\mathbf{u}} = 3\,\underline{i} - 4\,\underline{j}$

$$|\underline{\mathbf{u}}| = \sqrt{(x)^2 + (y)^2}$$
$$|\underline{\mathbf{u}}| = \sqrt{(3)^2 + (-4)^2}$$
$$|\underline{\mathbf{u}}| = \sqrt{9 + 16}$$
$$|\underline{\mathbf{u}}| = \sqrt{25}$$
$$|\underline{\mathbf{u}}| = 5$$

Q.3: if $\underline{\mathbf{u}} = 2\,i - 7\,\underline{j}$, $\underline{\mathbf{v}} = i - 6\,\underline{j}$, $\mathbf{w} = -i + \underline{j}$. Find the following vectors.

(i) $\underline{\mathbf{u}} + \underline{\mathbf{v}} - \underline{\mathbf{w}}$

$= (2\,\underline{i} - 7\,\underline{j}) + (\underline{i} - 6\,\underline{j}) - (-i + \underline{j})$

$= 2\,\underline{i} - 7\,\underline{j} + \underline{i} - 6\,\underline{j} + \underline{i} - \underline{j}$

$= 4\,\underline{i} - 14\,\underline{j}$

(ii) $2\underline{\mathbf{u}} - 3\underline{\mathbf{v}} + 4\underline{\mathbf{w}}$

$= 2(2\,\underline{i} - 7\,\underline{j}) - 3(\underline{i} - 6\,\underline{j}) + 4(-\underline{i} + \underline{j})$

$= 4\,\underline{i} - 14\,\underline{j} - 3\,\underline{i} + 18\,\underline{j} - 4\,\underline{i} + 4\,\underline{j}$

$= -3\,\underline{i} + 8\,\underline{j}$

(iii) $\frac{1}{2}\underline{\mathbf{u}} + \frac{1}{2}\underline{\mathbf{v}} + \frac{1}{2}\underline{\mathbf{w}}$

$= \frac{1}{2}(u + v + w) = \frac{1}{2}(2\,\underline{i} - 7\,\underline{j} + \underline{i} - 6\,\underline{j} - i + \underline{j})$

$= \frac{1}{2}(2\,\underline{i} - 12\,\underline{j}) = \underline{i} - 6\,\underline{j}$

Q.4: Find the sum of the vectors AB and CD, given the four points A(1, −1), B(2,0), C(−1,3), D(−2,2) .

SOLUTION: Given: A(1, −1), B(2,0), C(−1,3), D(−2,2)

$\overrightarrow{AB} = (x_2 - x_1)\,\underline{i} + (y_2 - y_1)\,\underline{j}$

$\overrightarrow{AB} = (2-1)\,\underline{i} + (0+1)\,\underline{j} = \underline{i} + \underline{j}$

$\overrightarrow{CD} = (x_2 - x_1)\,\underline{i} + (y_2 - y_1)\,\underline{j}$

$\overrightarrow{CD} = (-2+1)\,\underline{i} + (2-3)\,\underline{j} = -\underline{i} - \underline{j}$

NOW $\overrightarrow{AB} + \overrightarrow{CD} = \underline{i} + \underline{j} - \underline{i} - \underline{j} = 0\underline{i} + 0\underline{j} = \underline{0}$ (Null Vector)

Q.5: find the vector from the point A to the origin where AB = 4 i – 2 j and B is the point (−2,5).

SOLUTION: $\overrightarrow{AO} = ?$ here o = (0,0)

Given that: $\overrightarrow{AB} = 4\,\underline{i} - 2\,\underline{j}$, B(−2,5) $\Rightarrow \overrightarrow{OB} = -2\underline{i} + 5\underline{j}$

AS we know that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\Rightarrow \overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{AB}$

$\Rightarrow \overrightarrow{OA} = (-2\underline{i} + 5\underline{j}) - (4\,\underline{i} - 2\,\underline{j})$

$= -2\underline{i} + 5\underline{j} - 4\underline{i} + 2\underline{j}$

$= -6\underline{i} + 7\underline{j}$

$\Rightarrow -\overrightarrow{AO} = -6\underline{i} + 7\underline{j} \quad \Rightarrow \overrightarrow{AO} = 6\underline{i} - 7\underline{j}$ $(\because \overrightarrow{OA} = -\overrightarrow{AO})$

Q.9: if "O" is the origin and $\overrightarrow{OP} = \overrightarrow{AB}$, find the point when A and B are (−3,7) and (1,0) respectively. (2019 I S.Q)

SOLUTION: NOW we find point P(x, Y). Given that (−3,7) and (1,0) and here O(0,0).

As $\overrightarrow{OP} = \overrightarrow{AB}$

$\Rightarrow (x - 0)\,\underline{i} + (y - 0)\,\underline{j} = (1 + 3)\,\underline{i} + (0 - 7)\,\underline{j}$

$$x\underline{i} + y\underline{j} = 4\underline{i} - 7\underline{j}$$

Comparing both sides, we have

x = 4 and y = −7

$$Thus\ P(x, y) = P(4, -7)$$

Q.6: Find a unit vector in the direction of the vector given below.

(i) $\underline{v} = 2\underline{i} - \underline{j}$

$|\underline{v}| = \sqrt{(2)^2 + (-1)^2} = \sqrt{5}$

NOW $\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|}$

$= \frac{2\underline{i} - \underline{j}}{\sqrt{5}} = \frac{2}{\sqrt{5}}\underline{i} - \frac{1}{\sqrt{5}}\underline{j}$

$= \frac{1}{\sqrt{5}}(2\underline{i} - \underline{j})$

(ii) $\underline{v} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$

$|\underline{v}| = \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

NOW $\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|}$

$= \frac{\frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}}{1} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$

(iii) $\underline{v} = -\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$

$|\underline{v}| = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$

NOW $\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|}$

$= \frac{-\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}}{1} = -\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$

Q.7: If A, B, C are respectively the points (2,-4), (4,0), (1,6).Use vector method to find the coordinates of the point D

(i) ABCD is a parallelogram

SOLUTION: we find $D(x, y)$

Given: A (2, −4),B (4,0),C (1,6)

Since ABCD is a parallelogram, so

$$\overrightarrow{AB} = \overrightarrow{DC} \quad and\ \overrightarrow{AB} \parallel \overrightarrow{DC}$$

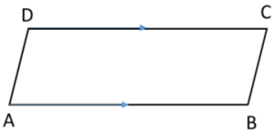
$$\overrightarrow{AB} = [4,0] - [2,-4] = [4 - 2, 0 + 4] = [2,4]$$

$$\overrightarrow{DC} = [1,6] - [x, y] = [1 - x, 6 - y]$$

$\because \overrightarrow{AB} = \overrightarrow{DC} \quad \Rightarrow [2,4] = [1 - x, 6 - y]$

$\Rightarrow 2 = 1 - x, \quad 4 = 6 - y \Rightarrow x = 1 - 2, y = 6 - 4$

$\Rightarrow x = -1, y = 2 \;\; so\; D(x,y) = (-1,2)$



$(ii) \because A(2,-4), B(4,0) and\; C(1,6)$

Let $D(x,y) be\; the\; req.pt$

$as\; ADBC\; is\; ||gram.$

So from $fig \because \overrightarrow{AD} = \overrightarrow{CB} \quad and\; \overrightarrow{AD} || \overrightarrow{CB}$

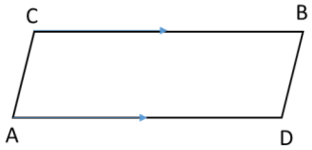
$$\overrightarrow{AD} = [x,y] - [2,-4] = [x-2, y+4]$$

$$\overrightarrow{CB} = [4,0] - [1,6] = [4,-1,0,-6] = [3,-6]$$

$$\because \overrightarrow{AD} = \overrightarrow{CB} \Rightarrow [x-2, y+4] = [3,-6]$$

$$\Rightarrow x - 2 = 3, \; y + 4 = -6 \Rightarrow x = 5, y = -10$$

$$\Rightarrow so\; D(x,y) = (5,-10)$$



Q.8: If B, C, D are respectively the points (4,1), (-2,3), (-8,0).Use vector method to find the coordinates of the point:

(i) A, if ABCD is a parallelogram

SOLUTION: we find $A(x,y)$

Given: B (4,1),C (−2,3),D (−8,0)

Since ABCD is a parallelogram, so

$$\overrightarrow{AB} = \overrightarrow{DC}$$

$(x_2 - x_1)\,\underline{\boldsymbol{i}} + (y_2 - y_1)\,\underline{\boldsymbol{j}} = (x_2 - x_1)\,\underline{\boldsymbol{i}} + (y_2 - y_1)\,\underline{\boldsymbol{j}}$

(4−x) $\underline{\boldsymbol{i}}$ + (1−y) $\underline{\boldsymbol{j}}$ = (−2+8)i + (3−0) $\underline{\boldsymbol{j}}$

(4−x) $\underline{\boldsymbol{i}}$ + (1− y) $\underline{\boldsymbol{j}}$ = 6i + 3$\underline{\boldsymbol{j}}$

Comparing both sides

4− x = 6 & 1− y = 3

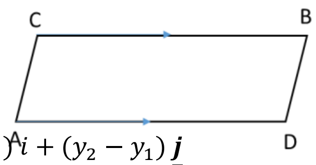
⟹ x = −2 & y =−2 Thus $A(-2,-2)$

(ii) E, AEBD is a parallelogram

SOLUTION: we find E(x,y)

Given: A (−2, −2),B (4,1),D (−8,0)

Since ADBC is a parallelogram, so



$\overrightarrow{AE} = \overrightarrow{BD}$

$(x_2 - x_1)\, i + (y_2 - y_1)\,\underline{\boldsymbol{j}} = (x_2 - x_1)\, i + (y_2 - y_1)\,\underline{\boldsymbol{j}}$

(x+2) $\underline{\boldsymbol{i}}$ + (y +2) $\underline{\boldsymbol{j}}$ = (4+8) $\underline{\boldsymbol{i}}$ + (1− 0) $\underline{\boldsymbol{j}}$

(x+2) $\underline{\boldsymbol{i}}$ + (y+2) $\underline{\boldsymbol{j}}$ = 12i + $\underline{\boldsymbol{j}}$

Comparing both sides

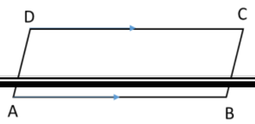
X+2 = 12 & y+ 2 = 1

⟹ x = 10 & y = −1 Thus $E(10,-1)$

Q.10: Use vectors, to show that ABCD is a parallelogram, when the points A,B,C and D are respectively (0,0), (a,0), $(\boldsymbol{b},\boldsymbol{c}), (\boldsymbol{b}-\boldsymbol{a},\boldsymbol{c})$.

SOLUTION: Given that: (0,0), (a,0), $(b,c), (b-a,c)$

$\overrightarrow{AB}$ = (a−0) $\underline{\boldsymbol{i}}$ + (0−0) $\underline{\boldsymbol{j}}$ = a$\underline{\boldsymbol{i}}$+0$\underline{\boldsymbol{j}}$



$\overrightarrow{DC} = (b - b + a)\,\underline{i}$ + (c−c) $\underline{j}$ = a$\underline{i}$+0$\underline{j}$

$\overrightarrow{AD}$ = (b−a−0) $\underline{i}$ + (c−0) $\underline{j}$ = (b−a) $\underline{i}$ + c$\underline{j}$

$\overrightarrow{BC}$ = (b−a) $\underline{i}$ + (c−0) $\underline{j}$ = (b−a) $\underline{i}$ + c$\underline{j}$

As $\overrightarrow{AB} = \overrightarrow{DC}$ & $\overrightarrow{AD} = \overrightarrow{BC}$

So ABCD is a parallelogram.

Q.11: if $\overrightarrow{AB} = \overrightarrow{CD}$. Find the coordinates of point A, when points B, C, D are (1,2), (−2,5), (4,11) respectively.

SOLUTION: NOW we find point A(x, Y). Given that B(1,2), C(−2,5), D(4,11)

As **if $\overrightarrow{AB} = \overrightarrow{CD}$**

⟹ (1−x) $\underline{i}$ +(2−y) $\underline{j}$ = (4+2) $\underline{i}$ + (11−5) $\underline{j}$

(1−x) $\underline{i}$+(2−y) $\underline{j}$ = 6$\underline{i}$ +6$\underline{j}$

Comparing both sides, we have

1− x = 6 and 2−y = 6

X = 5 and y = -4

$Thus \quad A(x,y) = A(5,-4)$

Q.12: Find the position vectors of the point of division of the line segments joining the following pair of points, in the given ratio:

(i) point "C" with position vector 2$\underline{i}$ – 3 $\underline{j}$ and point D with position vector 3$\underline{i}$ + 2$\underline{j}$ in the ratio 4 : 3

SOLUTION: Given that $\overrightarrow{OC}$ = a = 2$\underline{\boldsymbol{i}}$ – 3$\underline{\boldsymbol{j}}$, $\overrightarrow{OD}$ = b = 3$\underline{\boldsymbol{i}}$ + 2$\underline{\boldsymbol{j}}$, p : q = 4 : 3

Let $\overrightarrow{OP} = \underline{\boldsymbol{r}}$ = ? , Let P be the point which divides CD in ratio 4 : 3

Using ratio formula, we have

$$\underline{\boldsymbol{r}} = \frac{qa+pb}{p+q} = \frac{3(2\underline{i}-3\underline{j})+4(3\underline{i}+2\underline{j})}{3+4} = \frac{6\underline{i}-9\underline{j}+12\underline{i}+8\underline{j}}{7}$$

$$= \frac{18\underline{i}-\underline{j}}{7} = \frac{18}{7}\underline{i} - \frac{1}{7}\underline{j}$$

(ii) point "E" with position vector 5i and point F with position vector 4i + $\underline{j}$ in the ratio 2 : 5.

SOLUTION: Given that $\overrightarrow{OE}$= a = 5$\underline{\boldsymbol{j}}$, $\overrightarrow{OF}$= b = 4$\underline{\boldsymbol{i}}$ + $\underline{\boldsymbol{j}}$, p : q = 2 : 5

Let $\overrightarrow{OP}$= $\underline{r}$ = ?

Let P be the point which divides CD in ratio 2 : 5

Using ratio formula, we have

$$\underline{\boldsymbol{r}} = \frac{qa+pb}{p+q} = \frac{5(5\underline{\boldsymbol{j}})+2(4\underline{\boldsymbol{i}}+\underline{\boldsymbol{j}})}{2+5}$$

$$= \frac{25j + 8\underline{\boldsymbol{i}} + 2\underline{\boldsymbol{j}}}{7} = \frac{8\underline{\boldsymbol{i}} + 27\underline{\boldsymbol{j}}}{7}$$

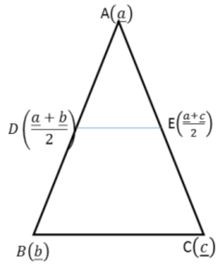
$$= \frac{8}{7}\underline{\boldsymbol{i}} + \frac{27}{7}\underline{\boldsymbol{j}}$$

Q.14: Prove that line segment joining the mid points of two sides of a triangle is parallel to third side & half as along.

SOLUTION: consider a triangle ABC, such that $\overrightarrow{OA}=\underline{a}$, $\overrightarrow{OB}=\underline{b}$, $\overrightarrow{OC}=\underline{c}$

Let D & E be the mid points of sides $\overrightarrow{AB}$ & $\overrightarrow{AC}$ respectively.

$\overrightarrow{OD}=\frac{a+b}{2}$; $\overrightarrow{OE}=\frac{a+c}{2}$



NOW $\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\underline{c}-\underline{b}$ → (i)

AND $\overrightarrow{DE}=\overrightarrow{OE}-\overrightarrow{OD}=\frac{a+b}{2}-\frac{a+c}{2}=\frac{(a+b)-(a+c)}{2}=\frac{a+b-a-c}{2}=\frac{c-b}{2}=\frac{1}{2}(\underline{c}-\underline{b})$ (ii)

From (i) & (ii) it is clear that BC & DE are parallel and $\overrightarrow{DE}$ is half to $\overrightarrow{BC}$.

Q.15: prove that line segments joining the mid points of sides of a quadrilateral taken in order form a parallelogram.

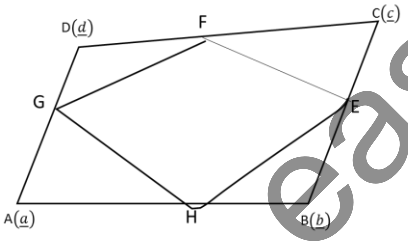
Q.15: prove that line segments joining the mid points of sides of a quadrilateral taken in order form a parallelogram.

SOLUTION: consider the quadrilateral ABCD, such that $\overrightarrow{OA}=\underline{a}$, $\overrightarrow{OB}=\underline{b}$, $\overrightarrow{OC}=\underline{c}$, $\overrightarrow{OD}=\underline{d}$

Let E, F, G, H be the mid points of sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{AD}$ respectively. D(d)

$\overrightarrow{OE}=\frac{a+b}{2}$, $\overrightarrow{OF}=\frac{b+c}{2}$, $\overrightarrow{OG}=\frac{c+d}{2}$, $\overrightarrow{OH}=\frac{a+d}{2}$

NOW $\overrightarrow{EF}=\overrightarrow{OF}-\overrightarrow{OE}=\frac{b+c}{2}-\frac{a+b}{2}$



$$=\frac{(\underline{b}+\underline{c})-(\underline{a}+\underline{b})}{2}=\frac{\underline{b}+\underline{c}-\underline{a}-\underline{b}}{2}$$

$$=\frac{\underline{c}-\underline{a}}{2}$$

$\overrightarrow{HG}=\overrightarrow{OG}-\overrightarrow{OH}=\frac{c+d}{2}-\frac{a+d}{2}$

$$=\frac{(\underline{c}+\underline{d})-(\underline{a}+\underline{d})}{2}=\frac{\underline{c}+d-\underline{a}-\underline{d}}{2}=\frac{\underline{c}-\underline{a}}{2}$$

$\overrightarrow{EH}=\overrightarrow{OH}-\overrightarrow{OE}=\frac{d+d}{2}-\frac{a+b}{2}$

$$=\frac{(\underline{a}+\underline{d})-(\underline{a}+\underline{b})}{2}=\frac{\underline{a}+\underline{d}-\underline{a}-\underline{b}}{2}=\frac{\underline{d}-\underline{b}}{2}$$

$\overrightarrow{FG}=\overrightarrow{OG}-\overrightarrow{OF}=\frac{c+d}{2}-\frac{b+c}{2}$

$$=\frac{(\underline{c}+\underline{d})-(\underline{b}+\underline{c})}{2}=\frac{\underline{c}+\underline{d}-\underline{b}-\underline{c}}{2}=\frac{\underline{d}-\underline{b}}{2}$$

As $\overrightarrow{EF}=\overrightarrow{HG}$ and $\overrightarrow{EH}=\overrightarrow{FG}$

$\Longrightarrow \overrightarrow{EF}\parallel\overrightarrow{HG}$ and $\overrightarrow{EH}\parallel\overrightarrow{FG}$

Thus EFGH is a parallelogram.

Concept of vector in space

If three dimensional space XOX' *and* ZOZ' *are called* Coordinate axis.

The planes made by XOY, YOZ *and* ZOX *are called* XY Plane. $YZ-$ *plane and* ZX *plane. respectively*

These planes are mutually orthogonal to each other.

Note:

If P is point in space then it will have three coordinates along $x-axis$, $y-axis$ *and* $z-axis$ *respectively.*

if a,b,c *are distance along* $x-axis, y-axis$ *and* $z-axis$ *respectively*

Then coordinates of point p are $p(a,b,c)$ *as shown in fig. let* $\vec{u}$ *be vector in space whoes position vector is* $\overrightarrow{OP}$ *then* $\vec{u}=[x,y,z]$

Now (i) Addition:

$$let\ \vec{u}=[x_1,y_1,z_1]\ ,\vec{v}=[x_2,y_2,z_2]$$
$$\vec{u}+\vec{v}=[x_1,y_1,z_1]+[x_2,y_2,z_2]$$
$$\vec{u}+\vec{v}=[x_1+x_2,y_1+y_2,z_1+z_2]$$

(ii) Difference:

$$let\ \vec{u}=[x_1,y_1,z_1]\ ,\vec{v}=[x_2,y_2,z_2]$$
$$\vec{u}-\vec{v}=[x_1,y_1,z_1]-[x_2,y_2,z_2]$$
$$\vec{u}-\vec{v}=[x_1-x_2,y_1-y_2,z_1-z_2]$$

(iii) Scalar multiplication:

if k *be any scalr and* $\vec{u}=[x,y,z]$ *then* $k\vec{u}$

$$=k[x,y,z]$$
$$=[\boldsymbol{kx},\boldsymbol{ky},\boldsymbol{kz}]$$

Magnitude of vector (in space)

let $\vec{u}=[x,y,z]$ *be a vector in space then its length or norm or magntude denoted by* $|\vec{u}|$ *and defines as*

$$|\vec{u}|=\sqrt{x^2+y^2+z^2}$$

Properties of a vector:

(i) Commutative property:

for any two vectors $\vec{u}$ *and* $\vec{v}$

$$\vec{u}+\vec{v}=\vec{v}+\vec{u}$$

Proof:

$$let\ \vec{u}=[x_1,y_1,z_1]\ ,\ \vec{u}=[x_2,y_2,z_2]$$
$$\vec{u}+\vec{v}=[x_1,y_1,z_1]+[x_2,y_2,z_2]$$
$$\vec{u}+\vec{v}=[x_1+x_2,y_1+y_2,z_1+z_2]$$
$$=[x_1+x_2,y_1+y_2,z_1+z_2]$$
$$=[x_2+x_1,y_2+y_1,z_2+z_1]$$
$$=[x_2,y_2,z_2]+[x_1,y_1,z_1]$$
$$=\vec{v}+\vec{u}$$

(in real numbers commutatives law hold)

Thus $\vec{u}+\vec{v}=\vec{v}+\vec{u}$

(ii) Associative Property:

for any three vectors $\vec{u}$, $\vec{v}$ *and* $\vec{w}$

$$(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$$

Proof:

Let

$(\vec{u}+\vec{v})+\vec{w}=(x_1,y_1,z_1+x_2,y_2,z_2)+(x_3,y_3z_3)$

$=(x_1+x_2,y_1+y_2,z_1+z_2)+(x_3,y_3,z_3)$

$=(x_1,y_1,z_1)+(x_2+x_3,y_2+y_3,z_2+z_3)$

$=(x_1,y_1,z_1)+\left((x_2,y_2,z_2)+(x_3,y_3,z_3)\right)$

$=\vec{u}+(\vec{v}+\vec{w})$

Hence $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$

Inverse property:

For a vector $\vec{u}$, $\vec{u}+(-\vec{u})=(-\vec{u})+\vec{u}=0$

Proof:

$let\ \vec{u}=[x,y,z]\ then$

$-\vec{u}=[-x,-y,-z]$

$then\ \vec{u}+(-\vec{u})=[x,y,z]+[-x.-y,-z]$

$=[x-x,y-y,z-z]$

$(0,0,0)=0$

Similarly:

$(-\vec{u})+\vec{u}=[-x,-y,-z]+[x,y,z]$

$=[-x+x,-y+y,-z+z]=[0,0,0]=o$

Thus $for\ \vec{u}\ its\ additive\ inverse\ is\ -\vec{u}$

***(iv) Distrubutive Property*:**

if k be any scalar and $\vec{u},\vec{v}$ *be two vectors then*

$k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}$

***Proof*:**

$\boldsymbol{let}\ \vec{u}=[\boldsymbol{x_1},\boldsymbol{y_1},\boldsymbol{z_1}],\ \boldsymbol{v}[\boldsymbol{x_2},\boldsymbol{y_2},\boldsymbol{z_2}]$

$k(\vec{u}+\vec{v})=k[(\boldsymbol{x_1},\boldsymbol{y_1},\boldsymbol{z_1})+(\boldsymbol{x_2},\boldsymbol{y_2},\boldsymbol{z_2})]$

$=k[x_1+x_2,y_1+y_2,z_1+z_2]$

$=[kx_1+kx_2,ky_1+ky_2,kz_1+kz_2]$

$=k(\boldsymbol{x_1},\boldsymbol{y_1},\boldsymbol{z_1})+k(\boldsymbol{x_2},\boldsymbol{y_2},\boldsymbol{z_2})$

$=k\vec{u}+\boldsymbol{k}\vec{v}$

***(v) Scalar Multiplication property*:**

for a, b being scalars and $\vec{u}$ *be a vector. then*

$a(b\vec{u})=(ab)\,\vec{u}$

Proof:

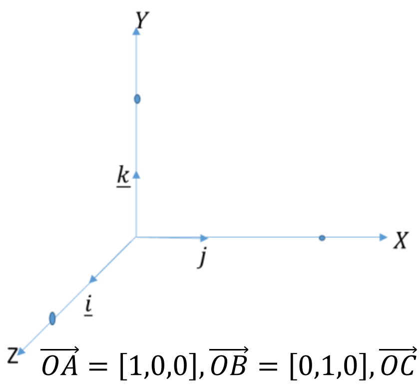
Let $\vec{u}=[x,y,z]$

⇨ $a\,(b\vec{u})=a\left(b(x,y,z)\right)=a(bx,by,bz)$

⇨ $=[abx,aby,abz]=(ab)\,\vec{u}$

Another Notation for represented vectors in space:

Let $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ *be vectors along* $x-axis$, $y-axis$ *and* $z-axis$ *respectively.*



As

$\overrightarrow{OA}=[1,0,0],\overrightarrow{OB}=[0,1,0],\overrightarrow{OC}=[0,0,1]$

$then\ \underline{i}=[1,0,0],\underline{j}=[0,1,0],\underline{k}=[0,0,1]$

where $\underline{i},\underline{j},$ *and* $\underline{k}$ *are called unit vectors along* $x-axis, y-axis$ *and* $z-axis.$

Now if $\vec{u}=[x,y,z]$ *then* $\vec{u}$ *can be written as*

$\vec{u}=[x,o,o]+[o,y,o]+[0,0,z]$

$=x[1,0,0]+y[0,1,0]+z[0,0,1]$

$\vec{u}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$

Distance between two points in space:

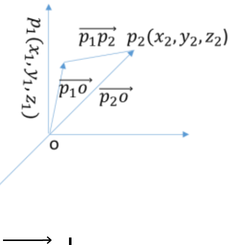
if $\overrightarrow{OP_1}$ *nad* $\overrightarrow{OP_2}$ *are the position vectors of the* ***points*** $p_1(x_1,y_1,z_1)$ *and* $p_2(x_2,y_2,z_2)$

The vectors $\overrightarrow{p_1p_2}$ *is given by*

$\overrightarrow{p_1p_2}=\overrightarrow{Op_2}+\overrightarrow{Op_1}$

$=[x_2,y_2,z_2]-[x_1,y_1,z_1]$

$=[x_2-x_1,y_1-y_2,z_1-z_2]$



By

$Distance\ b\backslash w\ p_1\ and\ p_2=|\overrightarrow{p_1p_2}|$

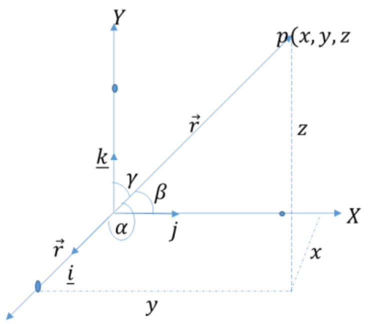
$=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

This is called distance formula b/w two points p_1 *and* p_2 *in R*

Distance angles and direction cosine of a vector:

Let $C=\overrightarrow{OP}=[x,y,z]=x\,\hat{\imath}+y\hat{\jmath}+z\,\hat{k}$ *be a vector*

s. that it makes angles α,β *and* γ *along coordinates axes the* α,β *and* γ *are called dire ction angles* Of vectors $\vec{r}$ *while* $\cos\alpha$, $\cos\beta$ *and* $\cos\gamma$ *are called direction cosines of* $\vec{r}$



Important result

Prove that $\boldsymbol{cos^2\alpha+cos^2\beta+cos^2\gamma=1}$

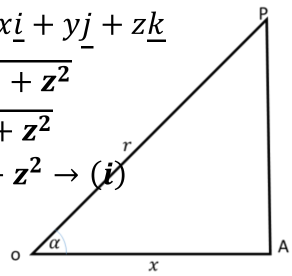
Proof:

$let\ \vec{r}=[x_1,y_1,z_1]=x\underline{i}+y\underline{j}+z\underline{k}$

⇨ $|\vec{r}|=\sqrt{\boldsymbol{x^2+y^2+z^2}}$

⇨ $\boldsymbol{r}=\sqrt{\boldsymbol{x^2+y^2+z^2}}$

⇨ $\boldsymbol{r^2=x^2+y^2+z^2}\rightarrow(\boldsymbol{i})$



$\boldsymbol{in\ right\ \Delta OAP}$

$\boldsymbol{cos\alpha}=\frac{\overline{OA}}{\overline{OP}}=\frac{z}{r}$

$\boldsymbol{similarly\ cos\beta}=\frac{\boldsymbol{y}}{\boldsymbol{r}}=\boldsymbol{cos\gamma}=\frac{\boldsymbol{z}}{\boldsymbol{r}}$

So $\boldsymbol{cos^2\alpha+cos^2\beta+cos^2\gamma}=\frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2}$

$=\frac{\boldsymbol{x^2+y^2+z^2}}{\boldsymbol{r^2}}=\frac{\boldsymbol{r^2}}{\boldsymbol{r^2}}=\boldsymbol{1}$

$\boldsymbol{cos^2\alpha+cos^2\beta+cos^2\gamma}$

$=\boldsymbol{1}$ ***hence proved.***

Exercise NO 7.2

Q.1: Let A =(2,5) , B = (−1,1) , C = (2, −6). Find

(i) $\overrightarrow{AB}$ = ?

$\overrightarrow{AB} = (x_2 - x_1)\,\underline{i} + (y_2 - y_1)\,\underline{j}$

$\overrightarrow{AB} = (-1-2)\,\underline{i} + (1-5)\,\underline{j}$

$\overrightarrow{AB} = -3\underline{i} - 4\underline{j}$

(ii) $2\overrightarrow{AB} - \overrightarrow{CB}$ = ?

$2\overrightarrow{AB} - \overrightarrow{CB} = 2[(x_2 - x_1)\,\underline{i} + (y_2 - y_1)\,\underline{j}] - [(x_2 - x_1)\,\underline{i} + (y_2 - y_1)\,\underline{j}]$

$2\overrightarrow{AB} - \overrightarrow{CB} = 2[(-1-2)\,\underline{i} + (1-5)\,\underline{j}] - [(-1-2)\,\underline{i} + (1+6)\,\underline{j}]$

$2\overrightarrow{AB} - \overrightarrow{CB} = 2[-3\underline{i} - 4\underline{j}] - [-3\underline{i} + 7\underline{j}] = -6\underline{i} - 8\underline{j} + 3\underline{i} - 7\underline{j} = -3\underline{i} - 15\underline{j}$

(iii) $2\overrightarrow{CB} - 2\overrightarrow{CA}$ = ?

$2\overrightarrow{CB} - 2\overrightarrow{CA} = 2(\overrightarrow{CB} - \overrightarrow{CA}) = 2[\{(x_2 - x_1)\,\underline{i} + (y_2 - y_1)\,\underline{j}\} - \{(x_2 - x_1)\,\underline{i} + (y_2 - y_1)\,\underline{j}\}]$

$2\overrightarrow{CB} - 2\overrightarrow{CA} = 2(\overrightarrow{CB} - \overrightarrow{CA}) = 2[\{(-1-2)\,\underline{i} + (1+6)\,\underline{j}\} - \{(-2-2)\,\underline{i} + (5+6)\,\underline{j}\}] = 2[\{(-3\underline{i} + 7\underline{j})\} - \{(0\underline{i} + 11\underline{j})\}]$

$2\overrightarrow{CB} - 2\overrightarrow{CA} = 2(\overrightarrow{CB} - \overrightarrow{CA}) = -6\underline{i} + 14\underline{j} - 0\underline{i} - 11\underline{j} = -6\underline{i} - 3\underline{j}$

Q.2: Let $\underline{u} = \underline{i} + 2j - \underline{k}$, $\underline{v} = 3\underline{i} - 2j + 2\underline{k}$, $\underline{w} = 5\underline{i} - j + 3\underline{k}$. Find the indicated vectors or numbers:

(i) u + 2v + w = ?

$\underline{u} + 2\underline{v} + \underline{w} = (\underline{i} + 2\underline{j} - \underline{k}) + 2(3\underline{i} - 2\underline{j} + 2\underline{k}) + (5\underline{i} - \underline{j} + 3\underline{k}) = \underline{i} + 2\underline{j} - \underline{k} + 6\underline{i} - 4\underline{j} + 4\underline{k} + 5\underline{i} - \underline{j} + 3\underline{k} = 12\underline{i} - 3\underline{j} + 6\underline{k}$

(ii) v − 3w = ?

$\underline{v} - 3\underline{w} = (3\underline{i} - 2\underline{j} + 2\underline{k}) - 3(5\underline{i} - \underline{j} + 3\underline{k}) = 3\underline{i} - 2\underline{j} + 2\underline{k} - 15\underline{i} + 3\underline{j} - 9\underline{k} = -12\underline{i} + \underline{j} - 7\underline{k}$

(iii) $|3\underline{v} + \underline{w}|$ =?

$3\underline{v} + \underline{w} = 3(3\underline{i} - 2\underline{j} + 2\underline{k}) + 5\underline{i} - \underline{j} + 3\underline{k} = 9\underline{i} - 6\underline{j} + 6\underline{k} + 5\underline{i} - \underline{j} + 3\underline{k} = 14\underline{i} - 7\underline{j} + 9\underline{k}$

$|3\underline{v} + \underline{w}| = \sqrt{(14)^2 + (-7)^2 + (9)^2} = \sqrt{196 + 49 + 81} = \sqrt{326}$

Q.3: Find the magnitude of the vector v and write the direction cosine of $\underline{v}$:

(i) $\underline{v} = 2\underline{i} + 3\underline{j} + 4\underline{k}$

MAGNITUDE

$|\underline{v}| = \sqrt{(x)^2 + (y)^2 + (z)^2}$

$|\underline{v}| = \sqrt{(2)^2 + (3)^2 + (4)^2}$

$|\underline{v}| = \sqrt{4 + 9 + 16} = \sqrt{29}$

DIRECTION COSINE

Direction cosine = $\frac{Direction\ ratio}{magnitude}$

∵ direction ratio = 2, 3, 4

So, Direction cosine = $\frac{2,3,4}{\sqrt{29}}$

Direction cosine = $\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$

(ii) $\underline{v} = \underline{i} - \underline{j} - \underline{k}$

MAGNITUDE

$|\underline{v}| = \sqrt{(x)^2 + (y)^2 + (z)^2}$

$|\underline{v}| = \sqrt{(1)^2 + (-1)^2 + (-1)^2}$

$|\underline{v}| = \sqrt{1 + 1 + 1} = \sqrt{3}$

DIRECTION COSINE

Direction cosine = $\frac{Direction\ ratio}{magnitude}$

∵ direction ratio = 1, −1, −1

So, Direction cosine = $\frac{1,-1,-1}{\sqrt{3}}$

Direction cosine = $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$

(iii) $\underline{v} = 4\underline{i} - 5\underline{j}$

MAGNITUDE

$|\underline{v}| = \sqrt{(x)^2 + (y)^2 + (z)^2}$

$|\underline{v}| = \sqrt{(4)^2 + (-5)^2 (0)^2}$

$|\underline{v}| = \sqrt{16 + 25 + 0} = \sqrt{41}$

DIRECTION COSINE

Direction cosine = $\frac{Direction\ ratio}{magnitude}$

∵ direction ratio = 4, −5, 0

So, Direction cosine = $\frac{4,-5,0}{\sqrt{29}}$

Direction cosine = $\frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}}, \frac{0}{\sqrt{29}}$

Q.4: Find α, so that

$$\left|\alpha\underline{i} + (\alpha+1)\underline{j} + 2\underline{k}\right| = 3$$

SOLUTION:

$$\left|\alpha\underline{i} + (\alpha+1)\underline{j} + 2\underline{k}\right| = 3$$

$$\sqrt{(\alpha)^2 + (\alpha+1)^2 + (2)^2} = 3$$

$$\sqrt{\alpha^2 + \alpha^2 + 1 + 2\alpha + 4} = 3$$

Squaring on both side

$$\alpha^2 + \alpha^2 + 1 + 2\alpha + 4 = 9$$

$$2\alpha^2 + 2\alpha + 5 = 9$$

$$2\alpha^2 + 2\alpha + 5 - 9 = 0$$

$$2\alpha^2 + 2\alpha - 4 = 0$$

Dividing by 2

$$\alpha^2 + \alpha - 2 = 0$$

$$\alpha^2 + 2\alpha - \alpha - 2 = 0$$

$$\alpha(\alpha+2) - 1(\alpha+2) = 0$$

$$(\alpha+2)(\alpha-1) = 0$$

Either α+2 = 0 or α−1 = 0

Q.5: Find a unit vector in the direction of $\underline{v} = i + 2\underline{j} - \underline{k}$

SOLUTION:

Let $\hat{V}$ be unit vector in the direction of v

Then, $\underline{v} = \underline{i} + 2\underline{j} - \underline{k}$

$$|\underline{v}| = \sqrt{(1)^2 + (2)^2 + (1)^2}$$

$$|\underline{v}| = \sqrt{1+4+1} = \sqrt{6}$$

Required unit vector is:

$$\hat{V} = \frac{\underline{v}}{\underline{v}} = \frac{\underline{i}+2\underline{j}-\underline{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\underline{i} + \frac{2}{\sqrt{6}}\underline{j} - \frac{1}{\sqrt{6}}\underline{k}$$

Q.6: if $\underline{a}$= 3$\underline{i}$−$\underline{j}$-4$\underline{k}$, $\underline{b}$ = −2$\underline{i}$−4$\underline{j}$−3$\underline{k}$, $\underline{c}$ = $\underline{i}$+2$\underline{j}$−$\underline{k}$. find a unit vector parallel to 3$\underline{a}$−2$\underline{b}$+4$\underline{c}$.

SOLUTION:

Let $\underline{v}$ = 3$\underline{a}$−2$\underline{b}$+4$\underline{c}$ = 3(3$\underline{i}$−$\underline{j}$-4k)−2(−2$\underline{i}$−4$\underline{j}$−3k)+4($\underline{i}$+2$\underline{j}$−$\underline{k}$)

$\underline{v}$ = 9$\underline{i}$−3$\underline{j}$−12$\underline{k}$+4$\underline{i}$+8$\underline{j}$+6$\underline{k}$+4$\underline{i}$+8$\underline{j}$−4$\underline{k}$ = 17$\underline{i}$+13$\underline{j}$ −10$\underline{k}$

$$|\underline{v}| = \sqrt{(17)^2 + (13)^2 + (-10)^2} = \sqrt{289+169+100} = \sqrt{558}$$

Let u be a unit vector parallel to v, then

$$\underline{u} = \frac{\underline{v}}{|\underline{v}|} = \frac{17\underline{i}+13\underline{j}-10\underline{k}}{\sqrt{558}} = \frac{17}{\sqrt{558}}\underline{i} + \frac{13}{\sqrt{558}}\underline{j} - \frac{10}{\sqrt{558}}\underline{k}$$

Q.7: Find a vector whose:

i) Magnitude is 4 and is parallel to 2$\underline{i}$−3$\underline{j}$+6$\underline{k}$ (2019 I S.Q)

SOLUTION:

U = ? Given: $|u| = 4$, v = 2$\underline{i}$−3$\underline{j}$+6$\underline{k}$

As u ∥ v

So u = v

$$\Rightarrow \frac{u}{|u|} = \frac{v}{|v|}$$

$$U = |u|\frac{v}{|v|} = (4)\frac{2\underline{i}-3\underline{j}+6\underline{k}}{\sqrt{2^2+(-3)^2+6^2}} = \frac{4[2\underline{i}-3\underline{j}+6\underline{k}]}{\sqrt{49}} = \frac{8\underline{i}-12\underline{j}+24\underline{k}}{7}$$

$$U = \frac{8}{7}\underline{i} - \frac{12}{7}\underline{j} + \frac{24}{7}\underline{k}$$

ii) magnitude is 2 and is parallel to −$\underline{i}$ + $\underline{j}$ + $\underline{k}$

SOLUTION:

U = ? Given: $|u| = 2$, v = −$\underline{i}$ + $\underline{j}$ + $\underline{k}$

As u ∥ v

So u = v

$$\Rightarrow \frac{u}{|u|} = \frac{v}{|v|}$$

$$U = |u|\frac{v}{|v|} = (2)\frac{-\underline{i}+\underline{j}+\underline{k}}{\sqrt{2^2+(-3)^2+6^2}} = \frac{2[-\underline{i}+\underline{j}+\underline{k}]}{\sqrt{49}} = \frac{-2\underline{i}+2\underline{j}+2\underline{k}}{7}$$

$$U = -\frac{2}{7}\underline{i} + \frac{2}{7}\underline{j} + \frac{2}{7}\underline{k}$$

Q.8: if $\underline{u}$ = 2$\underline{i}$+3$\underline{j}$+4$\underline{k}$, $\underline{v}$ = -$\underline{i}$+3$\underline{j}$-$\underline{k}$, $\underline{w}$ = $\underline{i}$+6$\underline{j}$+z$\underline{k}$ represents the sides of a triangle. Find the value of Z.

SOLUTION:

As u, v, w represents the sides of triangle

So $\underline{u}$ + $\underline{v}$ = $\underline{w}$

2$\underline{i}$+3$\underline{j}$+4$\underline{k}$ - $\underline{i}$+3$\underline{j}$-k = $\underline{i}$+6$\underline{j}$+z$\underline{k}$

$\underline{i}$+6$\underline{j}$+8$\underline{k}$ = $\underline{i}$+6$\underline{j}$+z$\underline{k}$

By comparing

⟹ Z = 3

Q.9: The position vectors of the points A , B , C and , D are $2\underline{i} - \underline{j} + \underline{k}$, $3\underline{i} + \underline{j}$, $2i + 4\underline{j} - 2\underline{k}$ and $-\underline{i} - 2\underline{j} + \underline{k}$ respectively. Show that AB is parallel to CD.

SOLUTION:

Given that: OA = $2\underline{i} - \underline{j} + \underline{k}$, OB = $3\underline{i} + \underline{j}$, OC = $2\underline{i} + 4\underline{j} - 2\underline{k}$, OD = $-\underline{i} - 2\underline{j} + \underline{k}$

AB = OB−OA = $(3\underline{i} + \underline{j}) - (2\underline{i} - \underline{j} + \underline{k}) = 3\underline{i} + \underline{j} - 2i + \underline{j} - \underline{k} = \underline{i} + 2\underline{j} - \underline{k}$

CD = OD−OC = $\left(-\underline{i} - 2\underline{j} + \underline{k}\right) - \left(2\underline{i} + 4\underline{j} - 2\underline{k}\right) = -\underline{i} - 2\underline{j} + \underline{k} - 2\underline{i} - 4\underline{j} + 2\underline{k} = -3\underline{i} - 6\underline{j} + 3\underline{k} = -3AB$

⟹ AB ∥ CD Hence AB is parallel to CD.

Q.10: We say that vectors v and w are parallel if there is a scalar c such that v = cw. The vector point in the same direction if c > 0 and the vector point in the opposite direction if c < 0.

a) Find two vectors of length 2 parallel to vector $\underline{v}$ = $2i - 4\underline{j} + 4\underline{k}$

SOLUTION:

Given: $|\underline{u}| = 2$, $\underline{v} = 2\underline{i} - 4\underline{j} + 4\underline{k}$ $\underline{u}$ = ?

As $\underline{u}$ ∥ $\underline{v}$

So $\underline{u}$ = ± $\underline{v}$

$$\Rightarrow \frac{\underline{u}}{|\underline{u}|} = \pm\frac{\underline{v}}{|\underline{v}|}$$

$$\underline{u} = \pm|\underline{u}|\frac{\underline{v}}{|\underline{v}|}$$

$$\underline{u} = \pm(2)\frac{2\underline{i}-4\underline{j}+4\underline{k}}{\sqrt{2^2+(-4)^2+4^2}}$$

$$\underline{u} = \pm\frac{2[2\underline{i}-4\underline{j}+4\underline{k}]}{6}$$

$$\underline{u} = \pm\frac{2\underline{i}-4\underline{j}+4\underline{k}}{3}$$

$$\underline{u} = \pm\left(\frac{2}{3}\underline{i} - \frac{4}{3}\underline{j} + \frac{4}{3}\underline{k}\right)$$

b) Find the constant a so that the vectors $\underline{v}$ = $\underline{i} - 3\underline{j} + 4\underline{k}$ & $\underline{w} = a\underline{i} + 9\underline{j} - 12\underline{k}$ are parallel.

SOLUTION:

As $\underline{v}$ ∥ $\underline{w}$

$\therefore \frac{a}{1} = \frac{9}{-3}$ (Direction ratio are same)

⟹ a = −3

c) Find a vectors of length 5 in the direction opposite that of $\underline{v}$ = $\underline{i} - 2\underline{j} + 3\underline{k}$ (2019 S.Q)

SOLUTION: $\underline{u}$ =?

Given: $|\underline{u}| = 5$, $\underline{v}$ = $\underline{i} - 2\underline{j} + 3\underline{k}$

As $\underline{u}$ ∥ $\underline{v}$

So $\underline{u} = -\underline{v}$

$\Rightarrow \frac{\underline{u}}{|\underline{u}|} = -\frac{\underline{v}}{|\underline{v}|}$

$\underline{u} = -|\underline{u}|\frac{\underline{v}}{|\underline{v}|}$

$\underline{u} = -(5)\frac{\underline{i}-2\underline{j}+3\underline{k}}{\sqrt{1^2+(-2)^2+3^2}}$

$\underline{u} = -\frac{5[\underline{i}-2\underline{j}+3\underline{k}]}{\sqrt{14}}$

$\underline{u} = \frac{-5\underline{i}+10\underline{j}-15\underline{k}}{\sqrt{14}}$

$\underline{u} = \frac{-5}{\sqrt{14}}\underline{i} + \frac{10}{\sqrt{14}}\underline{j} - \frac{15}{\sqrt{14}}\underline{k}$

d) Find a & b so that the vectors

$3\underline{i} - \underline{j} + 4\underline{k}$, $a\underline{i} + b\underline{j} - 2\underline{k}$ are parallel.

SOLUTION: As $\underline{v} \parallel \underline{w}$

$\therefore \frac{a}{3} = \frac{b}{-1} = \frac{-2}{4} \Rightarrow \frac{a}{3} = \frac{b}{-1} = \frac{-1}{2}$

$\Rightarrow \frac{a}{3} = \frac{-1}{2}$ and $\frac{b}{-1} = \frac{-1}{2}$

$\Rightarrow a = -\frac{3}{2}$ and $b = \frac{1}{2}$

Q.11: Find the direction cosines for the given vector.

i) $3\underline{i} - \underline{j} + 2\underline{k}$

SOLTION:

Let $\underline{v} = 3\underline{i} - \underline{j} + 2\underline{k}$

$|\underline{v}| = \sqrt{(3)^2 + (-1)^2 + (2)^2}$

$|\underline{v}| = \sqrt{9 + 1 + 4} = \sqrt{14}$

Direction ratio of v are 3 , −1 , 2

Direction cosines of v = $\frac{Direction\ ratio}{magnitude}$

Direction cosines of v = $\frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}}$

ii) $6\underline{i} - 2\underline{j} + \underline{k}$ (2017S.Q)

SOLTION:

Let $\underline{v} = 6\underline{i} - 2\underline{j} + \underline{k}$

$|\underline{v}| = \sqrt{(6)^2 + (-2)^2 + (1)^2}$

$|\underline{v}| = \sqrt{36 + 4 + 1} = \sqrt{41}$

Direction ratio of v are 6 , −2 , 1

Direction cosines of v = $\frac{Direction\ ratio}{magnitude}$

Direction cosines of v = $\frac{6}{\sqrt{41}}, \frac{-2}{\sqrt{41}}, \frac{1}{\sqrt{41}}$

iii) $\overrightarrow{PQ}$, where p = (2,1,5) & Q = (1,3,1)

$\overrightarrow{PQ} = (x_2 - x_1)\,\underline{i} + (y_2 - y_1)\,\underline{j} + (z_2 - z_1)\underline{k}$

$\overrightarrow{PQ} = (1-2)\,\underline{i} + (3-1)\,\underline{j} + (1-5)\,\underline{k}$

$\overrightarrow{PQ} = -\underline{i} + 2\underline{j} - 4\underline{k}$

$|\overrightarrow{PQ}| = \sqrt{(-1)^2 + (2)^2 + (-4)^2}$

$|\overrightarrow{PQ}| = \sqrt{1 + 4 + 16} = \sqrt{21}$

Direction ratio of v are −1 , 2 , −4

Direction cosines of v = $\frac{Direction\ ratio}{magnitude}$

Direction cosines of v = $\frac{-1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{-4}{\sqrt{21}}$

Q.12: Which of the following triples can be the direction angles of a single vector:

i) 45° , 45° , 60° (2018S.Q)

SOLUTION:

Let α = 45° , β = 45° , γ = 60°

If α , β , γ are direction angles of a single vector, then

$\boldsymbol{cos2\alpha + cos2\beta + cos2cos^2\alpha + cos^2\beta + cos^2\gamma = 1}$

L.H.S = $cos^2\alpha + cos^2\beta + cos^2\gamma$

= $cos^2 45° + cos^2 45° + cos^2 60°$

= $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2$

= $\frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{2+2+1}{4} = \frac{5}{4} \neq$ R.H.S

Thus , **β , γ** are not the direction angles of a single vector.

ii) 30° , 45° , 60°

SOLUTION:

Let α = 30° , β = 45° , γ = 60°

If α , β , γ are direction angles of a single vector, then

$\boldsymbol{cos^2\alpha + cos^2\beta + cos^2\gamma = 1}$

L.H.S = $cos^2\alpha + cos^2\beta + cos^2\gamma$

= $cos^2 30° + cos^2 45° + cos^2 60°$

= $\left(\frac{3}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2$

= $\frac{3}{4} + \frac{1}{2} + \frac{1}{4} = \frac{3+2+1}{4} = \frac{3}{2} \neq$ R.H.S

Thus , **β , γ** are not the direction angles of a single vector.

iii) 45° , 60° , 60°

SOLUTION:

Let α = 45° , β = 60° , γ = 60°

If α , β , γ are direction angles of a single vector, then

$\boldsymbol{cos^2\alpha + cos^2\beta + cos^2\gamma = 1}$

L.H.S = $cos^2\alpha + cos^2\beta + cos^2\gamma$

= $cos^2 45° + cos^2 60° + cos^2 60°$

= $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$

= $\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{2+1+1}{4} = \frac{4}{4} = 1$ R.H.S

Thus, **β**, are the direction angles of a single vector.

The Scalar product of two vectors (Dot product)

Definition:

The scalar product of two non-zero vectors $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u}.\vec{v}$ (read as $\vec{u}$ dot $\vec{v}$) and defined as ;

$\vec{u}.\vec{v} = |\vec{u}||\vec{v}|cos\theta$

Where θ is angle from $\vec{u}$ to $\vec{v}$ and $0 \leq \theta \leq \pi$

Note: the dot product is also referred to the scalar product or the inner product.

Deduction of the important results:

For unit vector $\underline{i}, \underline{j}, \underline{k}$ we have.

(a) $\underline{i}.\underline{i} = |\underline{i}||\underline{i}|cos0^0 = 1$

$\underline{j}.\underline{j} = |\underline{j}||\underline{j}|cos0^0 = 1$

$\underline{k}.\underline{k} = |\underline{k}||\underline{k}|cos0^0 = 1$

(b) $\underline{i}.\underline{j} = |\underline{i}||\underline{j}|cos90^0 = 0$

$\underline{j}.\underline{k} = |\underline{j}||\underline{k}|cos90^0 = 0$

$\underline{k}.\underline{i} = |\underline{k}||\underline{i}|cos90^0 = 0$

(c) prove that $\underline{u}.\underline{v} = \underline{v}.\underline{u}$

$\underline{u}.\underline{v} = |\underline{u}||\underline{v}|cos\theta$

$$\underline{u}.\underline{v} = |\underline{u}||\underline{v}|\cos(-\theta)$$
$$\underline{u}.\underline{v} = |\underline{v}||\underline{u}|cos\theta$$
$$\underline{u}.\underline{v} = \underline{v}.\underline{u}$$

Hence dot product of two vectors is commutative.

Perpendicular (orthogonal) vectors

Two non-zero vectors $\underline{u}$ *and* $\underline{v}$ *are perpendicular if and only if* $\underline{u}.\underline{v} = 0$

$$(\because angle\ b\backslash w\ \vec{u}\ and\ \vec{v}\ is\ \frac{\pi}{2}\ so\ \cos\frac{\pi}{2} = 0)$$
$$thus\ \vec{u}.\vec{v} = |\vec{u}||\vec{v}|cos\theta\ \cos\frac{\pi}{2} \Rightarrow \vec{u}.\vec{v} = 0$$

Properties of Dot Product:

Let $\underline{u}, \underline{v}$ *and* $\underline{w}$ *be vectors and let* $c \in R$

$$then\ (i)\ \underline{u}.\underline{v} = 0 \Rightarrow \underline{u} = 0\ or\ \underline{v} = 0$$
$$(ii)\ \underline{u}.\underline{v} = \underline{v}.\underline{u} (Commutative\ Property)$$
$$(iii)\ \underline{u}.(\underline{v} + \underline{w}) = \underline{u}.\underline{v} + \underline{u}.\underline{w}\ (dis.property)$$
$$(iv)\ (cu).v = c(u.v)\ (c\ is\ scalar)$$
$$(v)\ \underline{u}.\underline{v} = |\underline{u}|^2$$

Analytical Expression of dot product $\underline{u}.\underline{v}$:

(Dot product of vectors in their components from)

$$let\ \underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}\ and$$
$$\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$$

Be two vectors

Now

$$\underline{u}.\underline{v} = (a_1\underline{i} + b_1\underline{j} + c_1\underline{k}).(a_2\underline{i} + b_2\underline{j} + c_2\underline{k})$$
$$= (a_1.a_2)(\underline{i}.\underline{i}) + a_1b_2(\underline{i}.\underline{j}) + a_1c_2(\underline{i}.\underline{k}) + b_1a_2(\underline{j}.\underline{i})$$
$$+b_1b_2(\underline{i}.\underline{j}) + b_1c_2(\underline{j}.\underline{k}) + c_1a_2(\underline{k}.\underline{i}) + c_1b_2(\underline{k}.\underline{j}) + c_1c_2(\underline{k}.\underline{k})$$
$$\Rightarrow \underline{u}.\underline{v} = a_1a_2 + b_1b_2 + c_1c_2$$

Angle $b\backslash w$ **two vectors:**

For two vectors $\underline{u}$ *and* $\underline{v}$

$$(a)\quad \vec{u}.\vec{v} = |\vec{u}||\vec{v}|cos\theta \quad \because\ o \le \theta \le \pi$$
$$\because cos\theta = \frac{\vec{u}.\vec{v}}{|\underline{u}||\underline{v}|}$$
$$(b)\ if\ \underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}\ and$$
$$\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$$
$$\Rightarrow \underline{u}.\underline{v} = a_1a_2 + b_1b_2 + c_1c_2$$
$$|\underline{u}| = \sqrt{a_1^2 + b_1^2 + c_1^2}\quad ,\quad |\underline{v}| = \sqrt{a_2^2 + b_2^2 + c_2^2}$$

Since

$$cos\theta = \frac{\vec{u}.\vec{v}}{|\underline{u}||\underline{v}|}$$
$$\Rightarrow \cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Corollaries:

if $\theta = 0^0$ *or* π *the vector* $\underline{u}$ *and* $\underline{v}$ *are collinear.*

(ii) $\boldsymbol{if\ \theta = \frac{\pi}{2},\ cos\theta = 0} \Rightarrow \underline{u}.\underline{v} = 0$

Then vector $\underline{u}$ *and* $\underline{v}$ are $\perp$ *or othogonal.*

Projection of a vector upon another vector:

Let $\overrightarrow{OA} = \underline{u}$ *and* $\overrightarrow{OB} = \underline{v}$

let θ *be the angle between then such that* $0 \le \theta \le \pi$ *Draw* $\overline{BM} \perp \overline{OA}$ *then* $\overline{OM}$ *is called the projection of* $\underline{v}$ *along* $\underline{u}$ *Now in* ΔOMB.

$$cos\theta = \frac{|\overline{OM}|}{|\overline{OB}|} \Rightarrow |\overline{OM}| = |\overline{OB}|cos\theta$$
$$|\overline{OM}| = |\overline{v}|cos\theta \quad \because cos\theta = \frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$$
$$\because\ projction\ of\ \underline{v}\ along\ \underline{u} = \frac{\underline{u}.\underline{v}}{|\underline{u}|}$$

Similarly,

$$\because\ projction\ of\ \underline{u}\ along\ \underline{v} = \frac{\underline{u}.\underline{v}}{|\underline{v}|}$$

Exercise NO 7.3

Question No1

Find the cosine of angle $\boldsymbol{\theta}$ ***between*** $\underline{\boldsymbol{u}}$ ***and*** $\underline{\boldsymbol{v}}$

i) $\underline{u}$ **= 3**$\underline{i}$**+**$\underline{j}$**−**$\underline{k}$ **,** $\underline{v}$ **= 2**$\underline{i}$**−**$\underline{j}$**+**$\underline{k}$ **(2016 S.Q)**

SOLUTION:

$\underline{u}$**= 3**$\underline{i}$**+**$\underline{j}$**−**$\underline{k}$ **,** $\underline{v}$ **= 2**$\underline{i}$**−**$\underline{j}$**+**$\underline{k}$

If θ is the angle between u and v, then

$$\cos\theta = \frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$$
$$\cos\theta = \frac{(3\underline{i}+\underline{j}-\underline{k}).(2\underline{i}-\underline{j}+\underline{k})}{\sqrt{3^2+1^2+(-1)^2}.\sqrt{2^2+(-1)^2+1^2}}$$
$$\cos\theta = \frac{6-1-1}{\sqrt{9+1+1}.\sqrt{4+1+1}}$$
$$\cos\theta = \frac{4}{\sqrt{11}.\sqrt{6}} = \frac{4}{\sqrt{66}}$$

ii) u = $\underline{i}$**−3**$\underline{j}$**+4**$\underline{k}$ **,** $\underline{v}$ **= 4**$\underline{i}$**−**$\underline{j}$**+3**$\underline{k}$

SOLUTION:

$\underline{u}$ = $\underline{i}$−3$\underline{j}$+4$\underline{k}$, $\underline{v}$ = 4$\underline{i}$−$\underline{j}$+3$\underline{k}$

If θ is the angle between u and v, then

$$\cos\theta = \frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$$
$$\cos\theta = \frac{(\underline{i}-3\underline{j}+4\underline{k}).(4\underline{i}-\underline{j}+3\underline{k})}{\sqrt{1^2+(-3)^2+4^2}.\sqrt{(4)^2+(-1)^2+3^2}}$$
$$\cos\theta = \frac{4+3+12}{\sqrt{1+9+16}.\sqrt{16+1+9}}$$
$$\cos\theta = \frac{19}{\sqrt{26}.\sqrt{26}} = \frac{19}{26}$$

iii) $\underline{u}$ **=** $[-3, 5]$**,** $\underline{v}$ **=** $[6, -2]$

SOLUTION:

$\underline{u}$ = −3$\underline{i}$+5$\underline{j}$, $\underline{v}$= 6$\underline{i}$−2$\underline{j}$

If θ is the angle between u and v, then

$$\cos\theta = \frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$$
$$\cos\theta = \frac{(-3\underline{i}+5\underline{j}).(6\underline{i}-2\underline{j})}{\sqrt{(-3)^2+(5)^2}.\sqrt{(6)^2+(-2)^2}}$$
$$\cos\theta = \frac{-18-10}{\sqrt{9+25}.\sqrt{36+4}}$$

$\cos\theta = \frac{-28}{\sqrt{34}.\sqrt{40}} = \frac{-28}{\sqrt{1360}}$

$$\cos\theta = \frac{-28}{\sqrt{16}\,.\sqrt{85}} = \frac{-28}{4\sqrt{85}} = \frac{-7}{\sqrt{85}}$$

iv) $\underline{u}$ = $[2,-3,1]$, $\underline{v}$ = $[2,4,1]$

SOLUTION:

$\underline{u} = 2\underline{i} - 3\underline{j} + \underline{k}$, $\underline{v} = 2\underline{i} + 4\underline{j} + \underline{k}$

If θ is the angle between u and v, then

$\cos\theta = \frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$

$\cos\theta = \frac{(2\underline{i}-3\underline{j}+\underline{k}).(2\underline{i}+4\underline{j}+\underline{k})}{\sqrt{2^2+(-3)^2+1^2}.\sqrt{(2)^2+(4)^2+1^2}}$

$\cos\theta = \frac{4-12+1}{\sqrt{4+9+1}.\sqrt{4+16+1}}$

$\cos\theta = \frac{-7}{\sqrt{14}.\sqrt{21}} = \frac{-7}{\sqrt{294}}$

$\cos\theta = \frac{-7}{\sqrt{49\times6}} = \frac{-7}{7\sqrt{6}} = \frac{-1}{\sqrt{6}}$

Q.2: Calculate the projection of a along b and b along a when:

i) $\underline{a} = \underline{i} - \underline{k}$, $\underline{b} = \underline{j} + \underline{k}$

SOLUTION:

Projection of $\underline{a}$ along $\underline{b}$ = $\underline{a}.\hat{b} = \underline{a}\,.\frac{\underline{b}}{|\underline{b}|} = \frac{\underline{a}.\underline{b}}{|\underline{b}|} =$

$\frac{(\underline{i}-k).(\underline{j}+k)}{\sqrt{1^2+1^2}} = \frac{0+0+1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

Projection of $\underline{b}$ along $\underline{a}$ = $\underline{b}.\,\hat{a} = \underline{b}\,.\frac{\underline{a}}{|\underline{a}|} = \frac{\underline{b}.\underline{a}}{|\underline{a}|} =$

$\frac{(\underline{j}+k).(\underline{i}-k)}{\sqrt{1^2+(-1)^2}} = \frac{0+0-1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$

ii) $\underline{a} = 3\underline{i} + \underline{j} - \underline{k}$, $\underline{b} = -2\underline{i} - \underline{j} + \underline{k}$ (2018 S.Q)

SOLUTION:

Projection of $\underline{a}$ along $\underline{b}$ = $\underline{a}.\,\hat{b} = \underline{a}\,.\frac{\underline{b}}{|\underline{b}|} = \frac{\underline{a}.\underline{b}}{|\underline{b}|} =$

$\frac{(3\underline{i}+\underline{j}-\underline{k}).(-2\underline{i}-\underline{j}+\underline{k})}{\sqrt{(-2)^2+1^2+1^2}} = \frac{-6-1-1}{\sqrt{4+1+1}} = -\frac{8}{\sqrt{6}}$

Projection of $\underline{b}$ along $\underline{a}$ = $\underline{b}.\,\hat{a} = \underline{b}.\frac{\underline{a}}{|\underline{a}|} = \frac{\underline{b}.\underline{a}}{|\underline{a}|} =$

$\frac{(-2\underline{i}-\underline{j}+\underline{k}).(3\underline{i}+\underline{j}-\underline{k})}{\sqrt{3^2+(1)^2+(-1)^2}} = \frac{-6-1-1}{\sqrt{9+1+1}} = -\frac{8}{\sqrt{11}}$

Q.3(7.3): Find a real number 'α' so that the vectors u and v are perpendicular.

i) $\underline{u} = 2\alpha\underline{i} + \underline{j} - \underline{k}$, $\underline{v} = i + \alpha j + 4\underline{k}$

SOLUTION:

Since u and v are perpendicular, so

$\underline{u}.\,\underline{v} = 0$

$(2\alpha\underline{i}+\underline{j}-k)(\,i+\alpha\underline{j}+4\underline{k}) = 0$

$2\alpha+\alpha-4 = 0$

$3\alpha = 4 \implies \alpha = \frac{4}{3}$

ii) $\underline{u} = \alpha\underline{i} + 2\alpha\underline{j} - \underline{k}$, $\underline{v} = \underline{i} + \alpha\underline{j} + 3\underline{k}$

SOLUTION:

Since u and v are perpendicular, so

$\underline{u}\,.\,\underline{v} = 0$

$(\alpha\underline{i}+2\alpha\underline{j}-k)(\,\underline{i}+\alpha\underline{j}+3\underline{k}) = 0$

$\alpha+2\alpha^2-3 = 0$

$2\alpha^2+\alpha-3 = 0$

$2\alpha^2+3\alpha-2\alpha-3 = 0$

$\alpha(2\alpha+3)-1(2\alpha+3) = 0$

$(2\alpha+3)(\alpha-1) = 0$

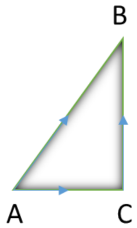
$2\alpha+3 = 0$ or $\alpha-1 = 0$

$\alpha = -\frac{3}{2}$ or $\alpha = 1$

Q.4: Find the number 'Z' so that the triangle with vertices A(1,−1,0), B(−2,2,1), C(0,2,Z) is a right triangle with right angle at C.

SOLUTION:

A(1,−1,0), B(−2,2,1), C(0,2,Z)



$\overrightarrow{CA} = (x_2-x_1)\,\underline{i} + (y_2-y_1)\,\underline{j} + (z_2-z_1)\underline{k}$

$\overrightarrow{CA} = (1-0)\,\underline{i} + (-1-2)\,\underline{j} + (0-Z)\,\underline{k}$

$\overrightarrow{CA} = \underline{i} - 3j - Z\underline{k}$

$\overrightarrow{CB} = (x_2-x_1)\,\underline{i} + (y_2-y_1)\,\underline{j} + (z_2-z_1)\underline{k}$

$\overrightarrow{CB} = (-2-0)\,\underline{i} + (2-2)\,\underline{j} + (1-Z)\,\underline{k}$

$\overrightarrow{CB} = -2\underline{i} + 0\underline{j} + (1-Z)\,\underline{k}$

As m∠ACB = 90° , so $\overrightarrow{CB} \perp \overrightarrow{CA}$

$\implies \overrightarrow{CA}\,.\,\overrightarrow{CB} = 0$

$(\underline{i} - 3\underline{j} - Z\underline{k}).(-2\underline{i} + 0\underline{j} + (1-Z)\,\underline{k}) = 0$

$-2-0-Z(1-Z) = 0$

$-2-Z+Z^2 = 0$

$Z^2 - Z - 2 = 0$

$Z^2 - 2Z + Z - 2 = 0$

$Z(Z-2) + 1(Z-2) = 0$

$(Z-2)(Z+1) = 0$

$Z - 2 = 0$ or $Z + 1 \Rightarrow Z = -1$

Q.5: if $\underline{v}$ is a vector for which $\underline{v}\,.\,\underline{i} = 0$, $\underline{v}.\,\underline{j} = 0$, $\underline{v}\,.\,\underline{k} = 0$. Find $\underline{v}$. (2018 S.Q)

SOLUTION:

Let the required vector $\underline{v}$ = xi + yj + z$\underline{k}$ (i)

As $\underline{v}$. i = 0 ⟹ (xi + yj + z$\underline{k}$).i = 0 ⟹ x = 0

As $\underline{v}$. j = 0 ⟹ (xi + yj + z$\underline{k}$).j = 0 ⟹ y = 0

As $\underline{v}$. k = 0 ⟹ (xi + yj + zk). $\underline{k}$ = 0 ⟹ Z = 0

Putting the values in (i)

$\underline{v}$ = 0i + 0j + 0k = 0 (**Null vector**)

Q.6: Show that the vectors 3i−2$\underline{j}$+$\underline{k}$, i−3$\underline{j}$+5k$\underline{k}$and 2i+$\underline{j}$-4k form a right triangle.

SOLUTION:

(i) Let u = $3\underline{i}-2\underline{j}+\underline{k}$, v = $\underline{i}-3\underline{j}+5\underline{k}$, w = $2\underline{i}+\underline{j}-4\underline{k}$

Since v + w = $\underline{i}-3\underline{j}+5k+2\underline{i}-4\underline{k} = 3\underline{i}-2\underline{j}+\underline{k}$ = u

Therefore u , v and w are the sides of a triangle.

Since u . w = $(3\underline{i}-2\underline{j}+k).(\,2\underline{i}+\underline{j}-4\underline{k})$ = 6−2−4 = 0

⟹ u ⊥ w = 0

∴ u , v and w are the sides of a right triangle.

(ii) Show that the set of points p = (1,3,2) , Q = (4,1,4) , R = (6,5,5) form a right triangle.

SOLUTION:

Let p = (1,3,2) , Q = (4,1,4) , R = (6,5,5)

$\overrightarrow{PQ} = (x_2 - x_1)\,\underline{i} + (y_2 - y_1)\,\underline{j} + (z_2 - z_1)\underline{k}$

$\overrightarrow{PQ} = (4-1)\,\underline{i} + (1-3)\,\underline{j} + (4-2)\,\underline{k}$

$\overrightarrow{PQ} = 3\underline{i} - 2\,\underline{j} + 2\underline{k}$

$\overrightarrow{QR} = (x_2 - x_1)\,\underline{i} + (y_2 - y_1)\,\underline{j} + (z_2 - z_1)\underline{k}$

$\overrightarrow{QR} = (6-4)\,\underline{i} + (5-1)\,\underline{j} + (5-4)\,\underline{k}$

$\overrightarrow{QR} = 2\underline{i} + 4\underline{j} + \underline{k}$

$\overrightarrow{PR} = (x_2 - x_1)\,\underline{i} + (y_2 - y_1)\,\underline{j} + (z_2 - z_1)\underline{k}$

$\overrightarrow{PR} = (6-1)\,\underline{i} + (5-3)\,\underline{j} + (5-2)\,\underline{k}$

$\overrightarrow{PR} = 5\underline{i} + 2\underline{j} + 3\underline{k}$

$\because \overrightarrow{PQ} + \overrightarrow{QR} = 3\underline{i} - 2\,j + 2\underline{k} + 2\underline{i} + 4\underline{j} + \underline{k} = 5\underline{i} + 2\underline{j} + 3\underline{k} = \overrightarrow{PR}$

∴ P, Q, R are the vertices of a triangle.

$\because\ \overrightarrow{PQ} \cdot \overrightarrow{QR} = (3\underline{i} - 2\,\underline{j} + 2\underline{k}) \cdot (2\underline{i} + 4\underline{j} + \underline{k}) = 6 - 8 + 2 = 0$

∴ P, Q, R are the vertices of a right triangle.

Q.7: Show that mid point of hypotenuse of a right triangle is equidistance from its vertices.

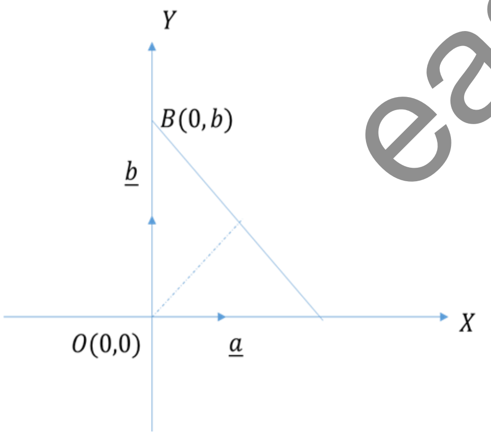
SOLUTION:

Consider a right triangle AOB such that O(0,0) , A(a,0) , B(0,b).

Let M be the mid point of hypotenuse $\overline{AB}$ such that

$M\left(\frac{a+0}{2}, \frac{b+0}{2}\right) = M\left(\frac{a}{2}, \frac{b}{2}\right)$

We want to prove: $|\overrightarrow{AM}| = |\overrightarrow{BM}| = |\overrightarrow{OM}|$



$\overrightarrow{AM} = \left(\frac{a}{2} - a\right)\underline{i} + \left(\frac{b}{2} - 0\right)\underline{j}$

$\overrightarrow{AM} = \left(-\frac{a}{2}\right)\underline{i} + \left(\frac{b}{2}\right)\underline{j}$

$|\overrightarrow{AM}| = \sqrt{\left(-\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$

$|\overrightarrow{AM}| = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \sqrt{\frac{a^2+b^2}{4}}$

$|\overrightarrow{AM}| = \frac{1}{2}\sqrt{a^2 + b^2}$

$\overrightarrow{BM} = \left(\frac{a}{2} - 0\right)\underline{i} + \left(\frac{b}{2} - b\right)\underline{j}$

$\overrightarrow{BM} = \left(\frac{a}{2}\right)\underline{i} - \left(\frac{b}{2}\right)\underline{j}$

$|\overrightarrow{BM}| = \sqrt{\left(\frac{a}{2}\right)^2 + \left(-\frac{b}{2}\right)^2}$

$|\overrightarrow{BM}| = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \sqrt{\frac{a^2+b^2}{4}}$

$|\overrightarrow{BM}| = \frac{1}{2}\sqrt{a^2 + b^2}$

$\overrightarrow{OM} = \left(\frac{a}{2} - 0\right)\underline{i} + \left(\frac{b}{2} - 0\right)\underline{j}$

$\overrightarrow{OM} = \left(\frac{a}{2}\right)\underline{i} + \left(\frac{b}{2}\right)\underline{j}$

$|\overrightarrow{OM}| = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$

$|\overrightarrow{OM}| = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \sqrt{\frac{a^2+b^2}{4}}$

$|\overrightarrow{OM}| = \frac{1}{2}\sqrt{a^2 + b^2}$

Clearly $|\overrightarrow{AM}| = |\overrightarrow{BM}| = |\overrightarrow{OM}|$, which is the required result.

Q.8: Prove that the perpendicular bisectors of the sides of a triangle are concurrent.

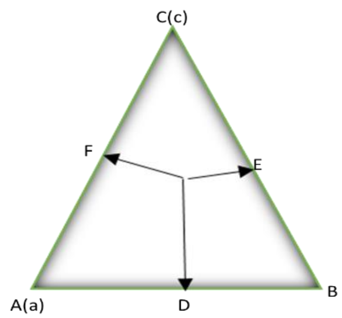
SOLUTION:

Consider the triangle ABC such that D , E , F are the mid points of sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AC}$ respectively. **C (c)**

Then $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$, $\overrightarrow{OC} = c$, $\overrightarrow{OD} = \frac{a+b}{2}$, $\overrightarrow{OE} = \frac{b+c}{2}$, $\overrightarrow{OF} = \frac{a+c}{2}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \underline{b} - \underline{a}$, $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \underline{c} - \underline{b}$, $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \underline{c} - \underline{a}$

Let $\overrightarrow{OD} \perp \overrightarrow{AB}$, $\overrightarrow{OE} \perp \overrightarrow{BC}$ be the right bisectors which meet at 'O'



As $\overrightarrow{OD} \perp \overrightarrow{AB}$, so

$\overrightarrow{OD} \cdot \overrightarrow{AB} = 0$

$\left(\frac{a+b}{2}\right) \cdot (\underline{b} - \underline{a}) = 0$

$(\underline{b} + \underline{a}) \cdot (\underline{b} - \underline{a}) = 0$

$b^2 - a^2 = 0 \rightarrow$ (i)

As $\overrightarrow{OE} \perp \overrightarrow{BC}$, so

$\overrightarrow{OE} \cdot \overrightarrow{BC} = 0$

$\left(\frac{b+c}{2}\right) \cdot (c - \underline{b}) = 0$

$(c + \underline{b}) \cdot (c - \underline{b}) = 0$

$c^2 - b^2 = 0 \rightarrow$(ii)

Adding (i) and (ii)

$b^2 - a^2 + c^2 - b^2 = 0$

$\Rightarrow \quad c^2 - a^2 = 0$

$\Rightarrow (\underline{c}+\underline{a}) \, . \, (\underline{c} - \underline{a}) = 0$

Divide both side by 2

$\Rightarrow \left(\frac{\underline{c}+\underline{a}}{2}\right) . \left(\underline{c} - \underline{a}\right) = \frac{0}{2}$

$\Rightarrow \overrightarrow{OF} \, . \, \overrightarrow{AC} = 0$

$\Rightarrow \overrightarrow{OF} \perp \overrightarrow{AC}$

So, $\overrightarrow{OF}$ is also a right bisector of $\overrightarrow{AC}$.

Thus all the right bisector are concurrent at 'O'

Q.9: Prove that the altitudes of a triangle are concurrent.

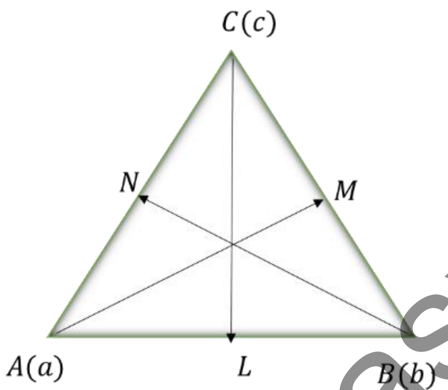
SOLUTION:

Consider the triangle ABC such that

Then $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$, $\overrightarrow{OC} = \underline{c}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \underline{b} - \underline{a}$, $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \underline{c} - \underline{b}$, $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \underline{c} - \underline{a}$

Let $\overrightarrow{AM} \perp \overrightarrow{BC}$, $\overrightarrow{CL} \perp \overrightarrow{AB}$ be the altitudes of a triangle which meet at 'O'



As $\overrightarrow{AM} \perp \overrightarrow{BC}$

so $\overrightarrow{OA} \perp \overrightarrow{BC}$

$\Rightarrow \overrightarrow{OA} \, . \, \overrightarrow{BC} = 0$

$\underline{a} \, . \, (\underline{c} - \underline{b}) = 0$

$\underline{a}.\underline{c} - \underline{a}.\underline{b} = 0 \rightarrow (i)$

As $\overrightarrow{CL} \perp \overrightarrow{AB}$

so $\overrightarrow{OC} \perp \overrightarrow{AB}$

$\Rightarrow \overrightarrow{OC} \, . \, \overrightarrow{AB} = 0$

$\underline{c} \, . \, (\underline{b} - \underline{a}) = 0$

$\underline{c}.\underline{b} - \underline{c}.\underline{a} = 0 \rightarrow$ (ii)

Adding (i) and (ii)

$\underline{a}.\underline{c} - \underline{a}.\underline{b} + \underline{c}.\underline{b} - \underline{c}.\underline{a} = 0$

$\Rightarrow \underline{c}.\underline{b} - \underline{a}.\underline{b} = 0$

$\Rightarrow (\underline{c} - \underline{a}).\underline{b} = 0$

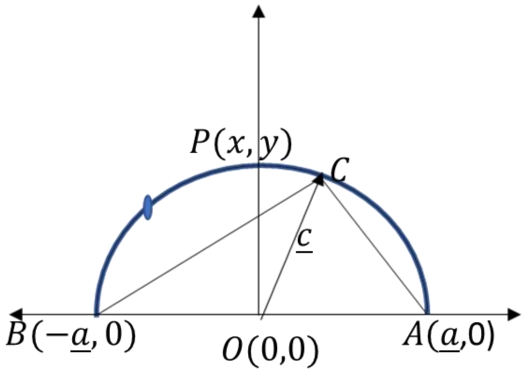
$\Rightarrow \overrightarrow{AC} \, . \, \overrightarrow{OB} = 0$

$\Rightarrow \overrightarrow{OB} \perp \overrightarrow{AC}$

$\Rightarrow \overrightarrow{BN} \perp \overrightarrow{AC}$ & passing through; O'

So altitude of the triangle are concurrent at 'o'

Q.10: Prove that the angle in a semi circle is a right angle. (2017 L.Q)



Consider a semicircle APB with center at origin O(0,0) and A , B are ends of diameter having position vectors $\underline{a}$, $-\underline{a}$ respectively.

Let C be any point on semicircle having position vector c.

Clearly $|\underline{a}| = |-\underline{a}| = |\underline{c}|$ (radii of semicircle) (i)

Now $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \underline{c} - \underline{a}$

And $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \underline{c} - (-\underline{a}) = \underline{c} + \underline{a}$

Consider $\overrightarrow{AC} \, . \, \overrightarrow{BC} = (\underline{c} - \underline{a}) \, . \, (\underline{c} + \underline{a})$

$\overrightarrow{AC} \, . \, \overrightarrow{BC} = \underline{c} \, . \, (\underline{c} + \underline{a}) - \underline{a} \, . \, (\underline{c} + \underline{a})$

$\overrightarrow{AC} \, . \, \overrightarrow{BC} = \underline{c} \, . \, \underline{c} + \underline{c} \, . \, \underline{a} - \underline{a} \, . \, \underline{c} - \underline{a} \, . \, \underline{a}$

$\overrightarrow{AC} \, . \, \overrightarrow{BC} = c^2 + \underline{a} \, . \, \underline{c} - \underline{a} \, . \, \underline{c} - a^2 \qquad \because \underline{a} \, . \, \underline{c} = \underline{c} \, . \, \underline{a}$

$\overrightarrow{AC} \, . \, \overrightarrow{BC} = c^2 - a^2$

$\overrightarrow{AC} \, . \, \overrightarrow{BC} = c^2 - c^2 \qquad \because |\underline{a}| = |\underline{c}|$

$\overrightarrow{AC} \, . \, \overrightarrow{BC} = 0$

$\Rightarrow \quad \overrightarrow{AC} \perp \overrightarrow{BC}$

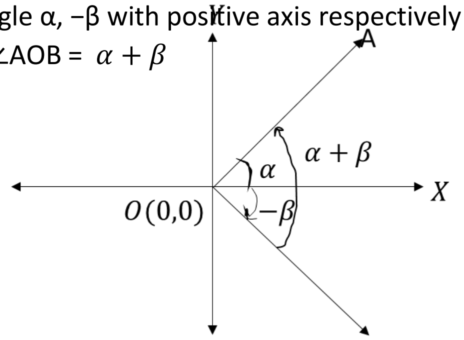
Thus the angle in a semicircle is a right angle

Q.11: Prove that $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

SOLUTION:

Consider two unit vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in XY-plane making angle α, −β with positive axis respectively

So that m∠AOB = $\alpha + \beta$



$\overrightarrow{OA} = \cos\alpha\,\underline{\boldsymbol{i}} + \sin\alpha\,\underline{\boldsymbol{j}}$

$\overrightarrow{OB} = \cos(-\beta)\,\underline{\boldsymbol{i}} + \sin(-\beta)\,\underline{\boldsymbol{j}} = \cos\beta\,\underline{\boldsymbol{i}} - \sin\beta\,\underline{\boldsymbol{j}}$

Taking dot product of $\overrightarrow{OA}$ and $\overrightarrow{OB}$

$\overrightarrow{OA}\,.\,\overrightarrow{OB} = (\cos\alpha\,\underline{\boldsymbol{i}} + \sin\alpha\,\underline{\boldsymbol{j}}).(\cos\beta\,\underline{\boldsymbol{i}} - \sin\beta\,\underline{\boldsymbol{j}})$

$|\overrightarrow{OA}|\,|\overrightarrow{OB}|\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$1\,.\,1\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad \because |\overrightarrow{OA}| = |\overrightarrow{OB}| = 1$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Q.12: Prove that in any triangle ABC

i) $b = c\cos A + a\cos C$

SOLUTION:

Consider a triangle ABC such that

$\overrightarrow{BC} = \underline{a}$, $\overrightarrow{CA} = \underline{b}$, $\overrightarrow{AB} = \underline{c}$

$\therefore \underline{a} + \underline{b} + \underline{c} = 0$

$\underline{b} = -\underline{a} - \underline{c}$

Taking dot product with b

$\underline{b}\,.\,\underline{b} = (-\underline{a} - \underline{c})\,.\,\underline{b}$

$b^2 = -\underline{a}.\underline{b} - c.\underline{b}$

$b^2 = -|\underline{a}||\underline{b}|\cos(\pi - C) - |\underline{c}||\underline{b}|\cos(\pi - A)$

$\because \cos(\pi - C) = (-\cos C)$

$\because \cos(\pi - A) = (-\cos A)$

$b^2 = -ab\,(-\cos C) - \text{cb}(-\cos A)$

$b^2 = ab\cos C + \text{cb}\cos A$

Taking b common

$b^2 = b[\text{a}\cos C + \text{c}\cos A]$

Cancel b on both sides

$\text{b} = a\cos C + c\cos A$

ii) $c = \text{a}\cos B + \text{b}\cos A$

SOLUTION:

Consider a triangle ABC such that

$\overrightarrow{BC} = \underline{a}$, $\overrightarrow{CA} = \underline{b}$, $\overrightarrow{AB} = \text{c}$

$\therefore \underline{a} + \underline{b} + \underline{c} = 0$

$\underline{c} = -\underline{a} - \underline{b}$

Taking dot product with c

$\underline{c}\,.\,\underline{c} = (-\underline{a} - \underline{b})\,.\,\underline{c}$

$c^2 = -\underline{a}.c - \underline{b}.\underline{c}$

$\because \cos(\pi - B) = (-\cos B)$

$\because \cos(\pi - A) = (-\cos A)$

$c^2 = -|\underline{a}||\underline{c}|\cos(\pi - B) - |\underline{b}||\underline{c}|\cos(\pi - A)$

$c^2 = -ac\,(-\cos B) - \text{bc}(-\cos A)$

$c^2 = ac\cos B + \text{bc}\cos A$

Taking c common

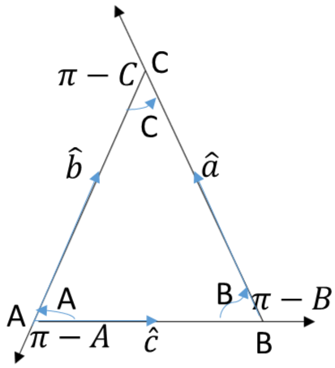
$c^2 = c[\text{a}\cos C + \text{c}\cos A]$

Cancel c on both sides

$\text{c} = a\cos B + c\cos A$

iii) $b^2 = c^2 + a^2 - 2ca\cos B$

SOLUTION:



Consider a triangle ABC such that

$\overrightarrow{BC} = \underline{a}$, $\overrightarrow{CA} = \underline{b}$, $\overrightarrow{AB} = \underline{c}$

$\therefore \underline{a} + \underline{b} + \underline{c} = 0$

$\underline{b} = -\underline{a} - \underline{c} = -(\underline{a}+\underline{c})$

Squaring on both sides

$(\underline{b})^2 = [-(\underline{a} + \underline{c})]^2$

$(\underline{b})^2 = (\underline{a} + \underline{c})^2$

$\underline{b}\,.\,\underline{b} = (\underline{a} + \underline{c}).(\underline{a} + \underline{c})$

$b^2 = \underline{a}.\underline{a} + \underline{a}.\underline{c} + \underline{c}.\underline{a} + \underline{c}.\underline{c}$

$b^2 = a^2 + c^2 + 2\underline{a}.\underline{c}$

$b^2 = a^2 + c^2 + 2|\underline{a}||\underline{c}|\cos(\pi - B)$

$\because \cos(\pi - B) = (-\cos B)$

$b^2 = a^2 + c^2 + 2ca\,(-\cos B)$

$b^2 = a^2 + c^2 - 2ac\cos B$

iii) $c^2 = a^2 + b^2 - 2ab\cos C$

SOLUTION:

Consider a triangle ABC such that

$\overrightarrow{BC} = \underline{a}$, $\overrightarrow{CA} = \underline{b}$, $\overrightarrow{AB} = \underline{c}$

$\therefore \underline{a} + \underline{b} + \underline{c} = 0$

$\underline{c} = -\underline{a} - \underline{b} = -(\underline{a}+\underline{b})$

squaring on both sides

$(\underline{c})^2 = [-(\underline{a} + \underline{b})]^2$

$(\underline{c})^2 = (\underline{a} + \underline{b})^2$

$\underline{c}\,.\,\underline{c} = (\underline{a} + \underline{b})(\underline{a} + \underline{b})$

$c^2 = \underline{a}.\underline{a} + \underline{a}.\underline{b} + \underline{b}.\underline{a} + \underline{b}.\underline{b}$

$c^2 = a^2 + b^2 + 2\underline{a}.\underline{b}$

$c^2 = a^2 + b^2 + 2|\underline{a}||\underline{b}|\cos(\pi - C)$

$\because \cos(\pi - C) = (-\cos C)$

$c^2 = a^2 + b^2 + 2ab\,(-\cos C)$

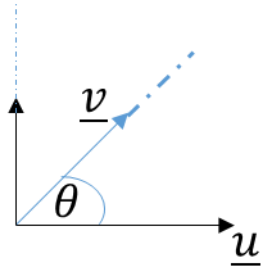
$c^2 = a^2 + b^2 - 2ab\cos C$

The cross product (vector product)

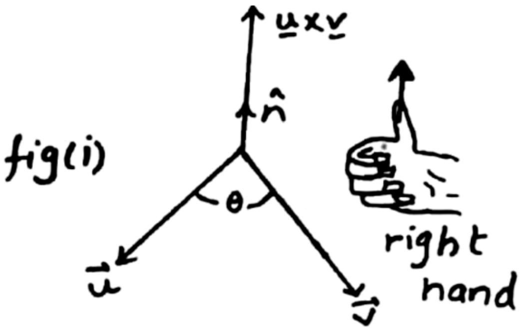
let $\underline{u}$ and $\underline{v}$ be two non − zero vectors. The cross or vector product of $\underline{u}$ and $\underline{v}$ written as $\underline{u} \times \underline{v}$ is define

as $\underline{u} \times \underline{v} = (|\underline{u}|\,|\underline{v}|\sin\theta)\hat{n}$

Where θ *is the angle* $b\backslash w$ *the vectors s. that* $o \le \theta \le \pi$ *and* $\hat{n}$ *is a unit vector* $\perp$ *ar to the plane of* $\underline{u}$ *and* $\underline{v}$ *written direction is given by the right hand rule.*

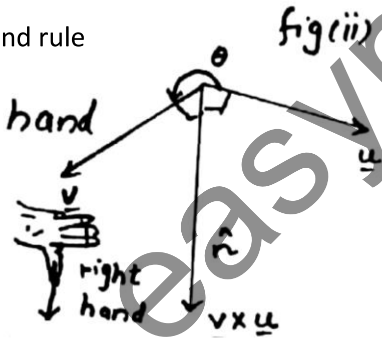


Right hand rule:



if the figure right hand point along the vecors $\underline{u}$ *and the curl towards the vector* $\underline{v}$, *then the thumbs will give* The direction of $\hat{n}$ which is $\underline{u} \times \underline{v}$

In fig.(ii) the right hand rule Shows the direction Of $\underline{v} \times \underline{u}$



Derivative of uses full results of cross products:

For unit vector $\underline{i}, \underline{j}, \underline{k}$ *we have.*

$$(a)\quad \underline{i} \times \underline{i} = |\underline{i}||\underline{i}| sino^0 \hat{n} = 0$$

$$\underline{j} \times \underline{j} = |\underline{j}|\,|\underline{j}|\, sino^0 \hat{n} = 0$$

$$\underline{k} \times \underline{k} = |\underline{k}||\underline{k}| sino^0 \hat{n} = 0$$

(b) $\underline{i} \times \underline{j} = |\underline{i}|\,|\underline{j}|\, sin9o^0 \hat{k} =$

$$\underline{j} \times \underline{k} = |\underline{j}|\,|\underline{k}| sin9o^0 \hat{\imath} = \hat{\imath}$$

$$\underline{k} \times \underline{i} = |\underline{k}||\underline{i}| sin9o^0 \hat{\jmath} = \hat{\jmath}$$

(c) Prove that $\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$

$$\underline{u} \times \underline{v} = |\underline{u}||\underline{v}| sin\theta \hat{n}$$

$$\underline{u}.\underline{v} = |\underline{u}||\underline{v}| \sin(-\theta) \hat{n}$$

$$\underline{u}.\underline{v} = |\underline{v}||\underline{u}| sin\theta \hat{n}$$

$$\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$$

$$\underline{u} \times \underline{u} = |\underline{u}||\underline{u}| sinO^0 \hat{n} = 0$$

Note:

The cross product of $\underline{i}, \underline{j}$ *and* $\underline{k}$ *are written in cyclic patteren*

It is helpful to remember.

Properties of cross product:

$$(\boldsymbol{i})\ \ \underline{u} \times \underline{v} = 0 \ \ if\ \underline{u} = 0 \ \ or\ \underline{v} = 0$$

$$(\boldsymbol{ii})\ \underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$$

$(\boldsymbol{iii})\ \underline{u} \times (\underline{v} \times \underline{w})$

$$= \underline{u} \times \underline{v} + \underline{u} \times \underline{w} \ \ (distributive\ property)$$

$$(\boldsymbol{iv})\ \underline{u} \times (k\underline{v}) = k(\underline{u}) \times \underline{v} = k(\underline{u} \times \underline{v})$$

$$(\boldsymbol{v})\ \ \underline{u} \times \underline{u} = \underline{0}$$

Analytical Expression of $\underline{u} \times \underline{v}$

Let

$$\underline{u} = a_1 \underline{i} + b_1 \underline{j} + c_1 \ \underline{k}\ and$$

$$\underline{v} = a_2 \underline{i} + b_2 \underline{j} + c_2 \ \underline{k}$$

$$\underline{u} \times \ \underline{v} = \left(a_1 \underline{i} + b_1 \underline{j} + c_1 \ \underline{k}\right) \times \left(a_2 \underline{i} + b_2 \underline{j} + c_2 \ \underline{k}\right)$$

$$= (a_1 a_2)(\underline{i} \times \underline{i}) + a_1 b_2 \left(\underline{i} \times \underline{j}\right) + a_1 c_2 (\underline{i} \times \underline{k}) + b_1 a_2 \left(\underline{j} \times \underline{i}\right)$$

$$+ b_1 b_2 \left(\underline{i} \times \underline{j}\right) + b_1 c_2 \left(\underline{j} \times \underline{k}\right) + c_1 a_2 (\underline{k} \times \underline{i}) + c_1 b_2 \left(\underline{k} \times \underline{j}\right) + c_1 c_2 (\underline{k} \times \underline{k})$$

$\Rightarrow \ a_1 b_2 \underline{k} + a_1 c_2 \underline{j} - b_1 a_2 \underline{k} + b_1 a_2 \underline{i} + c_1 a_2 \underline{j} - c_1 b_2 \underline{i}$

Rearrange we have,

$\Rightarrow \ \underline{u} \times \ \underline{v} = (b_1 c_2 - c_1 b_2)\underline{i} - (a_1 c_2 - c_1 a_2)\underline{j} + (a_1 b_2 - a_2 b_1)\underline{k}$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (b_1 c_2 - c_1 b_2)\underline{i} - (a_1 c_2 - c_1 a_2)\underline{j} + (a_1 b_2 - a_2 b_1)\underline{k}$$

$\Rightarrow \ (b_1 c_2 - c_1 b_2)\underline{i} - (a_1 c_2 - c_1 a_2)\underline{j} + (a_1 b_2 - a_2 b_1)\underline{k}$

Parallel vectors:

If $\underline{u}$ *and* $\underline{v}$ *are* $||$*vectors* $\theta = 0, sinO^0 = 0$ *then*

$$\underline{u} \times \ \underline{v} = |\underline{u}||\underline{v}| sin\theta \hat{n} = 0 \ \Rightarrow \underline{u} \times \ \underline{v} = 0$$

$$and\ if\ \underline{u} \times \ \underline{v} = 0\ then\ either\ sin\theta = 0\ or\ \underline{u} = 0\ or\ \underline{u} = 0$$

$(\boldsymbol{i})\ \boldsymbol{if\ sin\theta = 0} \ \ \Rightarrow \boldsymbol{\theta = 180^0 \ \ or\ O^0 \ \ which\ shows}$ **That** $\boldsymbol{vectors}$ $\underline{u}$ *and* $\underline{v}$ *are parallel.*

$\boldsymbol{if}\ \underline{u} = 0$ *or* $\underline{v}$ *then since zero vector has no specfic* $\boldsymbol{direction, so\ zero\ vector\ is\ ||to\ every\ vector.}$

Note:

Zero vector is both ||to $\perp$ *ar to every vector.*

Area of parallelogram:

Exercise NO 7.4

Q.1: Compute the cross product $\underline{a}\times\underline{b}$ and $\underline{b}\times\underline{a}$. Check your answer by showing that each $\underline{a}$ and $\underline{b}$ is perpendicular to $\underline{a}\times\underline{b}$ and $\underline{b}\times\underline{a}$.

i) $\underline{a} = 2\underline{i}+\underline{j}-\underline{k}, \underline{b} = \underline{i}-\underline{j}+\underline{k}$

SOLUTION:

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix}$$

$= \underline{i}\,(1-1) - \underline{j}\,(2+1) + \underline{k}\,(-2-1)$

$= 0\underline{i} - 3\underline{j} - 3\underline{k}$

Now $\underline{b}\times\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k}- \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$

$= \underline{i}\,(1-1) - \underline{j}\,(-1-2) + \underline{k}\,(1+2)$

$= 0\underline{i}+3\underline{j}+3\underline{k}$

As

$\underline{a}.\underline{a}\times\underline{b} = (2\underline{i}+\underline{j}-\underline{k}).(0\underline{i}-3\underline{j}-3\underline{k})$

$= 0-3+3 = 0$

So $\underline{a} \perp \underline{a}\times\underline{b}$

As

$\underline{a}.\underline{b}\times\underline{a} = (2\underline{i}+\underline{j}-\underline{k}).(0\underline{i}+3\underline{j}+3\underline{k})$

$= 0+3-3 = 0$

So $\underline{a}\perp\underline{b}\times\underline{a}$

As

$\underline{b}.\underline{a}\times\underline{b} = (2\underline{i}+\underline{j}-\underline{k}).(0\underline{i}-3\underline{j}-3\underline{k})$

$= 0-3+3 = 0$

So $\underline{b}\perp\underline{b}\times\underline{a}$

As

$\underline{b}.\underline{b}\times\underline{a} = (2\underline{i}+\underline{j}-\underline{k}).(0\underline{i}+3\underline{j}+3\underline{k})$

$= 0-3+3 = 0$

So $\underline{b}\perp\underline{b}\times\underline{a}$

ii) $\underline{a} = \underline{i}+\underline{j}, \underline{b} = \underline{i}-\underline{j}$

SOLUTION:

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$= \underline{i}\,(0+0) - \underline{j}\,(0-0) + \underline{k}\,(-1-1)$

$= 0\underline{i} - 0\underline{j} - 2\underline{k}$

Now $\underline{b}\times\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k}- \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix}$

$= \underline{i}\,(0-0) - \underline{j}\,(0-0) + \underline{k}\,(1+1)$

$= 0\underline{i} - 0\underline{j}+2\underline{k}$

As

$\underline{a}.\underline{a}\times\underline{b} = (\underline{i}+\underline{j}).(0\underline{i}-0\underline{j}-2\underline{k})$

$= 0-0-0 = 0$

So $\underline{a} \perp \underline{a}\times\underline{b}$

As

$\underline{a}.\underline{b}\times\underline{a} = (\underline{i}+\underline{j}).(0\underline{i}-0\underline{j}+2\underline{k})$

$= 0+0+0 = 0$

So $\underline{a}\perp\underline{b}\times\underline{a}$

As

$\underline{b}.\underline{a}\times\underline{b} = (\underline{i}-\underline{j}).(0\underline{i}-0\underline{j}-2\underline{k})$

$= 0+0-0 = 0$

So $\underline{b}\perp\underline{b}\times\underline{a}$

As

$\underline{b}.\underline{b}\times\underline{a} = (\underline{i}-\underline{j}).(0\underline{i}-0\underline{j}+2\underline{k})$

$= 0+0-0 = 0$

So $\underline{b}\perp\underline{b}\times\underline{a}$

iii) $\underline{a} = 3\underline{i}-2\underline{j}+\underline{k}, \underline{b} = \underline{i}+\underline{j}$

SOLUTION:

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$= \underline{i}\,(0-1) - \underline{j}\,(0-1) + \underline{k}\,(3+2)$

$= -1\underline{i}+\underline{j}+5\underline{k}$

Now $\underline{b}\times\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k}- \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{vmatrix}$

$= \underline{i}\,(1+0) - \underline{j}\,(1-0) + \underline{k}\,(-2-3)$

$= 1\underline{i}-\underline{j}-5\underline{k}$

As

$\underline{a}.\underline{a}\times\underline{b} = (3\underline{i}-2\underline{j}+\underline{k}).(-1\underline{i}+\underline{j}+5\underline{k})$

$= -3-2+5 = 0$

So $\underline{a} \perp \underline{a}\times\underline{b}$

As

$\underline{a}.\underline{b}\times\underline{a} = (3\underline{i}-2\underline{j}+\underline{k}).(1\underline{i}-\underline{j}-5\underline{k})$

$= 3+2-5 = 0$

So $\underline{a}\perp\underline{b}\times\underline{a}$

As

$\underline{b}.\underline{a}\times\underline{b} = (\underline{i}+\underline{j}).(-1\underline{i}+\underline{j}+5\underline{k})$

$= -1+1+0 = 0$

So $\underline{b}\perp\underline{b}\times\underline{a}$

As

$\underline{b}.\underline{b}\times\underline{a} = (\underline{i}+\underline{j}).(1\underline{i}-\underline{j}-5\underline{k})$

$= 1-1+0 = 0$

So $\underline{b}\perp\underline{b}\times\underline{a}$

Q.2: Find a unit vector perpendicular to the plane containing $\underline{a}$ and $\underline{b}$. Also find sine of the angle between them:

i) $\underline{a} = 2\underline{i}-6\underline{j}-3\underline{k}, \underline{b} = 4\underline{i}+3\underline{j}-\underline{k}$ (2019 S.Q)

SOLUTION:

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix}$$

$= \underline{i}\,(6+9) - \underline{j}\,(-2+12) + \underline{k}\,(6+24)$

$= 15\underline{i} - 10\underline{j} + 30\underline{k}$

$|\underline{a}\times\underline{b}| = \sqrt{15^2+(-10)^2+30^2}$

$|\underline{a} \times \underline{b}| = \sqrt{225+100+900} = \sqrt{1225} = 35$

Required unit vector = $\frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}$

$\hat{n} = \frac{15\underline{i}-10\underline{j}+30\underline{k}}{35} = \frac{15}{35}\underline{i} + \frac{10}{35}\underline{j} + \frac{30}{35}\underline{k} = \frac{3}{7}\underline{i} + \frac{2}{7}\underline{j} + \frac{6}{7}\underline{k}$

If θ is the angle between $\underline{a}$ and $\underline{b}$, then

$\sin\theta = \frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|} = \frac{35}{\sqrt{2^2+(-6)^2+(-3)^2}\,.\sqrt{4^2+3^2+(-1)^2}}$

$\sin\theta = \frac{35}{\sqrt{4+36+9}\,.\sqrt{16+9+1}} = \frac{35}{\sqrt{49}.\sqrt{26}} = \frac{35}{7.\sqrt{26}} = \frac{5}{\sqrt{26}}$

ii) $\underline{a} = -\underline{i} - \underline{j} - \underline{k}$, $\underline{b} = 2\underline{i} - 3\underline{j} + 4\underline{k}$

SOLUTION:

$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix}$

$= \underline{i}\,(-4-3) - \underline{j}\,(-4+2) + \underline{k}\,(3+2)$

$= -7\underline{i} + 2\underline{j} + 5\underline{k}$

$|\underline{a}\times\underline{b}| = \sqrt{(-7)^2 + 2^2 + 5^2}$

$|\underline{a}\times\underline{b}| = \sqrt{49+4+25} = \sqrt{78}$

Required unit vector = $\frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}$

$\hat{n} = \frac{-7\underline{i}+2\underline{j}+5\underline{k}}{\sqrt{78}} = \frac{-7}{\sqrt{78}}\underline{i} + \frac{2}{\sqrt{78}}\underline{j} + \frac{5}{\sqrt{78}}\underline{k}$

If θ is the angle between $\underline{a}$ and $\underline{b}$, then

$\sin\theta = \frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|} = \frac{\sqrt{78}}{\sqrt{(-1)^2+(-1)^2+(-1)^2}\,.\sqrt{2^2+(-3)^2+4^2}}$

$\sin\theta = \frac{\sqrt{78}}{\sqrt{1+1+1}\,.\sqrt{4+9+16}} = \frac{\sqrt{78}}{\sqrt{3}.\sqrt{29}} = \sqrt{\frac{78}{3}}.\frac{1}{\sqrt{29}} = \sqrt{26}.\frac{1}{\sqrt{29}} = \sqrt{\frac{26}{29}}$

iii) $\underline{a} = 2\underline{i} - 2\underline{j} + 4\underline{k}$, $\underline{b} = -\underline{i} + \underline{j} - 2\underline{k}$ (2016 S.Q)

SOLUTION:

$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix}$

$= \underline{i}\,(449) - \underline{j}\,(-4+4) + \underline{k}\,(2-2)$

$= 0\underline{i} + 0\underline{j} + 0\underline{k} = \underline{0}$ (null vector)

$|\underline{a}\times\underline{b}| = \sqrt{0^2+0+0^2}$

$|\underline{a}\times\underline{b}| = 0$

As $\underline{a}\times\underline{b} = 0$

$\Rightarrow$ $\underline{a}$ and $\underline{b}$ are parallel.

Also $\underline{a} = 2\underline{i} - 2\underline{j} + 4\underline{k} = -2\left(-\underline{i} + \underline{j} - 2\underline{k}\right) = -2b$

$\Rightarrow$ $\underline{a}$ and $\underline{b}$ are parallel but opposite in direction.

Thus the angle between them is 180°

$\therefore \theta = 180°$

$\Rightarrow \sin\theta = 0$

iv) $\underline{a} = \underline{i} + \underline{j}$, $\underline{b} = \underline{i} - \underline{j}$

SOLUTION:

$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$

$= \underline{i}\,(0+0) - \underline{j}\,(0-0) + \underline{k}\,(-1-1)$

$= 0\underline{i} - 0\underline{j} - 2\underline{k}$

$|\underline{a}\times\underline{b}| = \sqrt{0^2+(-0)^2+(-2)^2}$

$|\underline{a}\times\underline{b}| = \sqrt{0+0+4} = \sqrt{4} = 2$

Required unit vector = $\frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}$

$\hat{n} = \frac{0\underline{i}-0\underline{j}-2\underline{k}}{2} = \frac{0}{2}\underline{i} + \frac{0}{2}\underline{j} + \frac{-2}{2}\underline{k} = 0\underline{i} - 0\underline{j} - \underline{k} = -\underline{k}$

If θ is the angle between $\underline{a}$ and $\underline{b}$, then

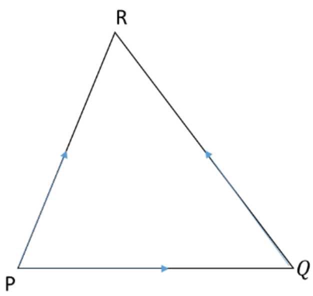
$\sin\theta = \frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|} = \frac{2}{\sqrt{1^2+1^2+0}\,.\sqrt{1^2+(-1)^2+0^2}}$

$\sin\theta = \frac{2}{\sqrt{1+1+0}\,.\sqrt{1+1+0}} = \frac{2}{\sqrt{2}.\sqrt{2}} = \frac{2}{(\sqrt{2})^2} = \frac{2}{2} = 1$

Q.3: Find the area of the triangle, determined by the points P, Q , R.

(i) P(0,0,0) ; Q(2,3,2) ; R(−1,1,4)

SOLUTION:



$\overrightarrow{PQ} = (x_2 - x_1)\,\underline{\boldsymbol{i}} + (y_2 - y_1)\,\underline{\boldsymbol{j}} + (z_2 - z_1)\underline{\boldsymbol{k}}$

$\overrightarrow{PQ} = (2-0)\,\underline{\boldsymbol{i}} + (3-0)\,\underline{\boldsymbol{j}} + (2-0)\underline{\boldsymbol{k}}$

$\overrightarrow{PQ} = 2\,\underline{\boldsymbol{i}} + 3\,\underline{\boldsymbol{j}} + 2\underline{\boldsymbol{k}}$

$\overrightarrow{PR} = (x_2 - x_1)\,\underline{\boldsymbol{i}} + (y_2 - y_1)\,\underline{\boldsymbol{j}} + (z_2 - z_1)\underline{\boldsymbol{k}}$

$\overrightarrow{PR} = (-1-0)\,\underline{\boldsymbol{i}} + (1-0)\,\underline{\boldsymbol{j}} + (4-0)\underline{\boldsymbol{k}}$

$\overrightarrow{PR} = -\,\underline{\boldsymbol{i}} + \underline{\boldsymbol{j}} + 4\underline{\boldsymbol{k}}$

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 2 \\ -1 & 1 & 4 \end{vmatrix}$$

$= \underline{i}\,(12-2) - \underline{j}\,(8+2) + \underline{k}\,(2+3)$

$= 10\underline{i} - 10\underline{j} + 5\underline{k}$

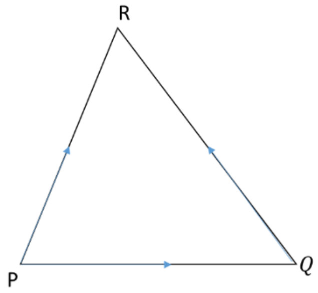
$|\overrightarrow{PQ}\times\overrightarrow{PR}| = \sqrt{10^2+(-10)^2+5^2}$

$|\overrightarrow{PQ}\times\overrightarrow{PR}| = \sqrt{100+100+25} = \sqrt{225} = 15$

Area of ΔPQR = $\frac{1}{2}|\overrightarrow{PQ}\times\overrightarrow{PR}| = \frac{1}{2}.15 = \frac{15}{2}$ square unit

(ii) P(1,-1,-1) ; Q(2,0,-1) ; R(0,2,1)

SOLUTION:



$\overrightarrow{PQ} = (x_2-x_1)\,\underline{\boldsymbol{i}} + (y_2-y_1)\,\underline{\boldsymbol{j}} + (z_2-z_1)\underline{\boldsymbol{k}}$

$\overrightarrow{PQ} = (2-1)\,\underline{\boldsymbol{i}} + (0+1)\,\underline{\boldsymbol{j}} + (-1+1)\underline{\boldsymbol{k}}$

$\overrightarrow{PQ} = \underline{\boldsymbol{i}} + \underline{\boldsymbol{j}} + 0\underline{\boldsymbol{k}}$

$\overrightarrow{PR} = (x_2-x_1)\,\underline{\boldsymbol{i}} + (y_2-y_1)\,\underline{\boldsymbol{j}} + (z_2-z_1)\underline{\boldsymbol{k}}$

$\overrightarrow{PR} = (0-1)\,\underline{\boldsymbol{i}} + (2+1)\,\underline{\boldsymbol{j}} + (1+1)\underline{\boldsymbol{k}}$

$\overrightarrow{PR} = -\underline{\boldsymbol{i}} + 3\,\underline{\boldsymbol{j}} + 2\underline{\boldsymbol{k}}$

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ -1 & 3 & 2 \end{vmatrix}$$

$= \underline{i}\,(2-0) - \underline{j}\,(2+0) + \underline{k}\,(3+1)$

$= 2\underline{i} - 2\underline{j} + 4\underline{k}$

$|\overrightarrow{PQ}\times\overrightarrow{PR}| = \sqrt{2^2+(-2)^2+4^2}$

$|\overrightarrow{PQ}\times\overrightarrow{PR}| = \sqrt{4+4+16} = \sqrt{24} = \sqrt{2\times 6} = 2\sqrt{6}$

Area of ΔPQR = $\frac{1}{2}|\overrightarrow{PQ}\times\overrightarrow{PR}| = \frac{1}{2}.2\sqrt{6} = \sqrt{6}$ square unit

Q.4: Find the area of the parallelogram whose vertices are:

i) A(0,0,0) ; B(1,2,3) ; C(2, −1,1) ; D(3,1,4)

SOLUTION:

$\overrightarrow{AB} = (x_2-x_1)\,\underline{\boldsymbol{i}} + (y_2-y_1)\,\underline{\boldsymbol{j}} + (z_2-z_1)\underline{\boldsymbol{k}}$

$\overrightarrow{AB} = (1-0)\,\underline{\boldsymbol{i}} + (2-0)\,\underline{\boldsymbol{j}} + (3-0)\underline{\boldsymbol{k}}$

$\overrightarrow{AB} = \underline{\boldsymbol{i}} + 2\,\underline{\boldsymbol{j}} + 3\underline{\boldsymbol{k}}$

$\overrightarrow{AD} = (x_2-x_1)\,\underline{\boldsymbol{i}} + (y_2-y_1)\,\underline{\boldsymbol{j}} + (z_2-z_1)\underline{\boldsymbol{k}}$

$\overrightarrow{AD} = (3-0)\,\underline{\boldsymbol{i}} + (1-0)\,\underline{\boldsymbol{j}} + (4-0)\underline{\boldsymbol{k}}$

$\overrightarrow{AD} = 3\,\underline{\boldsymbol{i}} + \underline{\boldsymbol{j}} + 4\underline{\boldsymbol{k}}$

$$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 3 & 1 & 4 \end{vmatrix}$$

$= \underline{i}\,(8-3) - \underline{j}\,(4-9) + \underline{k}\,(1-6)$

$= 5\underline{i} + 5\underline{j} - 5\underline{k}$

$|\overrightarrow{AB}\times\overrightarrow{AD}| = \sqrt{5^2+5^2+(-5)^2}$

$|\overrightarrow{AB}\times\overrightarrow{AD}| = \sqrt{25+25+25} = \sqrt{75} = 5\sqrt{3}$

Area of ΔPQR = $|\overrightarrow{AB}\times\overrightarrow{AD}| = 5\sqrt{3}$ sq. unit

ii) A(1,2, −1) ; B(4,2, −3) ; C(6, −5,2) ; D(9, −5,0)

SOLUTION:

$\overrightarrow{AB} = (x_2-x_1)\,\underline{\boldsymbol{i}} + (y_2-y_1)\,\underline{\boldsymbol{j}} + (z_2-z_1)\underline{\boldsymbol{k}}$

$\overrightarrow{AB} = (4-1)\,\underline{\boldsymbol{i}} + (2-2)\,\underline{\boldsymbol{j}} + (-3+1)\underline{\boldsymbol{k}}$

$\overrightarrow{AB} = 3\underline{\boldsymbol{i}} + 0\,\underline{\boldsymbol{j}} - 2\underline{\boldsymbol{k}}$

$\overrightarrow{AD} = (x_2-x_1)\,\underline{\boldsymbol{i}} + (y_2-y_1)\,\underline{\boldsymbol{j}} + (z_2-z_1)\underline{\boldsymbol{k}}$

$\overrightarrow{AD} = (9-1)\,\underline{\boldsymbol{i}} + (-5-2)\,\underline{\boldsymbol{j}} + (0+1)\underline{\boldsymbol{k}}$

$\overrightarrow{AD} = 8\,\underline{\boldsymbol{i}} - 7\,\underline{\boldsymbol{j}} + \underline{\boldsymbol{k}}$

$$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 0 & -2 \\ 8 & -7 & 1 \end{vmatrix}$$

$= \underline{i}\,(0-14) - \underline{j}\,(3+16) + \underline{k}\,(-21-0)$

$= -14\underline{i} - 19\underline{j} - 21\underline{k}$

$|\overrightarrow{AB}\times\overrightarrow{AD}| = \sqrt{(-14)^2+(-19)^2+(-21)^2}$

$|\overrightarrow{AB}\times\overrightarrow{AD}| = \sqrt{196+361+441} = \sqrt{998}$

Area of ΔPQR = $|\overrightarrow{AB}\times\overrightarrow{AD}| = \sqrt{998}$ sq. unit

iii) A(−1,1,1) ; B(−1,2,2) ; C(−3, 4, −5) ; D(−3,5, −4)

SOLUTION:

$\overrightarrow{AB} = (x_2-x_1)\,\underline{\boldsymbol{i}} + (y_2-y_1)\,\underline{\boldsymbol{j}} + (z_2-z_1)\underline{\boldsymbol{k}}$

$\overrightarrow{AB} = (-1+1)\,\underline{\boldsymbol{i}} + (2-1)\,\underline{\boldsymbol{j}} + (2-1)\underline{\boldsymbol{k}}$

$\overrightarrow{AB} = \underline{\boldsymbol{0i}} + \underline{\boldsymbol{j}} + \underline{\boldsymbol{k}}$

$\overrightarrow{AD} = (x_2-x_1)\,\underline{\boldsymbol{i}} + (y_2-y_1)\,\underline{\boldsymbol{j}} + (z_2-z_1)\underline{\boldsymbol{k}}$

$\overrightarrow{AD} = (-3+1)\,\underline{\boldsymbol{i}} + (5-1)\,\underline{\boldsymbol{j}} + (4-1)\underline{\boldsymbol{k}}$

$\overrightarrow{AD} = -2\,\underline{\boldsymbol{i}} + 4\underline{\boldsymbol{j}} - 5\underline{\boldsymbol{k}}$

$$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & 1 \\ -2 & 4 & -5 \end{vmatrix}$$

$= \underline{i}\,(-5-4) - \underline{j}\,(0+2) + \underline{k}\,(0+2)$

$= -9\underline{i} - 2\underline{j} + 2\underline{k}$

$|\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{(-9)^2 + (-2)^2 + 2^2}$

$|\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{81 + 4 + 4} = \sqrt{89}$

Area of ΔPQR = $|\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{89}$ sq. unit

Q.5: Which vectors if any, are perpendicular or parallel:

i) $\underline{u} = 5\underline{i} - \underline{j} + \underline{k}$; $\underline{v} = \underline{j} - 5\underline{k}$; $\underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$

SOLUTION:

As $\underline{u} . \underline{v} = (5\underline{i} - \underline{j} + \underline{k}).(\underline{j} - 5\underline{k}) = 0 - 1 - 5 = -6 \neq 0$

So $\underline{u}$ and $\underline{v}$ are not perpendicular.

As $\underline{u} . \underline{w} = (5\underline{i} - \underline{j} + \underline{k}).(-15\underline{i} + 3\underline{j} - 3\underline{k}) = -75 - 3 - 3 = -81 \neq 0$

So $\underline{u}$ and w are not perpendicular.

As $\underline{v} . \underline{w} = (\underline{j} - 5\underline{k}).(-15\underline{i} + 3\underline{j} - 3\underline{k}) = 0 + 3 + 15 = 18 \neq 0$

So $\underline{u}$ and w are not perpendicular.

Now $\underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$

$\underline{w} = -3(5\underline{i} - \underline{j} + \underline{k})$

$\underline{w} = -3\underline{u}$

$\Rightarrow \underline{u} \parallel \underline{w}$

ii) $\underline{u} = \underline{i} + 2\underline{j} - \underline{k}$; $\underline{v} = -\underline{i} + \underline{j} + \underline{k}$; $\underline{w} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}$

SOLUTION:

As $\underline{u} . \underline{v} = (\underline{i} + 2\underline{j} - \underline{k}).(-\underline{i} + \underline{j} + \underline{k}) = -1 + 2 - 1 = 0$

So $\underline{u}$ and $\underline{v}$ are perpendicular.

As $\underline{u} . \underline{w} = (\underline{i} + 2\underline{j} - \underline{k}).(-\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}) = -\frac{\pi}{2} - 2\pi - \frac{\pi}{2} = -3\pi \neq 0$

So $\underline{u}$ and w are not perpendicular.

As $\underline{v} . \underline{w} = (-\underline{i} + \underline{j} + \underline{k}).(-\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k}) = \frac{\pi}{2} - \pi + \frac{\pi}{2} = \frac{\pi - 2\pi + \pi}{2} = 0$

So $\underline{u}$ and w are perpendicular.

Now $\underline{w} = -\frac{\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k} = \frac{-\pi i - 2\pi j + \pi k}{2}$

$\underline{w} = -\frac{\pi}{2}(\underline{i} + 2\underline{j} - \underline{k})$

$\underline{w} = -\frac{\pi}{2}\underline{u}$

$\Rightarrow \underline{u} \parallel \underline{w}$ but opposite in direction.

Q.6: Prove that $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b}) = \underline{0}$

SOLUTION: (2015 S.Q)

L.H.S = $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b})$

$= (\underline{a} \times \underline{b}) + (\underline{a} \times \underline{c}) + (\underline{b} \times \underline{c}) + (\underline{b} \times \underline{a}) + (\underline{c} \times \underline{a}) + (\underline{c} \times \underline{b})$

$\because \underline{b} \times \underline{a} = -(\underline{a} \times \underline{b})$ & $\underline{c} \times \underline{a} = -(\underline{a} \times \underline{c})$ & $\underline{c} \times \underline{b} = -(\underline{b} \times \underline{c})$

$= (\underline{a} \times \underline{b}) + (\underline{a} \times \underline{c}) + (\underline{b} \times \underline{c}) - (\underline{a} \times \underline{b}) - (\underline{a} \times \underline{c}) - (\underline{b} \times \underline{c})$

= Q.7: If $\underline{a} + \underline{b} + \underline{c} = 0$**, then prove that** $\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$

SOLUTION: As $\underline{a} + \underline{b} + \underline{c} = \underline{0}$

$\underline{0}$ = R.H.S

Taking cross product with $\underline{a}$

$\underline{a} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{0} \times \underline{a}$

$\underline{a} \times \underline{a} + \underline{a} \times \underline{b} + \underline{a} \times \underline{c} = \underline{0}$

$\because \underline{a} \times \underline{a} = 0$

$0 + \underline{a} \times \underline{b} - (\underline{c} \times \underline{a}) = 0$

$\underline{a} \times \underline{b} = \underline{c} \times \underline{a}$ **(1)**

Taking cross product with $\underline{b}$

$\underline{b} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{0} \times \underline{b}$

$\underline{b} \times \underline{a} + \underline{b} \times \underline{b} + \underline{b} \times \underline{c} = \underline{0}$

$\because \underline{b} \times \underline{b} = 0$

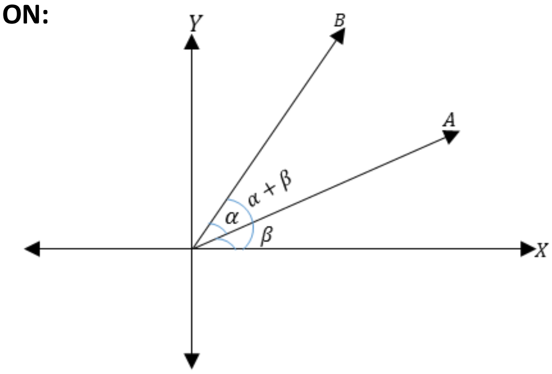
$\underline{b} \times \underline{a} + 0 - (\underline{c} \times \underline{b}) = 0$

$\underline{a} \times \underline{b} = \underline{c} \times \underline{a}$ **(2)**

From (1) & (2), we conclude that $\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$

Q.8: Prove that $\sin(\alpha - \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ **(2015 L.Q)**

SOLUTION:



Consider two unit vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in xy-plane making angle α, β with positive x-axis respectively.

Such that m∠AOB = $\alpha - \beta$

$\overrightarrow{OA} = \cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}$

$\overrightarrow{OB} = \cos\beta\,\underline{i} + \sin\beta\,\underline{j}$

Taking cross product of $\overrightarrow{OB}$ and $\overrightarrow{OA}$

$$\overrightarrow{OB}\times\overrightarrow{OA} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$|\overrightarrow{OB}|\,|\overrightarrow{OA}|\sin(\alpha-\beta)\,\underline{k} = \underline{i}\,(0-0) - \underline{j}\,(0-0) + \underline{k}$ $(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$$1.1\sin(\alpha-\beta)\,\underline{k} = \underline{k}\,(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Q.9: if $\underline{a}\times\underline{b} = 0$ and $\underline{a}\,.\,\underline{b} = 0$ what conclusion can be drawn about $\underline{a}$ or $\underline{b}$

SOLUTION:

$\because\ \underline{a}\times\underline{b} = 0$

- ⇨ $|\underline{a}||\underline{b}|\sin\theta = 0 \Rightarrow \sin\theta = 0$
- ⇨ $\theta = \sin^{-1}0 = 0$
- ⇨ $\theta = 0\ ,\pi$ *so $\underline{a}$ and $\underline{b}$ are paralllel.*

  And $\underline{a}\,.\,\underline{b} = 0$
- ⇨ $|\underline{a}||\underline{b}|\cos\theta = 0 \Rightarrow \cos\theta = 0$
- ⇨ $\theta = \cos^{-1}(0) = 90^0 \Rightarrow \theta = 90^0$

so $\underline{a}$ and $\underline{b}$ are perpendicular. at the same time $||$*and linear not possible so one vector should be zero or null.*

Triple product of vectors:

There are two types of triple product of vectors:

(a) Scalar triple product:

$$(\underline{u}\times\underline{v}).\underline{w}\quad or\ \underline{u}.(\underline{v}\times\underline{w})$$

(b) Vector triple product:

$$\underline{u}\times(\underline{v}\times\underline{w})$$

in this section we shall study the scalar triple product only.

Definition:

Let $\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$

$\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$ *and*

$\underline{w} = a_3\underline{i} + b_3\underline{j} + c_3\underline{k}$ *be three vectors, the scalar* Triple product of vectors $\underline{u},\underline{v}$ *and* $\underline{w}$ *id defined by* $(\underline{u}\times\underline{v}).\underline{w}$ *or* $\underline{u}.(\underline{v}\times\underline{w})$ *or* $w(\underline{u}\times\underline{v})$ *. the scalar*

product $\underline{u}.(\underline{v}\times\underline{w})$ *is written as;*

$$\underline{u}.(\underline{v}\times\underline{w}) = [\underline{u},\underline{v},\underline{w}]$$

Analytic Expression of $\underline{u}.(\underline{v}\times\underline{w})$

$\boldsymbol{let}\,\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$

$\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$ *and*

$\underline{w} = a_3\underline{i} + b_3\underline{j} + c_3\underline{k}$

$$\underline{v}\times\underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \boldsymbol{a_2} & \boldsymbol{b_2} & \boldsymbol{c_2} \\ \boldsymbol{a_3} & \boldsymbol{b_3} & \boldsymbol{c_3} \end{vmatrix}$$

$$= \underline{i}\big(b_2c_3 - c_2b_3\big) - \underline{j}(a_2c_3 - c_2a_3) + \underline{k}(a_2b_3 - b_2a_3)\big)$$

$$\underline{u}.(\underline{v}\times\underline{w}) = a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3))$$

$$\underline{u}.(\underline{v}\times\underline{w}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ \boldsymbol{a_2} & \boldsymbol{b_2} & \boldsymbol{c_2} \\ \boldsymbol{a_3} & \boldsymbol{b_3} & \boldsymbol{c_3} \end{vmatrix}$$

Which determine *fomula for scalar triple produc of* $\underline{u}\,,\underline{v}$ *and* $\underline{w}$

Prove that:

$$\underline{u}.(\underline{v}\times\underline{w}) = \underline{v}.(\underline{w}\times u) = \underline{w}.(\underline{u}\times\underline{v})$$

Proof: we know that

$\boldsymbol{let}\,\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$

$\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$ *and*

$\underline{w} = a_3\underline{i} + b_3\underline{j} + c_3\underline{k}$

$$\underline{u}.(\underline{v}\times\underline{w}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ \boldsymbol{a_2} & \boldsymbol{b_2} & \boldsymbol{c_2} \\ \boldsymbol{a_3} & \boldsymbol{b_3} & \boldsymbol{c_3} \end{vmatrix}$$

$$= -\begin{vmatrix} a_2 & b_2 & c_2 \\ \boldsymbol{a_1} & \boldsymbol{b_1} & \boldsymbol{c_1} \\ \boldsymbol{a_3} & \boldsymbol{b_3} & \boldsymbol{c_3} \end{vmatrix}\ \boldsymbol{interchange}\ R_1\ and\ R_2$$

$$= \begin{vmatrix} a_2 & b_2 & c_2 \\ \boldsymbol{a_3} & \boldsymbol{b_3} & \boldsymbol{c_3} \\ \boldsymbol{a_1} & \boldsymbol{b_1} & \boldsymbol{c_1} \end{vmatrix}\ \boldsymbol{interchange}\ R_1\ and\ R_2$$

$$\because\ \underline{u}.(\underline{v}\times\underline{w}) = \underline{v}.(\underline{w}\times u)$$

Now

$$\underline{v}.(\underline{w}\times u) = \begin{vmatrix} a_2 & b_2 & c_2 \\ \boldsymbol{a_3} & \boldsymbol{b_3} & \boldsymbol{c_3} \\ \boldsymbol{a_1} & \boldsymbol{b_1} & \boldsymbol{c_1} \end{vmatrix}$$

$$= -\begin{vmatrix} a_2 & b_2 & c_2 \\ \boldsymbol{a_1} & \boldsymbol{b_1} & \boldsymbol{c_1} \\ \boldsymbol{a_3} & \boldsymbol{b_3} & \boldsymbol{c_3} \end{vmatrix}\ \boldsymbol{interchange}\ R_1\ and\ R_2$$

$$= \begin{vmatrix} a_3 & b_3 & c_3 \\ \boldsymbol{a_1} & \boldsymbol{b_1} & \boldsymbol{c_1} \\ \boldsymbol{a_2} & \boldsymbol{b_2} & \boldsymbol{c_2} \end{vmatrix}\ \boldsymbol{interchange}\ R_1\ and\ R_2$$

$$\because\ \underline{v}.(\underline{w}\times u) = \underline{w}.(\underline{u}\times\underline{v})$$

Hence $\underline{u}.(\underline{v}\times\underline{w}) = \underline{v}.(\underline{w}\times u) = \underline{w}.(\underline{u}\times\underline{v})$

Note: (***i***) *Dot and cross may be interchnaged with out altering the value*

$$i.e\ (\underline{u}\times\underline{v}).\underline{w} = \underline{u}.(\underline{v}\times\underline{w}) = [\underline{u}\ \ \underline{v}\ \ \underline{w}]$$

$$(\underline{v}\times\underline{w}).\underline{u} = \underline{v}.(\underline{w}\times\underline{u}) = [\underline{v}\ \ w\ \ \underline{u}]$$

$$(\underline{w}\times\underline{u}).\underline{v} = \underline{w}.(\underline{u}\times\underline{v}) = [\underline{w}\ \ \underline{u}\ \ \underline{v}]$$

(ii) the value of the product changes if the order is non cycle.

(*iii*) $\underline{u}\,.\,\underline{v}\,.\,\underline{w}$ *and* $\underline{u}.(\underline{v}\times\underline{w})$ *are meaningless.*

the volume of parallelepiped

consider $\overrightarrow{OA} = \underline{u}$ *,* $\overrightarrow{OB} = \underline{v}$ *and* $\overrightarrow{OC} = \underline{w}$ *be the*

$adjasecent\ adges\ of\ parallel\ piped\ \ OAFCDEGB$

$let\ \theta\ be\ the\ angle\ b\backslash w\ \ \underline{w}\ \ and\ (\underline{u}\times\underline{v})$

$As\ |\underline{u}\times\underline{v}| = area\ of\ ||gram\ OAGB$

$area\ of\ base\ of\ ||parallelepiped$

$$Resolve\ \underline{w}\ in\ components\ in\ \Delta COM, cos\theta = \frac{|\overrightarrow{OM}|}{|\overrightarrow{OC}|}$$

$$\Rightarrow |\overrightarrow{OM}| = |\overrightarrow{OC}|cos\theta$$

$$\Rightarrow |\overrightarrow{OM}| = wcos\theta$$

$$As\ \ |\overrightarrow{OM}| = height\ of\ ||piped$$

$$As\ Volume\ ||piped = (area\ of\ base)(height)$$

$$= |\underline{u}\times\underline{v}|.|\underline{w}|\ cos\theta$$

$$\Rightarrow\ Volume\ of\ ||piped = \underline{w}.(\underline{u}\times\underline{v})$$

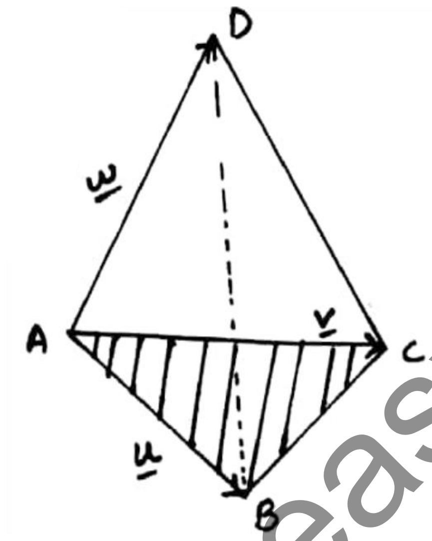
$$\because\ \underline{w}.(\underline{u}\times\underline{v}) = \underline{u}.(\underline{v}\times\underline{w})\ \left(\begin{matrix}\because \underline{w}.(\underline{u}\times\underline{v})\\ |\underline{u}\times\underline{v}|.|\underline{w}|\ cos\theta\end{matrix}\right)$$

$$so\ Volume\ of\ ||piped = \underline{u}.(\underline{v}\times\underline{w})$$

The Volume of Tetrahedron:

Volume of tetrahedron $ABCD$

$$= \frac{1}{3}(\Delta ABC)(height\ of\ D\ above\ the\ place\ ABC)$$



Thus, $\ \ Volume = \frac{1}{6}(\underline{u}\times\underline{v}).\underline{w} = \frac{1}{6}[\underline{u}\ \ \underline{v}\ \ \underline{w}]$

Properties of scalar triple product:

1. $if\ \ \underline{u}, \underline{v}\ \ and\ \ \underline{w}\ \ are\ coplaner, then\ the\ volume\ of\ ||piped\ so\ formed\ is\ zero. i.e\ the\ vector\ \ \underline{u}, \underline{v}, \underline{w}$

$$are\ coplaner\ \Leftrightarrow (\underline{u}\times\underline{v}).\underline{w} = 0$$

$if\ any\ two\ vectors\ of\ scalartriple\ product\ are\ equal, then\ its\ value\ is\ zero.\ \ i.e$

$$[\underline{u}\ \ \underline{v}\ \ \underline{w}] = [\underline{u}\ \ \underline{v}\ \ \underline{w}] = 0$$

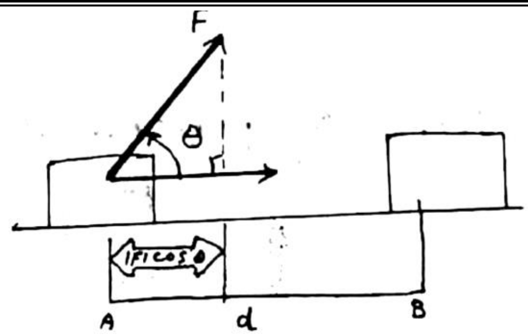
Application of vectors in physics and Engineering

$(a) Work\ done$:

$if\ \ d\ is\ distplacement\ from\ A\ to\ B\ when\ froce\ \vec{F}\ is\ appplied\ on\ the\ particle\ then,$

$$Workdone = (force)(Displacement)$$

$$\Rightarrow\ \underline{W} = \vec{F}.\overrightarrow{AB} = \underline{F}.\underline{d}\qquad \underline{d} = \overrightarrow{AB}$$



Exercise NO 7.5

Q.1: find the volume of the parallelepiped for which the given vectors are three edges:

i) $\underline{u} = 3\underline{i} + 2\underline{k}$

$\underline{v} = \underline{i} + 2\underline{j} + \underline{k}$

$\underline{w} = -\underline{j} + 4\underline{k}$

Volume of a parallelepiped = $[\underline{u}\ \underline{v}\ \underline{w}]$

$$= \begin{vmatrix}3 & 0 & 2\\ 1 & 2 & 1\\ 0 & -1 & 4\end{vmatrix}$$

$$= 3\begin{vmatrix}2 & 1\\ -1 & 4\end{vmatrix} - 0\begin{vmatrix}1 & 1\\ 0 & 4\end{vmatrix} + 2\begin{vmatrix}1 & 2\\ 0 & -1\end{vmatrix}$$

= 3(8+1) −0(4−0) +2(−1−0)

= 27−0−2 = 25 cubic unit

ii) $\underline{u} = 3\underline{i} - 4\underline{j} - \underline{k}$

$\underline{v} = \underline{i} - \underline{j} - 2\underline{k}$

$\underline{w} = 2\underline{i} - 3\underline{j} + \underline{k}$

Volume of a parallelepiped = $[\underline{u}\ \underline{v}\ \underline{w}]$

$$= \begin{vmatrix}1 & -4 & -1\\ 1 & -1 & -2\\ 2 & -3 & 1\end{vmatrix}$$

$$= 1\begin{vmatrix}-1 & -2\\ -3 & 1\end{vmatrix} - (-4)\begin{vmatrix}1 & -2\\ 2 & 1\end{vmatrix} - 1\begin{vmatrix}1 & -1\\ 2 & -3\end{vmatrix}$$

= 1(−1−6)+4(1+4)−1(−3+2)

= −7+20+1 = 14 cubic unit

iii) $\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k}$

$\underline{v} = 2\underline{i} - \underline{j} - \underline{k}$

$\underline{w} = \underline{j} + \underline{k}$

volume of a parallelepiped = $[\underline{u}\ \underline{v}\ \underline{w}]$

$$= \begin{vmatrix}1 & -2 & 3\\ 2 & -1 & -1\\ 0 & 1 & 1\end{vmatrix}$$

$$= 1\begin{vmatrix}-1 & -1\\ 1 & 1\end{vmatrix} - (-2)\begin{vmatrix}2 & -1\\ 0 & 1\end{vmatrix} + 3\begin{vmatrix}2 & -1\\ 0 & 1\end{vmatrix}$$

= 1(−1+1)+2(2+0) +3(2 +0)

$= 0+4+6 = 10$ cubic unit

Q.2: verify that $\underline{a}\,.\,\underline{b}\times\underline{c} = \underline{b}\,.\,\underline{c}\times\underline{a} = \underline{c}\,.\,\underline{a}\times\underline{b}$. if

$\underline{a} = 3\underline{i} - \underline{j} + 5\underline{k}\,;\ \underline{b} = 4\underline{i} - 2\underline{j} + 4\underline{k}\,;\ \underline{c} = 2\underline{i} + 5\underline{j} + \underline{k}$

$$\underline{a}\,.\,\underline{b}\times\underline{c} = \begin{vmatrix} 3 & -1 & 5 \\ 4 & 3 & -2 \\ 2 & 5 & 1 \end{vmatrix}$$

$= 3\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 4 & -2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}$

$= 3(3+10)+1(4+4)+5(20-6)$

$= 39+8+70 = 117$

$$\underline{a}\,.\,\underline{b}\times\underline{c} = \begin{vmatrix} 4 & 3 & -2 \\ 2 & 5 & 1 \\ 3 & -1 & 5 \end{vmatrix}$$

$= 4\begin{vmatrix} 5 & 1 \\ -1 & 5 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix}$

$= 4(25+1)-3(10-3)-2(-2-15)$

$= 104-21+34 = 117$

$$\underline{a}\,.\,\underline{b}\times\underline{c} = \begin{vmatrix} 2 & 5 & 1 \\ 3 & -1 & 5 \\ 4 & 3 & -2 \end{vmatrix}$$

$= 2\begin{vmatrix} -1 & 5 \\ 3 & -2 \end{vmatrix} - 5\begin{vmatrix} 3 & 5 \\ 4 & -2 \end{vmatrix} + 1\begin{vmatrix} 3 & -1 \\ 4 & 3 \end{vmatrix}$

$= 2(2-15)-5(-6-20)+1(9+4)$

$= -26+130+13 = 117$

Q.3: Prove that the vectors $\underline{i} - 2\underline{j} + 3\underline{k}$; $-2\underline{i} + 3\underline{j} - 4\underline{k}$ and $\underline{i} - 3\underline{j} + 5\underline{k}$ are coplanar.

SOLUTION:

Let $\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k}\,;\quad \underline{v} = -2\underline{i} + 3\underline{j} - 4\underline{k}\,;\quad \underline{w} = \underline{i} - 3\underline{j} + 5\underline{k}$

$[\underline{u}\,\underline{v}\,\underline{w}] = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 1\begin{vmatrix} 3 & -4 \\ -3 & 5 \end{vmatrix} - (-2)\begin{vmatrix} -2 & -4 \\ 1 & 5 \end{vmatrix} + 3\begin{vmatrix} -2 & 3 \\ 1 & -3 \end{vmatrix} = 1(15-12) + 2(-10+4) + 3(6-3) = 3 - 12 + 9 = 0$

Thus the vectors are perpendicular.

Q.4: Find the constant 'α' such that the vectors are coplanar.

i) $\underline{i} - \underline{j} + \underline{k}$; $\underline{i} - 2\underline{j} - 3\underline{k}$ **and** $3\underline{i} - \alpha\underline{j} + 5\underline{k}$

SOLUTION: (2017 S.Q)

Let $\underline{u} = \underline{i} - \underline{j} + \underline{k}\,;\ \underline{v} = \underline{i} - 2\underline{j} - 3\underline{k};\ \underline{w} = 3\underline{i} - \alpha\underline{j} + 5\underline{k}$

The vectors $\underline{u}, \underline{v}, \underline{w}$ are coplanar if $[\underline{u}\,\underline{v}\,\underline{w}] = 0$

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & -3 \\ 3 & -\alpha & 5 \end{vmatrix} = 0$$

$1(-10-3\alpha) + 1(5+9) + 1(-\alpha+6) = 0$

$-10-3\alpha+14-\alpha+6 = 0$

$-4\alpha + 10 = 0$

$\Rightarrow \alpha = \frac{10}{4} \qquad \Rightarrow \alpha = \frac{5}{2}$

ii) $\underline{i} - 2\alpha\underline{j} - \underline{k}$; $\underline{i} - \underline{j} + 2\underline{k}$ **and** $\alpha\,\underline{i} - 2\underline{j} + \underline{k}$

SOLUTION:

Let $\underline{u} = \underline{i} - 2\alpha\underline{j} - \underline{k}\,;\ \underline{v} = \underline{i} - \underline{j} + 2\underline{k};\ \underline{w} = \underline{i} - 2\underline{j} + \underline{k}$

The vectors $\underline{u}, \underline{v}, \underline{w}$ are coplanar if $[\underline{u}\,\underline{v}\,\underline{w}] = 0$

$$\begin{vmatrix} 1 & -2\alpha & -1 \\ 1 & -1 & 2 \\ \alpha & -1 & 1 \end{vmatrix} = 0$$

$1(-1+2) - 1(2\alpha - 1) - \alpha(4\alpha - 1) = 0$

$\Rightarrow 1 + 2\alpha + 1 - 4\alpha^2 - \alpha = 0 \Rightarrow -4\alpha^2 + \alpha + 2 = 0$

$$4\alpha^2 - \alpha - 2 = 0$$

Using quadratic formula $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$\alpha = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-2)}}{2(4)} = \frac{1 \pm \sqrt{1+32}}{8}$$

$$\Rightarrow \alpha = \frac{1 \pm \sqrt{33}}{8}$$

Q.5:(a): Find the value of i) $2\underline{i} \times 2\underline{j}\,.\,\underline{k}$ ii) $3\underline{j}\,.\,\underline{k} \times \underline{i}$ iii) $[\underline{k}\ \underline{i}\ \underline{j}]$ (2015 S.Q) iv) $[\underline{i}\ \underline{i}\ \underline{k}]$

SOLUTION:

i) $2\underline{i} \times 2\underline{j}\,.\,\underline{k} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2(2-0) - 0(0-0) + 0(0-0) = 4 - 0 - 0 = 4$

ii) $3\underline{j}\,.\,\underline{k} \times \underline{i} = \begin{vmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0(0-0) - 3(0-1) + 0(0-1) = 0 + 3 + 0 = 3$

iii) $[\underline{k}\ \underline{i}\ \underline{j}] = \underline{k}\,.\,\underline{j} \times \underline{j} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0(0-0) - 0(0-0) + 1(1-0) = 0 - 0 + 1 = 1$

iv) $[\underline{i}\ \underline{i}\ \underline{k}] = \underline{i}\,.\,\underline{i} \times \underline{k} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(0-0) - 0(1-0) + 0(0-0) = 0 - 0 - 0 = 0$

(b) Prove that

$\underline{u}\,.\,(\underline{v}\times\underline{w}) + \underline{v}\,.\,(\underline{w}\times\underline{u}) + \underline{w}\,.\,(\underline{u}\times\underline{v}) = 3\underline{u}\,.\,(\underline{v}\times\underline{w})$

SOLUTION:

L.H.S

$= \underline{u}\,.\,(\underline{v}\times\underline{w}) + \underline{v}\,.\,(\underline{w}\times\underline{u}) + \underline{w}\,.\,(\underline{u}\times\underline{v})$

$\because \underline{u}\,.\,(\underline{v}\times\underline{w}) = \underline{v}\,.\,(\underline{w}\times\underline{u}) = \underline{w}\,.\,(\underline{u}\times\underline{v})$

$= \underline{u}\,.\,(\underline{v}\times\underline{w}) + \underline{u}\,.\,(\underline{v}\times\underline{w}) + \underline{u}\,.\,(\underline{v}\times\underline{w})$

$= 3\underline{u}\,.\,(\underline{v}\times\underline{w})$

= R.H.S

Q.6: Find the volume of the tetrahedron with the vertices.

i) (0,1,2) , (3,2,1) , (1,2,1) , and (5,5,6)

SOLUTION:

Let A(0,1,2) , B(3,2,1) , C(1,2,1) , D(5,5,6)

$\overrightarrow{AB} = (3-0)\underline{i} + (2-1)\underline{j} + (1-2)\underline{k} = 3\underline{i} + \underline{j} - \underline{k}$

$\overrightarrow{AC} = (1-0)\underline{i} + (2-1)\underline{j} + (1-2)\underline{k} = \underline{i} + \underline{j} - \underline{k}$

$\overrightarrow{AD} = (5-0)\underline{i} + (5-1)\underline{j} + (6-2)\underline{k} = 5\underline{i} + 4\underline{j} + 4\underline{k}$

Volume of the tetrahedron ABCD = $\frac{1}{6}[\overrightarrow{AB}\ \ \overrightarrow{AC}\ \ \overrightarrow{AD}\ ]$

$$= \frac{1}{6}\begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ 5 & 4 & 4 \end{vmatrix}$$

$$= \frac{1}{6}[3(4+4) - 1(4+5) + (-1)(4-5)]$$

$$= \frac{1}{6}[24 - 9 + 1] = \frac{1}{6}.16 = \frac{8}{3} \text{ Cubic unit}$$

ii) (2,1,8) , (3,2,9) , (2,1,4) , and (3,3,10) (2015 S.Q)(2018,19L.Q)

SOLUTION:

Let A**(2,1,8)** , B**(3,2,9)** , C**(2,1,4)** , D**(3,3,10)**

$\overrightarrow{AB} = (3-2)\underline{i} + (2-1)\underline{j} + (9-8)\underline{k} = \underline{i} + \underline{j} + \underline{k}$

$\overrightarrow{AC} = (2-2)\underline{i} + (1-1)\underline{j} + (4-8)\underline{k} = 0\underline{i} + 0\underline{j} - 4\underline{k}$

$\overrightarrow{AD} = (3-2)\underline{i} + (3-1)\underline{j} + (10-8)\underline{k} = \underline{i} + 2\underline{j} + 2\underline{k}$

Volume of the tetrahedron ABCD = $\frac{1}{6}[\overrightarrow{AB}\ \ \overrightarrow{AC}\ \ \overrightarrow{AD}\ ]$

$$= \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= \frac{1}{6}[1(0+8) - 1(0+4) + 1(0-0)]$$

$$= \frac{1}{6}[8 - 4 + 0] = \frac{1}{6}.4 = \frac{2}{3} \text{ Cubic unit}$$

Q.7: Find the work done, if the point at which the constant force $\vec{F} = 4\underline{i} + 3\underline{j} + 5\underline{k}$ is applied to an object, moves from $P_1(3,1,-2)$ to $P_2(2,4,6)$.

SOLUTION: Given: $\vec{F} = 4\underline{i} + 3\underline{j} + 5\underline{k}$, $P_1(3,1,-2)$, $P_2(2,4,6)$

Displacement: $\underline{d} = \overrightarrow{P_1P_2} = (2-3)\underline{i} + (4-1)\underline{j} + (6+2)\underline{k} = -\underline{i} + 3\underline{j} + 8\underline{k}$

We know that Work done = $\vec{F}\,.\,\underline{d} = (4\underline{i} + 3\underline{j} + 5\underline{k}).(-\underline{i} + 3\underline{j} + 8\underline{k}) = -4 + 9 + 40 = 45$ Units

Q.8: A particle, acted by constant forces $4\underline{i} + \underline{j} - \underline{k}$ and $3\underline{i} - \underline{j} - \underline{k}$ is displaces from A(1,2,3) to B(5,4,1). Find the work done.

SOLUTION: Given: let $\vec{F_1} = 4\underline{i} + \underline{j} - \underline{k}$; $\vec{F_2} = 3\underline{i} - \underline{j} - \underline{k}$ and A(1,2,3) , B(5,4,1)

Total force: $\vec{F} = \vec{F_1} + \vec{F_2} = 4\underline{i} + \underline{j} - \underline{k} + 3\underline{i} - \underline{j} - \underline{k} = 7\underline{i} + 0\underline{j} - 4\underline{k}$

Displacement: $\underline{d} = \overrightarrow{AB} = (5-1)\underline{i} + (4-2)\underline{j} + (1-3)\underline{k} = 4\underline{i} + 2\underline{j} - 2\underline{k}$

We know that Work done = $\vec{F}\,.\,\underline{d} = (3\underline{i} - \underline{j} - \underline{k}).(4\underline{i} + 2\underline{j} - 2\underline{k}) = 28 + 0 + 8 = 36$ units

Q.9: A particle is displace from the point A(5,-5,-7) , to the point B(6,2,-2) under the action of constant forces defined by

$$\mathbf{10\underline{i} - 6\underline{j} + 11\underline{k}\ ,\ 4\underline{i} + 5\underline{j} + 9\underline{k}\ ,\ -2\underline{i} + \underline{j} - 9\underline{k}.}$$

Show that the total work done by the forces is 67 units.

SOLUTION: Given: Let $\vec{F_1} = 10\underline{i} - 6\underline{j} + 11\underline{k}$, $\vec{F_2} = 4\underline{i} + 5\underline{j} + 9\underline{k}$ $\vec{F_3} = -2\underline{i} + \underline{j} - 9\underline{k}$ and A(5,-5,-7) , B(6,2,-2)

Total force : $\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3} = 10\underline{i} - 6\underline{j} + 11\underline{k} + 4\underline{i} + 5\underline{j} + 9\underline{k} + (-2\underline{i} + \underline{j} - 9\underline{k}) = 12\underline{i} + 0\underline{j} + 11\underline{k}$

Displacement: $\underline{d} = \overrightarrow{AB} = (6-5)\underline{i} + (2+5)\underline{j} + (-2+7)\underline{k} = \underline{i} + 7\underline{j} + 5\underline{k}$

We know that Work done = $\vec{F}\,.\,\underline{d} = (12\underline{i} + 0\underline{j} + 11\underline{k}).(\underline{i} + 7\underline{j} + 5\underline{k}) = 12 + 0 + 55 = 67$ units

Q.10: A force of magnitude 6 units acting parallel to $2\underline{i} - 2\underline{j} + \underline{k}$ displaces the point of application from (1,2,3) to (5,3,7). Find the work done.

SOLUTION: Let $\vec{F} = ?$, $|\vec{F}| = 6$ Given vector: $\underline{v} = 2\underline{i} - 2\underline{j} + \underline{k}$ and let A(1,2,3) , B(5,3,7)

As $\vec{F} \parallel \underline{v}$

So $\hat{F} = \hat{v}$ $\Rightarrow \frac{\vec{F}}{|\vec{F}|} = \frac{\underline{v}}{|\underline{v}|}$ $\Rightarrow \vec{F} = |\vec{F}|\frac{\underline{v}}{|\underline{v}|} = 6.\frac{2\underline{i} - 2\underline{j} + \underline{k}}{\sqrt{4+4+1}} = 6.\frac{2\underline{i} - 2\underline{j} + \underline{k}}{3} = 2\left(2\underline{i} - 2\underline{j} + \underline{k}\right) = 4\underline{i} - 4\underline{j} + 2\underline{k}$

Displacement: $\underline{d} = \overrightarrow{AB} = (5-1)\underline{i} + (3-2)\underline{j} + (7-3)\underline{k} = 4\underline{i} + \underline{j} + 4\underline{k}$

We know that

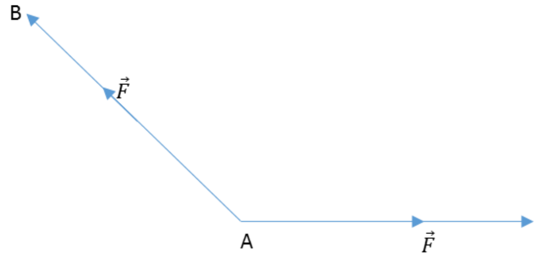
Work done = $\vec{F}\,.\,\underline{d}$

$$= \left(4\underline{i} - 4\underline{j} + 2\underline{k}\right).\left(4\underline{i} + \underline{j} + 4\underline{k}\right)$$

$= 16 - 4 + 8 = 20$ Units

Q.11: A force $\vec{F} = 3\underline{i} + 2\underline{j} - 4\underline{k}$ is applied at the point (1,-1,2). Find the moment of the force about the point (2,-1,3).

SOLUTION: Given: $\vec{F} = 3\underline{i} + 2\underline{j} - 4\underline{k}$ and Let A (1,-1,2), B(2,-1,3)



$\underline{r} = \overrightarrow{AB} = (1-2)\underline{i} + (-1+1)\underline{j} + (2-3)\underline{k} = -\underline{i} + -0\,\underline{j} - \underline{k}$

Moment of force about A = $\underline{r} \times \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix} = \underline{i}(0+2) - \underline{j}(4+3) + \underline{k}(-2-0) = 2\underline{i} - 7\,\underline{j} - 2\underline{k}$

Q.12: A force $\vec{F} = 4\underline{i} - 3\underline{k}$ passes through the point A(2,-2,5). Find the moment of $\underline{F}$ about the point B(1,-3,1).

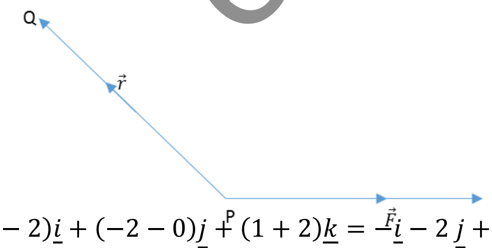
SOLUTION: Given: $\vec{F} = 4\underline{i} - 3\underline{k}$ and A **(2,-2,5)**, B**(1,-3,1)**

$\underline{r} = \overrightarrow{AB} = (2-1)\underline{i} + (-2+3)\underline{j} + (5-1)\underline{k} = \underline{i} + \underline{j} + 4\underline{k}$

Moment of force about B = $\underline{r} \times \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 4 \\ 4 & 0 & -3 \end{vmatrix} = \underline{i}(-3-0) - \underline{j}(-3-16) + \underline{k}(0-4) = -3\underline{i} + 19\,\underline{j} - 4\underline{k}$

Q.13: Given a force $\vec{F} = 2\underline{i} + \underline{j} - 3\underline{k}$ **acting at a point A(1,-2,1). Find the moment of $\underline{F}$ about the point B(2,0,-2). (19 S.Q**

SOLUTION: Given: $\vec{F} = 2\underline{i} + \underline{j} - 3\underline{k}$ and A **(1,-2,1)**, B**(2,0,-2)**

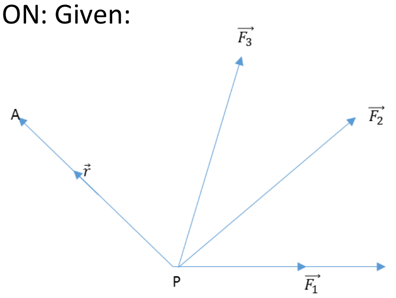


$\underline{r} = \overrightarrow{AB} = (1-2)\underline{i} + (-2-0)\underline{j} + (1+2)\underline{k} = -\underline{i} - 2\,\underline{j} + 3\underline{k}$

Moment of force about B = $\underline{r} \times \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix} = \underline{i}(6-3) - \underline{j}(3-6) + \underline{k}(-1+4) = 3\underline{i} + 3\,\underline{j} + 3\underline{k}$

Q.14: Find the moment about A(1,1,1) of each of the concurrent forces $\underline{i} - 2\,\underline{j}$, $3\underline{i} + 2\underline{j} - \underline{k}$, $5\underline{j} + 2\underline{k}$, where P(2,0,1) is their point of concurrency.

SOLUTION: Given:



Let $\vec{F_1} = \underline{i} - 2\,\underline{j}$, $\vec{F_2} = 3\underline{i} + 2\underline{j} - \underline{k}$ $\vec{F_3} = 5\underline{j} + 2\underline{k}$ and A(1,1,1), P(2,0,1)

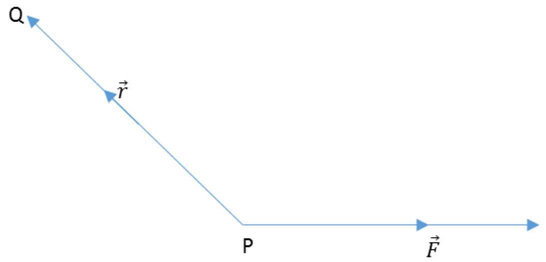
Total force: $\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3} = \underline{i} - 2\,\underline{j} + 3\underline{i} + 2\underline{j} - \underline{k} + 5\underline{j} + 2\underline{k} = 4\underline{i} + 5\underline{j} + \underline{k}$

$\underline{r} = \overrightarrow{AP} = (2-1)\underline{i} + (0-1)\underline{j} + (1-1)\underline{k} = \underline{i} - \underline{j} + 0\underline{k}$

Moment of force about B = $\underline{r} \times \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 4 & 5 & 1 \end{vmatrix} = \underline{i}(-1-0) - \underline{j}(1-0) + \underline{k}(5+4) = -\underline{i} - \underline{j} + 9\underline{k}$

Q.15: A force $\vec{F} = 7\underline{i} + 4\underline{j} - 3\underline{k}$ is applied at P(1,-2,3). Find its moment about the point Q(2,1,1).

SOLUTION: $\vec{F} = 7\underline{i} + 4\underline{j} - 3\underline{k}$



$\underline{r} = \overrightarrow{QP} = (1-2)\underline{i} + (-2-1)\underline{j} + (3-1)\underline{k}$

$= -\underline{i} - 3\underline{j} + 2\underline{k}$

Moment of force about B = $\underline{r} \times \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -3 & 2 \\ 7 & 4 & -3 \end{vmatrix} = \underline{i}(9-8) - \underline{j}(3-14) + \underline{k}(-4+21) = \underline{i} + 11\,\underline{j} + 17\underline{k}$