Cone:
A solid figure generated by a straight line passing through a fixed point and revolving about a fixed line is called cone.

Axis of Cone:
The fixed line passing through vertex and perpendicular to the center of base is called axis of cone.

Apex:
The fixed point of cone is called vertex or apex of cone.

Napes of Cone:
The cone has two parts called napes. The part above the vertex is upper nape. The part below the vertex is lower nape.

Types of Cone:
Cones are named according to shape of their base. If the base of cone is circle then it is called circular cone. If the base of cone is ellipse, then it called elliptic cone.
" A cone having its axis perpendicular circular base is called Right circular cone."

Conic Section:
Some standard conic sections are Circle, Ellipse parabola and Hyperbola**.**
We first study properties of a circle other conics will be taken up later.

Circle:
A set of all points in a plane which are equidistance from a fixed point is called circle. The fixed point is celled center and fixed distance is called radius of the circle.

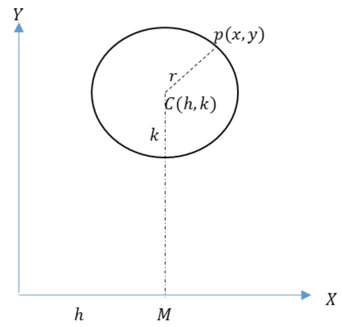
Points Circle:
If the plane passes through vertex of cone, the instruction is a single point or if $r = 0$

Equation of circle:
Let $P(x, y)$ be any point on the circle (h, k) from figure.

$$radius\ |CP| = r$$

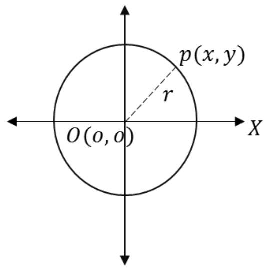
Using distance formula

$$r^2 = (x-h)^2 + (y-k)^2 \quad hence\ required.$$



$$Circle\ is\ (x-h)^2 + (y-k)^2 = r^2$$

Note:
If
$P(x, y) be\ any\ point\ on\ the\ circle\ having\ centre\ at\ orgin\ O(o, o) and\ radius\ r.$
$\because |OP| = r$
Using distance $(o, o)\ and\ radius\ r.$
$\because |OP| = r$
Using distance formula

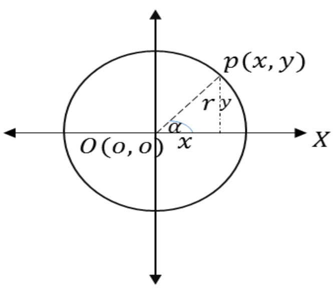


$r = \sqrt{(x-0)^2 + (y-0)^2}$
$r = \sqrt{x^2 + y^2}$
Squaring both sides
$r^2 = x^2 + y^2$
⇨ $x^2 + y^2 = r^2$
$\because A\ cicle\ having\ radius\ zero\ is\ called\ point\ circle.$
So,

$$x^2 + y^2 = r^2 \quad put\ r = 0$$

⇨ $x^2 + y^2 =$
$0 \quad which\ is\ equation\ of\ point\ circle.$

Parametric Equations of Circle:
Let
$P(x, y)\ be\ any\ point\ on\ the\ circle\ having\ centre$ a t origin and radius.
Draw $\perp ar\ PM\ f$rom point on $x - axis.$
$In\ \Delta POM$



$|OM| = x, \ \ |PM| = y$
$|OP| = r, and$
$m\angle MOP = \theta \quad so$
$cos\theta = \frac{|PM|}{|OP|} = \frac{x}{r}$
⇨ $x = rcos\theta$
$sin\theta = \frac{|PM|}{|OP|} = \frac{y}{r}$
⇨ $y = rsin\theta$
are called
$parametric\ eqs.\ of\ circle\ where\ \theta\ is\ parameter.$

General form of an Equation of a Circle:

Theorem:
The equation $x^2 + y^2 + 2gx + 2fy + c = 0 \rightarrow (i)$
Represents a circle $i, f, and\ c\ being\ constants.$
By$(i) x^2 + y^2 + 2gx + 2fy = -c$
Adding $g^2\ and\ f^2\ on\ both\ sides$

$$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$$

⇨ $(x+g)^2+(y+f)^2=g^2+f^2-c$

$(x+g)^2+(y+f)^2=g^2+f^2-c$

⇨ $[x-(-g)]^2+[y-(-f)]^2=$
$\left(\sqrt{g^2+f^2-c}\right)^2$

it is of the form

$(x-h)^2+(y-k)^2=r^2$

where centre $=(-g,-f)$

Radius $=\sqrt{g^2+f^2-c}$

The equation $x^2+y^2+2gx+2fy+c=0$

is called General equation of circle

Every equation of 2nd degree in x and y represents an eq. of circle if (i) coefficients of x^2 *and* y^2 *is* 1

(ii) it does not involve product xy.

1. A circle passing three non- collinear points.
2. A circle passing through two points and having its center on a given line.
3. A circle passing through two points and equation of tangent at one of these points is known.
4. A circle passing through two points and touching a given line.

Exercise6.1

Question # 1. In each of the following find an equation of the circle with

(a) Centre at $(5,-2)$ and radius 4.

Solution.

Centre: $C(h,k)=(5,-2)$

$\Rightarrow h=5, k=-2$

Radius: $r=4$

Required equation of circle is

$(x-h)^2+(y-k)^2=r^2$

$(x-5)^2+(y+2)^2=(4)^2$

$x^2+25-10x+y^2+4+4y=16$

$x^2+y^2-10x+4y+13=0$

(b) Centre at $\left(\sqrt{2},-3\sqrt{3}\right)$ and radius$2\sqrt{2}$.

Solution.

Centre: $C(h,k)=\left(\sqrt{2},-3\sqrt{3}\right)$

$\Rightarrow h=\sqrt{2},\ k=-3\sqrt{3}$

Radius: $r=2\sqrt{2}$

Required equation of circle is

$(x-h)^2+(y-k)^2=r^2$

$\left(x-\sqrt{2}\right)^2+\left(y+3\sqrt{3}\right)^2=\left(2\sqrt{2}\right)^2$

$x^2+2-2\sqrt{2}x+y^2+9(3)+6\sqrt{3}y=4(2)$

$x^2+y^2-2\sqrt{2}x+6\sqrt{3}y+21=0$

(c) End of a diameter at $(-3,\ 2)$ and$(5,-6)$.

Solution.

Let $A(5,-6)$ *and* $B(5,-6)$

Centre of circle= Mid-point $\overline{AB}$

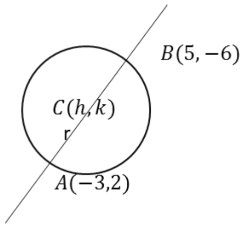
$=C\left(\frac{-3+5}{2},\frac{2-6}{2}\right)=C\left(\frac{2}{2},-\frac{4}{2}\right)=C(1,-2)$

Centre: $C(h,k)=(1,-2)$

$\Rightarrow h=1, k=-2$

Radius of circle= $|AC|=$
$\sqrt{(-3-1)^2+(2+2)^2}$

$=\sqrt{16+16}=\sqrt{32}=4\sqrt{2}$



Radius: $r=4\sqrt{2}$

Required equation of circle is

$(x-h)^2+(y-k)^2=r^2$

$(x+1)^2+(y+2)^2=\left(4\sqrt{2}\right)^2$

$x^2+1+2x+y^2+4+4y=32$

$x^2+y^2+2x+4y-27=0$

Question # 2. Find the centre and radius of the circle with the given equation.

(a) $x^2+y^2+12x-10y=0$

Sol:

Comparing it with

$x^2+y^2+2gx+2fy+c=0$

$\Rightarrow 2g=12\Rightarrow g=6$ also $\Rightarrow 2f=-10\Rightarrow f=-5$ and $c=0$

Centre=$(-g,-f)=(-6,5)$

Radius= $\sqrt{g^2+f^2-c}$

$=\sqrt{(-6)^2+(-5)^2-0}=\sqrt{36+25}=\sqrt{61}$

(b) $5x^2+5y^2+14x+12y-10=0$

Sol: (b)

$$x^2+y^2+\frac{14}{5}x+\frac{12}{5}y-2=0$$

Comparing it with

$x^2+y^2+2gx+2fy+c=0$

$\Rightarrow 2g=\frac{14}{5}\Rightarrow g=\frac{7}{5}$ also $\Rightarrow 2f=\frac{12}{5}\Rightarrow f=\frac{6}{5}$ and $c=-2$

Centre=$(-g,-f)=(-\frac{7}{5},-\frac{6}{5})$

Radius= $\sqrt{(-\frac{7}{5})^2+(-\frac{6}{5})^2+2}$

$=\sqrt{\frac{49}{25}+\frac{36}{25}+2}=\sqrt{\frac{49+36+50}{25}}=\sqrt{\frac{135}{25}}=\sqrt{\frac{27}{5}}$

$$=\frac{3\sqrt{3}}{5}$$

(c) $x^2+y^2-6x+4y+13=0$

Sol: (c)

Comparing it with

$x^2+y^2+2gx+2fy+c=0$

$\Rightarrow 2g=-6\Rightarrow g=-3$ also $\Rightarrow 2f=4\Rightarrow f=2$ and $c=13$

Centre=$(-g,-f)=(3,-2)$

Radius= $\sqrt{(3)^2+(-2)^2-13}$

$= \sqrt{9+4-13} = \sqrt{0} = \sqrt{61}$

Which represent a point circle.

Q#3 Write an equation of the circle that passes through given points.

(a) $A(4,5), B(-4,-3)$ and $C(8,-3)$

Sol: (a)

$A(4,5), B(-4,-3)$ and $C(8,-3)$

Let the required equation of circle is

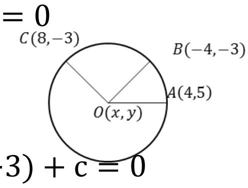
$$x^2 + y^2 + 2gx + 2fy + c = 0$$

At $A(4,5)$

$(4)^2 + (5)^2 + 2g(4) + 2f(5) + c = 0$

$16 + 25 + 8g + 10f + c = 0$

$8g + 10f + c + 41 = 0 \rightarrow (1)$



At $B(-4,-3)$

$(-4)^2 + (-3)^2 + 2g(-4) + 2f(-3) + c = 0$

$16 + 9 - 8g - 6f + c = 0$

$-8g - 6f + c + 25 = 0 \rightarrow (2)$

At $C(8,-3)$

$(8)^2 + (-3)^2 + 2g(8) + 2f(-3) + c = 0$

$64 + 9 + 16g - 6f + c = 0$

$16g - 6f + c + 73 = 0 \rightarrow (3)$

$Now\ Eq\ (1) - Eq(2)$

$8g + 10f + c + 41 = 0$

$\mp 8g \mp 6f \pm c \pm 25 = 0$

$16g + 16f + 16 = 0$

$\Rightarrow\ g + f + 1 = 0 \rightarrow (4)$

$Now\ Eq\ (3) - Eq(2)$

$16g - 6f + c + 73 = 0$

$\mp 8g \mp 6f \pm c \pm 25 = 0$

$24g + 48 = 0$

$\Rightarrow\ 24g = -48\ \Rightarrow\ g = -2$

Put in (4)

$g + f + 1 = 0 \Rightarrow -2 + f + 1 = 0\ \Rightarrow\ f = 1$

Put $f = 1$ and $g = -2$ in eq (2)

$-8g - 6f + c + 25 = 0\ \Rightarrow\ -8(-2) - 6(1) + c + 25 = 0$

$\Rightarrow\ 16 - 6 + c + 25 = 0 \Rightarrow c + 35 = 0 \Rightarrow c = -35$

Hence, required equation of circle is

$x^2 + y^2 + 2(-2)x + 2(1)y + (-35) = 0$

$\Rightarrow\ x^2 + y^2 - 4x + 2y - 35 = 0$

(b) $A(-7,7), B(5,-1)$ and $C(10,0)$

Sol: (b)

$A(-7,7), B(5,-1)$ and $C(10,0)$

Let the required equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

At $A(-7,7)$

$(-7)^2 + (7)^2 + 2g(-7) + 2f(7) + c = 0$

$49 + 49 - 14g + 14f + c = 0$

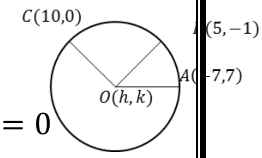
$-14g + 14f + c + 98 = 0\ldots..(1)$

At $B(5,-1)$

$(5)^2 + (-1)^2 + 2g(5) + 2f(-1) + c = 0$

$25 + 1 + 10g - 2f + c = 0$

$10g - 2f + c + 26 = 0\ldots..(2)$



At $C(10,0)$

$(10)^2 + (0)^2 + 2g(10) + 2f(0) + c = 0$

$100 + 0 + 20g + 0f + c = 0$

$20g + c + 100 = 0\ldots..(3)$

Now Eq (1)-Eq(2)

$-14g + 14f + c + 98 = 0$

$\pm 10g \mp 2f \pm c \pm 26 = 0$

$-24g + 16f + 72 = 0$

$\Rightarrow\ -3g + 2f + 9 = 0\ldots..(4)$

Now Eq (3)-Eq(2)

$20g + +0f + c + 100 = 0$

$\pm 10g \mp 2f \pm c \pm 26 = 0$

$10g + 2f + 74 = 0\ldots.(5)$

Now Eq (3)-Eq(2)

$-3g + 2f + 9 = 0$

$\pm 10g \pm 2f \pm 74 = 0$

$-13g - 65 = 0\ldots.(5)$

$\Rightarrow -13g = 65\ \Rightarrow\ g = -5$

Put in (4)

$-3g + 2f + 9 = 0 \Rightarrow\ -3(-5) + 2f + 9 = 0 \Rightarrow\ 15 + 2f + 9 = 0 \Rightarrow f = -12$

Put $g = -5$ in eq (3)

$20g + c + 100 = 0\ \Rightarrow\ 20(-5) + c + 100 = 0 = 0$

$\Rightarrow\ -100 + c + 100 = 0 \Rightarrow c = 0$

Hence, required equation of circle is

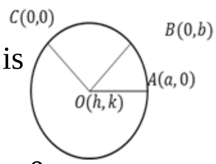
$x^2 + y^2 + 2(-5)x + 2(-12)y + (0) = 0$

$\Rightarrow\ x^2 + y^2 - 10x - 24y = 0$

(c) $A(a,0), B(0,b)$ and $C(0,0)$

Sol: (c)

$A(a,0), B(0,b)$ and $C(0,0)$



Let the required equation of circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$

At $A(a,0)$

$(a)^2 + (0)^2 + 2g(a) + 2f(0) + c = 0$

$a^2 + 0 + 2ag + c = 0$

$a^2 + 2ag + c = 0\ldots..(1)$

At $B(0,b)$

$(0)^2 + (b)^2 + 2g(0) + 2f(b) + c = 0$

$b^2 + 2bf + c = 0\ldots..(2)$

At $C(0,0)$

$(0)^2 + (0)^2 + 2g(0) + 2f(0) + c = 0$

$c = 0$ put in eq (1)

$a^2 + 2ag + c = 0\ \Rightarrow\ a^2 + 2ag + 0 = 0$

$g = -\frac{a^2}{2a} = -\frac{a}{2}$

Eq (2) $\Rightarrow b^2 + 2bf + 0 = 0 \Rightarrow f = -\frac{b^2}{2b} = -\frac{b}{2}$

Hence, required equation of circle is

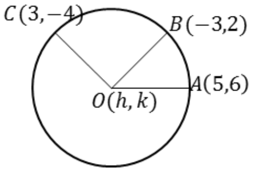
$x^2 + y^2 + 2\left(-\frac{a}{2}\right)x + 2\left(-\frac{b}{2}\right)y + 0 = 0$

$\Rightarrow\ x^2 + y^2 - ax - by = 0$

d) $\boldsymbol{A(5,6), B(-3,2), C(3,-4)}$

Solution:

Clearly from fig.



$$|AO|^2 = |BO|^2$$
$$(h-5)^2 + (k-6)^2 = (h+3)^2 + (k-2)^2$$
$$h^2 + 25 - 10h + k^2 + 36 - 12k$$
$$= h^2 + 9 + 6k + k^2 + 4 - 4k$$
$$-10h - 12k + 61 = 6h - 4k + 13$$

⇨ $-10h - 6h - 12k + 4k + 61 - 13 = 0$

⇨ $-16h - 8k + 48 = 0$

⇨ $2h + k - 6 = 0 (\div by -8) \rightarrow (i)$

Also $|AO|^2 = |CO|^2$

$$(h-5)^2 + (k-6)^2 = (h-3)^2 + (k+4)^2$$
$$h^2 + 25 - 10h + k^2 + 36 - 12k$$
$$= h^2 + 9 - 6h + k^2 + 16 + 8k$$
$$-10h + 6h - 12k - 8k + 61 - 25 = 0$$
$$-4h - 20k + 36 = 0$$

⇨ $-2h - 10k + 18 = 0 \rightarrow (ii)$

$by\ (i) + (ii)$

⇨
$$\begin{array}{r} 2h + k - 6 = 0 \\ -2h - 10k + 18 = 0 \\ \hline -9k + 12 = 0 \end{array}$$

⇨ $-9k = -12$

⇨ $k = \frac{4}{3}\ put\ in\ (i)$

$$2h + \frac{4}{3} - 6 = 0$$
$$6h + 4 - 18 = 0$$

⇨ $6h - 14 = 0$

⇨ $6h = 14$

⇨ $h = \frac{7}{3}$

As $r = |AO| = \sqrt{(h-5)^2 + (k-6)^2}$

$$r = \sqrt{\left(\frac{7}{3} - 5\right)^2 + \left(\frac{4}{3} - 6\right)^2}$$
$$r = \sqrt{\left(-\frac{8}{3}\right)^2 + \left(-\frac{14}{3}\right)^2}$$
$$r = \sqrt{\frac{64 + 196}{9}}$$

⇨ $r = \sqrt{\frac{260}{9}}$

⇨ $Now\ eq\ of\ circle\ is\ (x-h)^2 + (y-k)^2 = r^2$

⇨ $\left(x - \frac{7}{3}\right)^2 + \left(y - \frac{4}{3}\right)^2 = \frac{260}{9}$

⇨ $\left(\frac{3x-7}{3}\right)^2 + \left(\frac{3y-4}{3}\right)^2 = \frac{260}{9}$

⇨ $\frac{9x^2+49-42x}{9} + \frac{9y^2+16-24y}{9} = \frac{260}{9}$

⇨ $9x^2 + 49 - 42x + 9y^2 + 16 - 24y - 260 = 0$

⇨ $9x^2 + 9y^2 - 42x - 24y - 195 = 0$

$3x^2 + 3y^2 - 14x - 8y - 65 = 0 (\div by\ 3)$

⇨ $3(x^2 + y^2) - 14x - 8y - 65 = 0$

Question#4 Find equation of circle passing through

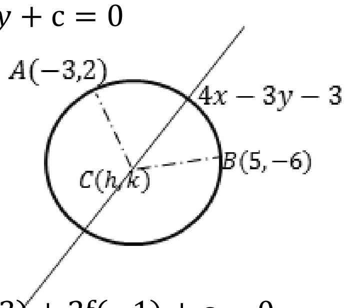
(a) $\boldsymbol{A(3,-1), B(0,1)}$ **and having centre at** $\boldsymbol{4x - 3y - 3 = 0}$

Sol: (a)

$A(3,-1), B(0,1)$

Let the required equation of circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$



At $A(3,-1)$

$(3)^2 + (-1)^2 + 2g(3) + 2f(-1) + c = 0$

$9 + 1 + 6g - 2f + c = 0$

$6g - 2f + c + 10 = 0$.....(1)

At $B(0,b)$

$(0)^2 + (1)^2 + 2g(0) + 2f(1) + c = 0$

$1 + 2f + c = 0$....(2)

As centre $(-g,-f)$ lies on the the line

$4x - 3y - 3 = 0$

$\Rightarrow\ 4(-g) - 3(-f) - 3 = 0$

$\Rightarrow -4g + 3f - 3 = 0$.....(3)

Now Eq (1)-Eq(2)

$$\begin{array}{r} 6g - 2f + c + 10 = 0 \\ 0g \pm 2f \pm c \pm 1 = 0 \\ \hline 6g - 4f + 9 = 0....(4) \end{array}$$

Multiply Eq(3) by 3 and Eq(4) by 2 and adding

$$\begin{array}{r} -12g + 9f - 9 = 0 \\ 12g - 8f + 18 = 0 \\ \hline f + 9 = 0 \end{array}$$

$\Rightarrow\ \ f = -9$ put in Eq (4)

$-4g + 3f - 3 = 0 \Rightarrow -4g + 3(-9) - 3 = 0 \Rightarrow -4g - 27 - 3 = 0$

$\Rightarrow g = -\dfrac{15}{2}$

Put in $f = -9$ in eq (2)

$1 + 2f + c = 0 \Rightarrow 1 + 2(-9) + c = 0$

$1 - 18 + c = 0$

$c = 17$

Hence, required equation of circle is

$x^2 + y^2 + 2\left(-\dfrac{15}{2}\right)x + 2(-9)y + (17) = 0$

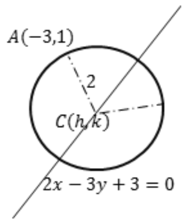
$\Rightarrow\ x^2 + y^2 - 15x - 18y + 17 = 0$

(b) Find an equation of circle passing through $A(-3,1)$ with radius 2 and centre at $2x - 3y + 3 = 0$

Sol: $A(-3,1)$ and radius 2

Let the required equation of circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$



At $A(-3,1)$

$(-3)^2 + (1)^2 + 2g(-3) + 2f(1) + c = 0$

$9 + 1 - 6g + 2f + c = 0$

$-6g + 2f + c + 10 = 0$…..(1)

As centre $(-g,-f)$ lies on the line $2x - 3y + 3 = 0$.

$\Rightarrow -2g + 3f + 3 = 0$…..(2)

Since radius is 2

$\Rightarrow\ \sqrt{g^2 + f^2 - c} = r$

$\Rightarrow\ \sqrt{g^2 + f^2 - c} = 2$

$\Rightarrow g^2 + f^2 - c = 4$

$\Rightarrow g^2 + f^2 - c - 4 = 0$…..(3)

Adding Eq(1) and Eq(3)

$-6g + 2f + c + 10 = 0$

$g^2 + f^2 - c - 4 = 0$

$-6g + 2f + g^2 + f^2 + 6 = 0$…..(4)

Eq(2) $\Rightarrow -2g = -3f - 3 \Rightarrow g = \frac{3f+3}{2}$

Put in Eq(4)

$-6g + 2f + g^2 + f^2 + 6 = 0$

$\Rightarrow -6\left(\frac{3f+3}{2}\right) + 2f + (\frac{3f+3}{2})^2 + f^2 + 6 = 0$

$\Rightarrow -3(3f + 3) + 2f + \dfrac{9f^2 + 9 + 18f}{4} + f^2 + 6 = 0$

$\Rightarrow -12(3f + 3) + 8f + 9f^2 + 9 + 18f + 4f^2 + 24 = 0$

$\Rightarrow -36f + 36 + 8f + 9f^2 + 9 + 18f + 4f^2 + 24 = 0$

$\Rightarrow 13f^2 - 10f - 3 = 0$

$\Rightarrow 13f^2 - 13f + 3f - 3 = 0$

$\Rightarrow 13f(f - 1) + 3(f - 1) = 0$

$\Rightarrow (f - 1)(13f + 3) = 0$

$\Rightarrow f = 1, -\frac{3}{13}$

When $f = 1$

$g = \frac{3f+3}{2} \Rightarrow g = \frac{3(1)+3}{2} = \frac{6}{2} = 3$

From Eq(3)

$\Rightarrow\ (3)^2 + (1)^2 - c - 4 = 0$

$\Rightarrow 9 + 1 - c - 4 = 0$

$\Rightarrow c = 6$

Hence required equation when $f = 1, g = 3$ and $c = 6$ is

$x^2 + y^2 + 2(3)x + 2(1)y + 6 = 0$

$\Rightarrow\ x^2 + y^2 + 6x + 2y + 6 = 0$

When $f = -\frac{3}{13}$

$g = \frac{3f+3}{2} \Rightarrow g = \frac{3(-\frac{3}{13})+3}{2} = \frac{\frac{-9+39}{13}}{2} = \frac{15}{13}$

From Eq(3)

$\Rightarrow\ \left(\dfrac{15}{13}\right)^2 + \left(-\dfrac{3}{13}\right)^2 - c - 4 = 0$

$\Rightarrow 225 + 9 - 169c - 676 = 0$

$\Rightarrow 169c = -442$

$\Rightarrow c = -\dfrac{442}{169} = -\dfrac{34}{13}$

Hence required equation when $f = -\frac{3}{13}, g = \frac{15}{13}$ and $c = -\frac{34}{13}$ is

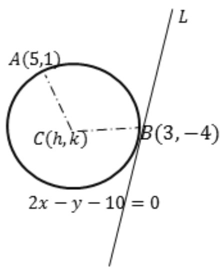
$x^2 + y^2 + 2\left(\dfrac{15}{13}\right)x + 2\left(-\dfrac{3}{13}\right)y + \left(-\dfrac{34}{13}\right) = 0$

$\Rightarrow 13x^2 + 13y^2 + 30x - 6y - 34 = 0$

(c)

$A(5,1)$ and tangent to the line $2x - y - 10 = 0$ at $B(3,-4)$

Solution: it is clear



$|AC|^2 = |BC|^2$

$(h - 5)^2 + (k - 1)^2 = (h - 3)^2 + (k + 4)^2$

$h^2 + 25 - 10h + k^2 + 1 - 2k = h^2 + 9 - 6h + k^2 + 16 + 8k$

$-10h + 2k + 26 = -6h + 8k + 25$

⇨ $-4h - 10k + 1 = 0 \rightarrow (i)$

$slope\ of\ 2x - y - 10 = 0 = -\dfrac{a}{b} = \dfrac{-2}{-1} = 2$

$slope\ of\ BC = \dfrac{k+4}{h-3}$

$\because both\ lines\ are\ \perp ar\ so$

⇨ $(2)\left(\frac{k+4}{h-3}\right) = -= 1$

⇨ $2k + 8 = -h + 3$

⇨ $h + 22k + 5 = 0$

⇨ $5k + 10k + 25 = 0 \rightarrow (ii)$

By $(i)+(ii)$

$$\Rightarrow \quad \begin{array}{r} 5h+10k+25=0 \\ -4h-10k+1=0 \\ \hline h+26=0 \end{array}$$

$\Rightarrow$ $h=-26\ put\ in\ (ii)$

$$-26+2k+5=0$$

$\Rightarrow$ $-26+2k+5=0$

$\Rightarrow$ $2k-21=0$

$\Rightarrow$ $k=\frac{21}{2}$

$$r=|AC|=\sqrt{(h-5)^2+(k-1)^2}$$

$\Rightarrow$ $r=|AC|=\sqrt{(-26-5)^2+\left(\frac{21}{2}-1\right)^2}$

$\Rightarrow$ $=\sqrt{(-31)^2+\left(\frac{19}{2}\right)^2}$

$\Rightarrow$ $=\sqrt{961+\frac{361}{4}}=\sqrt{\frac{3844+361}{4}}=\sqrt{\frac{4205}{2}}$

$\Rightarrow$ $Now\ Eq.of\ circle\ is$

$$(x-h)^2+(y-k)^2=r^2$$

$\Rightarrow$ $(x+26)^2+\left(y-\frac{21}{2}\right)^2=\left(\frac{\sqrt{4205}}{2}\right)^2$

$\Rightarrow$ $x^2+52x+676+y^2-21y+\frac{441}{4}=\frac{4205}{4}$

$\Rightarrow$ $x^2+y^2+52x-21y+\frac{441}{4}-\frac{4205}{4}=0$

$\Rightarrow$ $x^2+y^2+52x-21y+\left(-\frac{1014}{4}\right)=0$

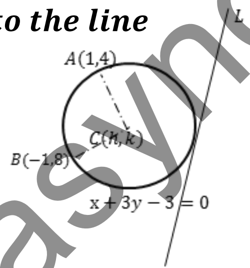
$\Rightarrow$ $x^2+y^2+52x-21y-265=0$

(d)

$A(1,4), B(-1,8)$ and tangent to the line $x+3y-3)=0$

Solution

It is clear from fig.



$|AC|^2=|BC|^2$

$(h-1)^2+(k-4)^2(h+1)^2+(k-8)^2$

$h^2+1-2h+k^2+16-8k=h^2+1+2h+k^2+64-16x$

$-2h-8k+16=2h+1-16k+64$

$$-4h+8k-48=0$$

$$h=2k+12=0$$

$$h=2k-12\rightarrow(i)$$

$$Now\ |AC|=|CL|\ where\ L\ is\ tangent\ line.$$

$$\sqrt{(h-1)^2(k-4)^2}=\frac{|h+3k-3|}{\sqrt{(1)^2+(3)^2}}$$

$$\sqrt{h^2-2h+1+k^2-8k+16}=\frac{|h+3k-3|}{\sqrt{10}}$$

Squaring both sides

$$h^2-2h+k^2-8k+17=\frac{h^2+9k^2+9+6hk-18k-6h}{10}$$

$\Rightarrow$ $10h^2-20h+10k^2-80k+170=h^2+9k^2+9+6hk-18k-6h$

$\Rightarrow$ $9h^2+k^2-14h-62k+161-6hk=0\rightarrow(ii)$

Put(i) and (ii)

$$9(2k-12)^2+k^2-14(2k-12)-62k+161-6(2k-12)k=0$$

$$9(4k^2+144-48k)+k^2-28k+168-62k+161-12k^2+72k=0$$

$\Rightarrow$ $36k^2+1296-432k+k^2-28k+168-62k+161-12k^2+72k=0$

$\Rightarrow$ $25k^2-450k+1625=0$

$\Rightarrow$ $k^2-18k+65=0$ $(\div\ by\ 25)$

$\Rightarrow$ $k^2-13k-5k+65=0$

$\Rightarrow$ $k(k-13)-5(k-13)=0$

$\Rightarrow$ $(k-5)(k-13)=0$

$\Rightarrow$ $k-5=0\quad k-13=0$

$\Rightarrow$ $k=5;\quad k=13$

$when\ k=13\ then\ h=2(13)-12=14$

$when\ k=5\ then\ h=2(5)-12=-2$

So

$$C_1(-2,5)and\ C_2(14,13)$$

Now $r_1=|AC_1|=\sqrt{(1+2)^2+(4-5)^2}=\sqrt{10}$

$r_2=|AC_2|=\sqrt{(1-14)^2+(4-13)^2}=\sqrt{250}$

Now $eq.of\ circle\ is$

$$(x+2)^2+(y-5)^2=\left(\sqrt{10}\right)^2$$

$\Rightarrow$ $x^2+4+4x+y^2+25-10y-10=0$

$\Rightarrow$ $x^2+y^2+4x\pm10y+19=0$

Also eq. of circle is

$$(x-14)^2+(y-13)^3=\left(\sqrt{250}\right)^2$$

$\Rightarrow$ $x^2-28x+196+y^2+169-26y-250=0$

$\Rightarrow$ $x^2+y^2-28x-26y+115=0$

Q#5) Find equation of circle of radius 'a' And lying in the second quadrant such that it is tangent to both axis.

Sol: Since circle is lying in second quadrant with radius "a" and tangent to the both axes, therefore

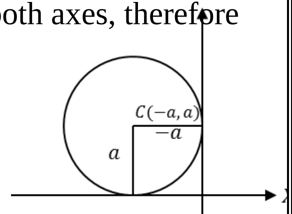
Centre: $(-a,a)$

Required equation of circle is

$(x-h)^2+(y-k)^2=r^2$

At $(-a,a)$

$(x+a)^2+(y-a)^2=a^2$



$\Rightarrow$ $x^2+a^2+2ax+y^2+a^2-2ay=a^2$

$\Rightarrow$ $x^2+y^2+2ax-2ay+a^2=0$

Which is required.

Q#6) Show that the lines $3x - 2y = 0$ and $2x + 3y - 13 = 0$ are tangent to the circle $x^2 + y^2 + 6x - 4y = 0$

Sol:

$x^2 + y^2 + 6x - 4y = 0$

Comparing with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 2g = 6 \Rightarrow g = 3$

Also $\Rightarrow 2f = -4 \Rightarrow f = -2$ and $c = 0$

Centre$=(-g, -f) = (-3, 2)$

Radius$= \sqrt{(-3)^2 + (2)^2 - 0}$

$= \sqrt{9+4} = \sqrt{13}$

Now distance from the centre and the line $3x - 2y = 0$ is the radius

$$r = \frac{|3(-3) - 2(2)|}{\sqrt{(3)^2 + (-2)^2}}$$

$r = \frac{|13|}{\sqrt{13}} = \sqrt{13}$.....(1)

Now distance from the centre and the line $2x + 3y - 13 = 0$ is the radius

$$r = \frac{|2(-3) + 3(2) - 13|}{\sqrt{(2)^2 + (3)^2}}$$

$r = \frac{|13|}{\sqrt{13}} = \sqrt{13} \rightarrow (2)$

Hence, from (1) and (2) it s clear that both lines are tangent to the given circle.

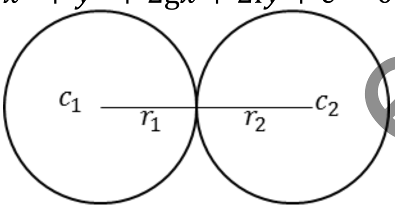
Q#7) Show that $x^2 + y^2 + 2x - 2y - 7 = 0$ and $x^2 + y^2 - 6x + 4y - 9 = 0$ touch externally.

Sol:

Let $C_1: x^2 + y^2 + 2x - 2y - 7 = 0$

Comparing with

$x^2 + y^2 + 2gx + 2fy + c = 0$



$\Rightarrow 2g = 2 \Rightarrow g = 1$ also $\Rightarrow 2f = -2 \Rightarrow f = -1$ and $c = -7$

Centre$=c_1(-g, -f) = (-1, 1)$

Radius$= \sqrt{(-1)^2 + (1)^2 + 7}$

$r_1 = \sqrt{1+1+7} = \sqrt{9} = 3$

Let $C_2: x^2 + y^2 - 6x + 4y - 9 = 0$

Comparing with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 2g = -6 \Rightarrow g = -3$ also $\Rightarrow 2f = 4 \Rightarrow f = 2$ and $c = 9$

Centre$=c_2(-g, -f) = (3, -2)$

Radius$= \sqrt{(3)^2 + (-2)^2 - 9}$

$r_2 = \sqrt{9+4-9} = \sqrt{4} = 2$

Two circles touching externally if

$|c_1c_2| = r_1 + r_2$

$\Rightarrow \sqrt{(-1-3)^2 + (1+2)^2} = 3 + 2$

$\Rightarrow \sqrt{16+9} = 5$

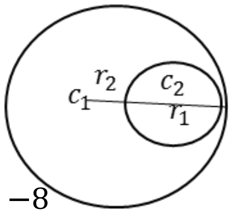
$\Rightarrow \sqrt{25} = 5$

$\Rightarrow 5 = 5$

$\Rightarrow$ Both circles touches externally.

Q#8) Show that $x^2 + y^2 + 2x - 8 = 0$ and $x^2 + y^2 - 6x + 6y - 46 = 0$ touch internally.

Sol:



Let $C_1: x^2 + y^2 + 2x - 8 = 0$

Comparing with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 2g = 2 \Rightarrow g = 1$

also $\Rightarrow 2f = 0 \Rightarrow f = 0$ and $c = -8$

Centre$= c_1(-g, -f) = (-1, 0)$

Radius$= \sqrt{(-1)^2 + (0)^2 + 8}$

$r_1 = \sqrt{1+8} = \sqrt{9} = 3$

Let $C_2: x^2 + y^2 - 6x + 6y - 46 = 0$

Comparing with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 2g = -6 \Rightarrow g = -3$ also $\Rightarrow 2f = 6 \Rightarrow f = 3$ and $c = -46$

Centre$=c_2(-g, -f) = (3, -3)$

Radius$= \sqrt{(3)^2 + (-3)^2 + 46}$

$r_2 = \sqrt{9+9+46} = \sqrt{64} = 8$

Two circles touching internally if

$|c_1c_2| = |r_1 - r_2|$

$\Rightarrow \sqrt{(-1-3)^2 + (0+3)^2} = |3 - 8|$

$\Rightarrow \sqrt{16+9} = |-5|$

$\Rightarrow \sqrt{25} = 5$

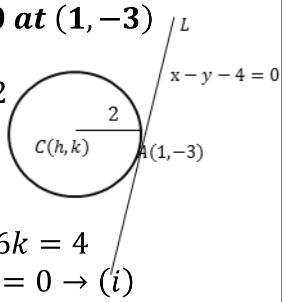
$\Rightarrow 5 = 5$

$\Rightarrow$ Both circles touches internally.

Question No.9

Find Equation of circles of line radius 2 and tangent to line $x - y - 4 = 0$ at $(1, -3)$

Solution:



$\because |AC| = 2$

⇨ $|AC|^2 = 4$

⇨ $(h-1)^2 + (k+3)^2 = 4$

$h^2 + 1 - 2h + k^2 + 9 + 6k = 4$

⇨ $h^2 + k62 - 2h + 6k + 6 = 0 \rightarrow (i)$

⇨ Slope of$x - y - 4 = 0 = \frac{-a}{b} = -\frac{1}{-1} = 1$

Slope of $AC = \frac{k+3}{h-1}$

$\because$ $both$ $lines$ are $\perp$ ar so

$$\left(\frac{k+3}{h-1}\right)(1) = (-1)$$

⇨ $k + 3 = -h + 1$

⇨ $h = -k - 2 = 0$ put $in(i)$

$(-k-2)^2 + k^2 - 2(-k-2) + 6k + 6 = 0$

$K^2 + 4 + 4k + k^2 + 2k + 4 + 6k + 6 = 0$

$$2k^2 + 12k + 14 = 0$$
$$k^2 + 6k + 7 = 0 \;\; (\div\, by\ 2)$$

⇨ $k = \frac{-6 \pm \sqrt{36-28}}{2} = \frac{-6\pm\sqrt{8}}{2}$

⇨ $k = \frac{-6\pm2\sqrt{2}}{2} = \frac{2(-3\pm\sqrt{2})}{2}$

⇨ $k = -3+\sqrt{2} \quad or\ k = -3-\sqrt{2}$

When $k = -3+\sqrt{2}\ then$

$$h = -(-3+\sqrt{2}) - 2 = 3 - \sqrt{2} - 2$$
$$h = 1 - \sqrt{2}$$

when $h = 1-\sqrt{2}\quad ,k = -3+\sqrt{2}, \quad r = 2$

$Eq. of\ circle\ \ is$

$$\left(x - 1 + \sqrt{2}\right)^2 + \left(y + 3 - \sqrt{2}\right)^2 = 4$$

Tangent and Normal:

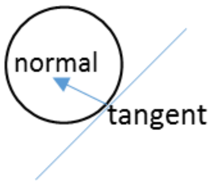
Tangent to a circle:

"A line which touches the circle without cutting it, is called tangent line"

Normal to a circle:

"A Line ⊥ $ar\ to\ the\ tangent\ line\ at\ a\ point\ of\ tandency$ $is\ called\ Normal\ line.$

$$for\ circle\ x^2 + y^2 = r^2$$



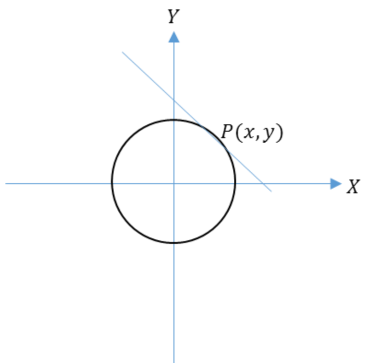
Equation of tangent:

Given equation of circle is

$$x^2 + y^2 = r^2 \rightarrow (i)$$

$\because\ point\ P(x,y)$ lies on the circle so (i) becomes. As

$$x_1^2 + y_1^2 + 7 = 0 \quad (\div\ by\ 2)$$



Diff $(i) w.r.t\ x, we\ get$

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(r^2)$$
$$2x + 2y\frac{dy}{dx} = 0$$
$$x + y\frac{dy}{dx} = 0$$

⇨ $\frac{dy}{dx} = -\frac{x}{y}$

$\because m = slope\ of\ tangent\ line\ at\ P(x_1,y_1)$

So $m = \left.\frac{dy}{dx}\right|_p = \frac{-x_1}{y_1}$

Using $y - y_1 = m(x - x_1)$

⇨ $y - y_1 = m(x - x_1)$

⇨ $y - y_1 = \frac{-x_1}{y_1}(x - x_1)$

⇨ $yy_1 - y_1^2 = -xx_1 + x_1^2$

⇨ $xx_1 + yy_1 = x_1^2 + y_1^2$

⇨ $xx_1 + yy_1 = r^2$

$(\because x_1 + y_1 = r^2) by\ (ii)$

$which\ is\ equation\ of\ tangent\ line.$

Equation of Normal

$$\because slope\ of\ normal\ line = \frac{y_1}{x_1}$$

so eq. of normal line at $P(x,y) is$

$$y - y_1 = \frac{y_1}{x_1}(x - x_1)$$

⇨ $x_1y - x_1y_1 = xy_1 - x_1y_1$

⇨ $x_1y = xy_1$

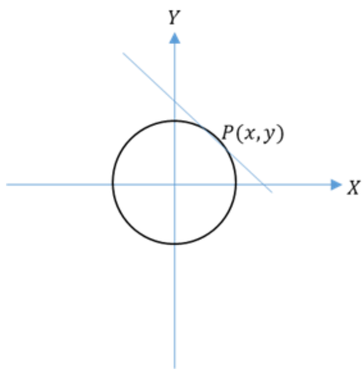
⇨ $\frac{y}{y_1} = \frac{x}{x_1}$

Which is eq. of normal line.

For circle

$$\boldsymbol{x^2 + y^2 + 2gx + 2fy + c = 0}$$

Equation of tangent:



Given equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$\because P(x,y) lies\ on\ the\ cicle\ so\ eq(i) become\ as$

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_{1_c} = 0 \rightarrow (ii)$$

$$Diff(i) w.r.t\ x\ we\ get$$

$$\frac{d}{dx}(x^2 + y^2 + 2gx + 2fy + c) = \frac{d}{dx}(o)$$

⇨ $2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$

⇨ $x + y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$

⇨ $x + y\frac{dy}{dx} + g + f\frac{dy}{dx} = 0$

⇨ $(y+f)\frac{dy}{dx} = -(x+g)$

or $\frac{dy}{dx} = -\left(\frac{x+g}{y+f}\right)$

$m = slope\ of\ tangent\ line\ at\ point\ P(x_1,y_1)$

$$m = -\left(\frac{x_1 + g}{y_1 + f}\right)$$

Using $y - y_1 = m(x - x_1)$

⇨ $y - y_1 = -\left(\frac{x_1+g}{y_1+f}\right)(x - x_1)$

⇨ $y - y_1(y_1 + f) = -(x - x_1)(x_1 + g)$

⇨ $(y - y_1)(y_1 + f) = -(x - x_1)(x_1 + g)$

⇨ $yy_1 - y_1^2 + fy - fy_1 = -xx_1 + x_1^2 - xg + x_1 g$

⇨ $xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$

$\quad adding\ gx_1 + fy_1 + c\ on\ both\ sides$

$$xx_1 + yy_1 + gx + gx_1 + fy + fy_1 + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

⇨ $xx_1 + yy_1 + g(x + x_1) + f(y - y_1) + c = 0$

$\quad \because x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0\ \ by(ii)$

Equation of Normal

Now slope of normal line is $\frac{y_1+f}{x_1+g}$

Equation of normal line is $y - y_1 = \left(\frac{y_1-f}{x_1+g}\right)(x - x_1)$

⇨ $\frac{x-x_1}{x_1+g} = \frac{y-y_1}{y_1+f}$

Theorem: the point

$\boldsymbol{P(x_1, y_1)}$ ***lies outside, on or inside the circle*** ,

$$\boldsymbol{x^2 + y^2 = r^2\ according\ as\ x_1^2 + y_1^2 \geq 0}$$

Proof:

$\quad Given\ eq.\ of\ circle\ is\quad x^2 + y^2 = r^2$

$\quad centre = (0,0), radius = r$

Now point

$P(x_1, y_1) will\ lie\ outside, on\ inside\ the\ circle$

$$x^2 + y^2 = r^2\ \ if\quad |CP| \underset{<}{\geq} r$$

⇨ $\sqrt{(x_1 - 0)^2 + (y_1 - 0)^2} \underset{<}{\geq} r$

Squaring both sides

⇨ $x_1^2 + y_1^2 \underset{<}{\geq} r^2$

⇨ $x_1^2 + y_1^2 - r^2 \underset{<}{\geq} 0$

Hence proved.

theorem:

the point $\boldsymbol{p(x_1, y_1)}$ ***lies outside, on or inside the circle***

$$\boldsymbol{x^2 + y^2 + 2gx + 2fy + c = 0\ \ accorrding\ as}$$

$$\boldsymbol{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \underset{<}{\geq} 0}$$

Proof:

Given eq. of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

$\quad centre = (-g, -f), radius = \sqrt{g^2 + f^2 - c}$

Now point

$P(x_1, y_1)\ will\ lie\ outside\ on\ or\ inside\ the\ circle\ x^2 + y^2 + 2gx + 2fy + c = 0$

$$if\ |CP| \underset{<}{\geq} \boldsymbol{r}$$

⇨ $\sqrt{(x_1 - (-g))^2 + (y_1 - (-f))^2} \underset{<}{\geq} \sqrt{g^2 + f^2 - c}$

Squaring both sides

⇨ $(x_1 + g)^2 + (y_1 + f)^2 \underset{<}{\geq} \boldsymbol{g^2 + f^2 - c}$

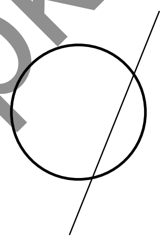
⇨ $x_1^2 + g^2 + 2gx_1 + 2fy_1 + g^2 + f^2 - g^2 - f^2 + c \underset{<}{\geq} \boldsymbol{0}$

⇨ $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \underset{<}{\geq} \boldsymbol{0\ \ hence\ prove}$

Theorem:

The line $\boldsymbol{y = mx + c}$ ***intersects the circle*** $\boldsymbol{x^2 + y^2 = a^2}$ ***in at the most two points.***

Proof:



Given $equation\ of\ circle\ is$

$$x^2 + y^2 = a^2 \rightarrow (i)$$

$$also\ given\ equation\ of\ line\ is$$

$$y = mx + c \rightarrow (ii)$$

Put $value\ of\ y\ in\ (i)$

⇨ $x^2 + (mx + c)^2 = a^2$

⇨ $x^2 + m^2x^2 + c^2 + 2mcx - a^2 = 0$

⇨ $\because (1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0 \rightarrow (iii)$

$this\ being\ quadratic\ in\ x, gives\ two\ value\ of\ x$

Thus the line $intersects\ the\ circle\ x^2 + y^2 = a^2$

At the most two points.

Nature of points:

For nature of points, we examine the discriminant $b^2 - 4ac\ of\ eq.\ The\ points\ ar$

(a) Real and distinct if

$$(2mc)^2 - 4(1 + m^2)(c^2 - a^2) > 0$$

(b) Real and coincident if

$$(2mc)^2 - 4(1 + m^2)(c^2 - a^2) = 0$$

(c) Imaginary if

$$(2mc)^2 - 4(1 + m^2)(c^2 - a^2) < 0$$

Condition of tangency

$$for\ condition\ of\ tangency;$$

$$b^2 - 4ac = 0$$

⇨ $(2mc)^2 - 4(1 + m^2)(c^2 - a^2) = 0$

⇨ $4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - m^2a^2) = 0$

Or$\frac{m^2}{c^2} - c^2 + a^2 - m^2c^2 + m^2a^2 = 0$

⇨ $a^2(1 + m^2) = c^2$

$$or\ c = \pm a\sqrt{1 + m^2}$$

Required condition of tangency

$$by\ (ii) y = mx \pm a\sqrt{1 + m^2}$$

which is equation of tangent to circle

$x^2 + y^2 = a^2$

Theorem:

Two tangent can be drawn to a circle from any point

$\boldsymbol{P(x_1, y_1). The\ tangents\ are\ real\ and\ distinct,}$ **Coincident**

$\boldsymbol{or\ imaginary\ according\ as\ the\ point\ lies\ outside, on\ or\ inside, on\ or\ inside\ the\ circle.}$

Proof:

Consider $eq. of\ circle\ x^2 + y^2 = a^2 \rightarrow (i)$

$$\because y = mx + a\sqrt{1 + m^2} \rightarrow (ii)$$

Is$eq. of\ either\ tangent\ to\ the\ circle\ x^2 + y^2 = a^2$

let the tangent line passes through point $P(x_1, y_1)$

Then equation (ii)*becomes*

$$y_1 = mx_1 + a\sqrt{1 + m^2}$$

$$y_1 - mx_1 = a\sqrt{1 + m^2}$$

squring both sides

$$(y_1 - mx_1)^2 = \left(a\sqrt{1 + m^2}\right)^2$$

⇨ $y_1^2 + m^2x_1^2 - 2mx_1y_1 = a^2(1 + m^2)$

⇨ $x_1^2m^2 - a^2m^2 - 2mx_1y_1 + y_1^2 - a^2 = 0$

⇨ $(x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 - a^2 = 0 \rightarrow (iii)$

This being quadratic in m. it will give two valves of m.

Thus two tangent can be drawn to a circle from any point $P(x_1, y_1)$.

Nature of tangent:

For nature of tangent, we examine the discriminant of $eq(iii) i.e$

(a) Real and distinct if $b^2 - 4ac > 0$

(b) Real and coincident if $b^2 - 4ac = 0$

(c) Imaginary if $b^2 - 4ac < 0$

So $(-2x_1y_1)^2 - 4(x_1^2 - a^2)(y_1^2 - a^2) \gtreqless 0$

⇨ $4x_1^2y_1^2 - 4(x_1^2y_1^2 - a^2x_1^2 - a^2y_1^2 + a^4) \gtreqless 0$

⇨ $x_1^2y_1^2 - x_1^2y_1^2 + a^2x_1^2 + a^2y_1^2 - a^4 \gtreqless 0$

⇨ $a^2(x_1^2 + y_1^2) - a^4 \gtreqless 0$

⇨ $x_1^2 + y_1^2 - a^2 \gtreqless 0$

Which is condition for a point

$P(x_1, y_1) lying\ out\ side\ on\ or\ inside\ the\ circle$

$$x^2 + y^2 = a^2$$

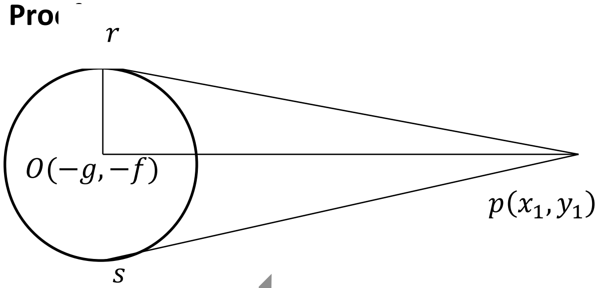
Theorem:

Find length of tangent to the circle

$$\boldsymbol{x^2 + y^2 + 2gx + 2fy + c = 0\ drawn\ from\ point}$$

$\boldsymbol{P(x, y) lying\ outside\ the\ circle.}$

Pro



For $x^2 + y^2 + 2gx + 2fy + c = 0$

$$Centre = 0(-g, -f)$$

Radius $= |TO| = \sqrt{g^2 + f^2 - c}$

⇨ $|TO|^2 = g^2 + f^2 - c$

Let

PT and PS be two tangent drwn from point $P(x_1, y_1)$*to the given circle.*

from figure

$$in\ rt\ \Delta\ PTO$$

⇨ $|PO|^2 = |PT|^2 + |TO|^2$

⇨ $|PT|^2 = |PO|^2 - |TO|^2 \rightarrow (i)$

$\because |PC| = \sqrt{(x_1 - (-g))^2 + (y_1 - (-f))^2}$

⇨ $|PC|^2 = (x_1 + g)^2 + (y_1 + f)^2$

Now (i)

$$|PT|^2 = (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$$

$$x_1^2 + g^2 + 2gx_1 + y_1^2 + f^2 + 2fy_1 - g^2 - f^2 + c$$

$$|PT|^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$$|PT|^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$$|PT| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

which is required lenght of tangent.

Exercise 6.2

Question # 1. Write down equation of the tangent and normal to the circle.

i). $\boldsymbol{x^2 + y^2 = 25}$ **at** $\boldsymbol{P(4, 3)}$ **and at** $\boldsymbol{(5cos\theta, 5sin\theta)}$

Solution.

Circle: $x^2 + y^2 = 25$

$x^2 + y^2 - 25 = 0$

Comparing it with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 2g = 0 \Rightarrow g = 0$ also $\Rightarrow 2f = 0 \Rightarrow f = 0$ and $c = -25$

Since equation of tangent at $P(x_1, y_2)$ is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

At $P(x_1, y_2) = P(4, 3)$

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

$\Rightarrow x(4) + y(3) + 0(x + 4) + 0(y + 3) - 25 = 0$

$\Rightarrow 4x + 3y - 25 = 0$

$\Rightarrow 4x + 3y = 25$

The equation of Normal at $P(x_1, y_2)$ is

$(y - y_1)(x_1 + g) = (x - x_1)(y_1 + f)$

At $P(x_1, y_2) = P(4, 3)$

$(y - 3)(4 + g) = (x - 4)(3 + f)$

$\Rightarrow (y - 3)(4 + 0) = (x - 4)(3 + 0)$

$\Rightarrow (y - 3)4 = (x - 4)3$

$\Rightarrow 4y - 12 = 3x - 12$

$\Rightarrow 3x - 4y = 0$

Now, we take $P(x_1, y_2) = P(5cos\theta, 5cos\theta)$

Since equation of tangent at $P(x_1, y_2)$ is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

At $P(x_1, y_2) = P(5cos\theta, 5sin\theta)$

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

$\Rightarrow x(5cos\theta) + y(5sin\theta) + 0(x + 5cos\theta) + 0(y + 5sin\theta) - 25 = 0$

$\Rightarrow 5xcos\theta + 5ysin\theta - 25 = 0$

$\Rightarrow xcos\theta + ysin\theta = 5$

The equation of Normal at $P(x_1, y_2)$ is

$(y - y_1)(x_1 + g) = (x - x_1)(y_1 + f)$

At $P(x_1, y_2) = (5cos\theta, 5sin\theta)$

$(y - 3)(4 + g) = (x - 4)(3 + f)$

$\Rightarrow (y - 5sin\theta)(5cos\theta + 0) = (x - 5cos\theta)(5sin\theta + 0)$

$\Rightarrow (y - 5sin\theta)5cos\theta = (x - 5cos\theta)5sin\theta$

$\Rightarrow (y - 5sin\theta)cos\theta = (x - 5cos\theta)sin\theta$

$\Rightarrow ycos\theta - 5sin\theta cos\theta = xsin\theta - 5sin\theta cos\theta$

$\Rightarrow xsin\theta - ycos\theta = 0$

(ii). $3x^2 + 3y^2 + 5x - 13y + 2 = 0$ at $P(1, \frac{10}{3})$

Solution. (ii)

Circle: $3x^2 + 3y^2 + 5x - 13y + 2 = 0$ $x^2 + y^2 - 25 = 0$

$\Rightarrow x^2 + y^2 + \frac{5}{3}x - \frac{13}{3}y + \frac{2}{3} = 0$

Comparing it with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 2g = \frac{5}{3} \Rightarrow g = \frac{5}{6}$ also $\Rightarrow 2f = \frac{-13}{3} \Rightarrow f = -\frac{13}{6}$ and $c = \frac{2}{3}$

Since equation of tangent at $P(x_1, y_2)$ is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

At $P(x_1, y_2) = P(1, \frac{10}{3})$

$$x(1) + y\left(\frac{10}{3}\right) + g(x + 1) + f\left(y + \frac{10}{3}\right) + c = 0$$

$$x(1) + y\left(\frac{10}{3}\right) + \frac{5}{6}(x + 1) - \frac{13}{6}\left(y + \frac{10}{3}\right) + \frac{2}{3} = 0$$

Multiply by 18

$\Rightarrow 18x + 60y + 15(x + 1) - 13(3y + 10) - 12 = 0$

$\Rightarrow 18x + 60y + 15x + 15 - 39y - 130 - 12 = 0$

$\Rightarrow 33x + 21y - 103 = 0$

The equation of Normal at $P(x_1, y_2)$ is

$(y - y_1)(x_1 + g) = (x - x_1)(y_1 + f)$

At $P(x_1, y_2) = P(1, \frac{10}{3})$

$$\left(y - \frac{10}{3}\right)\left(1 + \frac{5}{6}\right) = (x - 1)\left(\frac{10}{3} - \frac{13}{6}\right)$$

$$\Rightarrow \left(\frac{3y - 10}{3}\right)\left(\frac{6 + 5}{6}\right) = (x - 1)\left(\frac{20 - 13}{6}\right)$$

$\Rightarrow (3y - 10)11 = (x - 1)21$

$\Rightarrow 33y - 110 = 21x - 21$

$\Rightarrow 21x - 33y + 89 = 0$

Q#2) Write down the equation of tangent and normal to the circle $4x^2 + 4y^2 - 16x + 24y - 117 = 0$ at the point on the circle whose abscissa is -4.

Sol:

Circle: $4x^2 + 4y^2 - 16x + 24y - 117 = 0$....(1)

At $x = -4$

$4(-4)^2 + 4y^2 - 16(-4) + 24y - 117 = 0$

$\Rightarrow 64 + 4y^2 + 64 + 24y - 117 = 0$

$\Rightarrow 4y^2 + 24y + 11 = 0$

$\Rightarrow 4y^2 + 22y + 2y + 11 = 0$

$\Rightarrow 2y(2y + 11) + 1(2y + 11) = 0$

$\Rightarrow (2y + 11)(2y + 1) = 0$

$\Rightarrow y = -\frac{1}{2}, -\frac{11}{2}$

So, points are $P_1\left(-4, -\frac{11}{2}\right), P_2\left(-4, -\frac{1}{2}\right)$

take derivative of (i) w.r.t x

$$8x + 8y\frac{dy}{dx} - 16 + 24\frac{dy}{dx} = 0$$

$\Rightarrow \frac{dy}{dx}(8y + 24) = -8x + 16$

$$or\ \frac{dy}{dx} = \frac{-8x + 16}{8y + 24} = \frac{8(-x + 2)}{8(y + 3)}$$

$\Rightarrow \frac{dy}{dx} = \frac{-x+2}{y+3}$

Slope of tangent $at\ P_1 = \frac{dy}{dx}\Big|_{\left(-4,-\frac{11}{2}\right)} = \frac{-(-4)+2}{-\frac{11}{2}+3}$

$$\frac{dy}{dx}\Big|_{\left(-4,-\frac{11}{2}\right)} = \frac{6}{\frac{-11+6}{2}}$$

$$or\ \frac{dy}{dx}\Big|_{\left(-4,-\frac{11}{2}\right)} = -\frac{12}{2}$$

Equation of tangent

$\Rightarrow y + \frac{11}{2} = -\frac{12}{5}(x+4)$

$\Rightarrow 5y + \frac{55}{2} = -12x - 48$

$\Rightarrow 10y + \frac{55}{2} = -12x - 48$

$24x + 10y = -96 - 55$

$or\ 24x + 10y + 151 = 0$

Slope of normal $= \frac{5}{12}$

Equation of normal

$y + \frac{11}{2} = \frac{5}{12}(x+4)$

⇨ $12y + 66 = 5x + 20$

⇨ $5x - 12y - 46 = 0$

⇨ Now slope of tangent at P_2

$$= \frac{dy}{dx}\Big|_{\left(-4,-\frac{1}{2}\right)} = \frac{-(-4)+2}{-\frac{1}{2}+3}$$

$$\frac{dy}{dx}\Big|_{\left(-4,-\frac{1}{2}\right)} = \frac{6}{\frac{-1+6}{2}}$$

$$or\ \frac{dy}{dx}\Big|_{\left(-4,-\frac{1}{2}\right)} = \frac{12}{5}$$

Equation of tangent

$\Rightarrow y + \frac{1}{2} = \frac{12}{5}(x+4)$

$\Rightarrow 5y + \frac{5}{2} = 12x + 48$

$\Rightarrow 5y + \frac{5}{2} = 12x + 48$

$\Rightarrow 10y + 5 = 24x + 96$

$or\ 24x - 10y = 5 - 96$

$24x - 10y = -91$

$or\ 24x - 10y + 91 = 0$

$slope\ of\ normal = -\frac{5}{12}$

$equation\ of\ normal$

$y + \frac{1}{2} = -\frac{5}{12}(x+4)$

⇨ $12y + 6 = -5x - 20$

⇨ $5x + !2y = -26$

$or\ 5x + 12y + 26 = 0$

Q#3) Check the position of the point $(5,6)$ with respect to the circle

(i). $x^2 + y^2 = 81$

Sol: (i).

$x^2 + y^2 = 81$

$\Rightarrow x^2 + y^2 - 81 = 0$

At $(5,6)$

$L.H.S = (5)^2 + (6)^2 - 81$

$= 25 + 36 - 81$

$= -20 < 0$

Thus, the point $(5,6)$ lies inside the circle.

(ii). $2x^2 + 2y^2 + 12x - 8y + 1 = 0$

Sol: (ii).

$2x^2 + 2y^2 + 12x - 8y + 1 = 0$

At $(5,6)$

$L.H.S = 2(5)^2 + 2(6)^2 + 12(5) - 8(6) + 1$

$= 50 + 72 + 60 - 48 + 1$

$= 135 > 0$

Thus, the point $(5,6)$ lies outside the circle.

Q#4) Find the length of the tangent drawn from the point $(-5,4)$ to the circle $5x^2 + 5y^2 - 10x + 15y - 131 = 0$

Sol:

Circle: $5x^2 + 5y^2 - 10x + 15y - 131 = 0$

$\Rightarrow x^2 + y^2 - 2x + 3y - \frac{131}{5} = 0$

At point P$(-5,4)$, we calculate length of tangent from P

length of tangent from P=

$\sqrt{(-5)^2 + (4)^2 - 2(-5) + 3(4) - \frac{131}{5}}$

$= \sqrt{25 + 16 + 10 + 12 - \frac{131}{5}}$

$= \sqrt{63 - \frac{131}{5}} = \sqrt{\frac{315 - 131}{5}} = \sqrt{\frac{184}{5}}$

Q#5) Find the length of the chord cut off from the line $2x + 3y = 13$ by the circle$x^2 + y^2 = 26$.

Sol:

Circle: $x^2 + y^2 = 26$....(1)

And $2x + 3y = 13 \Rightarrow 2x = 13 - 3y$

$\Rightarrow \mathrm{x} = \frac{13-3y}{2}$ put in Eq(1)

$x^2 + y^2 = 26 \Rightarrow \left(\frac{13-3y}{2}\right)^2 + y^2 = 26$

$\Rightarrow \frac{169 + 9y^2 - 78y}{4} + y^2 = 26$

$\Rightarrow 169 + 9y^2 - 78y + 4y^2 = 104$

$\Rightarrow 13y^2 - 78y + 65 = 0$
$\Rightarrow y^2 - 6y + 5 = 0$
$\Rightarrow y^2 - 5y - y + 5 = 0$
$\Rightarrow y(y-5) - 1(y-5) = 0$
$\Rightarrow (y-5)(y-1) = 0$
$\Rightarrow$ y = 5, 1 put in given st. line
$x = \frac{13-3y}{2} \Rightarrow x = \frac{13-3(5)}{2} = \frac{13-15}{2} = -\frac{2}{2} = -1$
and $x = \frac{13-3y}{2} \Rightarrow x = \frac{13-3(1)}{2} = \frac{13-3}{2} = -\frac{10}{2} = -5$
Hence, the points are $A(-1,5)$ and $B(5,1)$
Length of chord $\overline{AB} = \sqrt{(-1-5)^2 + (5-1)^2}$
$= \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}$

Q#6) Find the coordinates of the points of intersection of the line $x + 2y = 6$ with the circle $x^2 + y^2 - 2x - 2y - 39 = 0$
Sol:
Circle: $x^2 + y^2 - 2x - 2y - 39 = 0$...(1)
And $x + 2y = 6 \Rightarrow x = 6 - 2y$
put in Eq(1)
$x^2 + y^2 - 2x - 2y - 39 = 0$
$\Rightarrow (6-2y)^2 + y^2 - 2(6-2y) - 2y - 39 = 0$
$\Rightarrow 36 + 4y^2 - 24y + y^2 - 12 + 4y - 2y - 39 = 0$
$\Rightarrow 5y^2 - 22y - 15 = 0$
$\Rightarrow 5y^2 - 25y + 3y - 15 = 0$
$\Rightarrow 5y(y-5) + 3(y-5) = 0$
$\Rightarrow (y-5)(5y+3) = 0$
$\Rightarrow y = 5, -\frac{3}{5}$ put in given st. line
$x = 6 - 2y \Rightarrow x = 6 - 2(5) = 6 - 10 = -4$
and $x = 6 - 2y \Rightarrow x = 6 - 2\left(-\frac{3}{5}\right) = \frac{30+6}{5} = \frac{36}{5}$
Thus, the points of intersection $P_1(-4,5)$ and $P_2(\frac{36}{5}, -\frac{3}{5})$.

Q#7) Find equation of the tangents to the circle $x^2 + y^2 = 2$
(i). Parallel to the line $x - 2y + 1 = 0$
Sol: (i)
Circle: $x^2 + y^2 = 2$
$\Rightarrow x^2 + y^2 = (\sqrt{2})^2$
Comparing it with $x^2 + y^2 = r^2$
$\Rightarrow r = \sqrt{2}$
Given the line $l_1: x - 2y + 1 = 0$
Slope=$m_1 = -\frac{1}{-2} = \frac{1}{2}$
Which is slope of tangent parallel to l_1.
Let required equation of the tangent is
$y = m_1 x \pm r\sqrt{1 + m_1^2}$
$= \left(\frac{1}{2}\right)x \pm \sqrt{2}\sqrt{1 + \left(\frac{1}{2}\right)^2}$
$= \frac{x}{2} \pm \sqrt{2}\sqrt{1 + \frac{1}{4}} = \frac{x}{2} \pm \sqrt{2\left(\frac{5}{4}\right)}$
$y = \frac{x}{2} \pm \sqrt{\frac{4}{2}}$ Multiply by 2
$\Rightarrow 2y = x \pm \sqrt{2}\sqrt{5}$
$\Rightarrow 2y = x \pm \sqrt{10}$
Hence $x - 2y + \sqrt{10} = 0$ and $x - 2y - \sqrt{10}$ are req. Tangent.
(ii).perpendicular to the line $3x + 2y = 6$
Sol: (ii)
Circle: $x^2 + y^2 = 2$
$\Rightarrow x^2 + y^2 = (\sqrt{2})^2$
Comparing it with $x^2 + y^2 = r^2$
$\Rightarrow r = \sqrt{2}$
Given the line $l_2: 3x + 2y - 6 = 0$
Slope=$m_2 = -\frac{3}{2} = \frac{-3}{2}$
Which is slope of tangent parallel to $l_1 = \frac{2}{3}$.
Let required equation of the tangent is
$y = m_2 x \pm r\sqrt{1 + m_1^2}$
$= \left(\frac{2}{3}\right)x \pm \sqrt{2}\sqrt{1 + \left(\frac{2}{3}\right)^2}$
$= \frac{2x}{3} \pm \sqrt{2}\sqrt{1 + \frac{4}{9}} = \frac{x}{2} \pm \sqrt{2\left(\frac{13}{9}\right)}$
$y = \frac{2x}{3} \pm \frac{\sqrt{26}}{3}$ Multiply by 3
$\Rightarrow 3y = 2x \pm \sqrt{26}$
$\Rightarrow 2x - 3y \pm \sqrt{26} = 0$
Hence $2x - 3y - \sqrt{26} = 0$ and $2x - 3y + \sqrt{26} = 0$
Are required tangent.
Question No.8 find equation of tangent drawn from
(i) $(0,5)$ to $x^2 + y^2 = 16$
Also find points of contact.
Solution:
Eq. of circle is $x^2 + y^2 = 16$
⇨ $x^2 + y^2 = (4)^2$
$a = 4$
Eq. of tangent is

$$y = mx + a\sqrt{1+m^2} \to (i)$$

At $(0,5) and\ a = 4$

$$5 = m(0) + 4\sqrt{1+m^2}$$

$$5 = 4\sqrt{1+m^2}$$

⇨ $25 = 16(1+m^2)$

⇨ $25 = 16 + 16m^2$

⇨ $16m^2 = 9$

⇨ Or$n^2 = \frac{9}{16}$

⇨ $m = \pm\frac{3}{4}$

$$Now\ (i) for\ m = \pm\frac{3}{4}\ and\ a = 4$$

$$y = \pm\frac{3}{4}x + 4\sqrt{1+\frac{9}{16}}$$

$$y = \pm\frac{3}{4} + 4\sqrt{\frac{25}{16}}$$

$$y = \pm\frac{3}{4}x + 4\left(\frac{5}{4}\right)$$

Or $y = \pm\frac{3}{4}x + 5$

⇨ $4y = \pm 3x + 20$

Or $\pm 3x - 4y + 20 = 0$

Hence $3x - 4y + 20\ and - 3x - 4y + 20 = 0\ or$

$3x + 4y - 20 = 0\ are\ required\ tangent.$

Points of contact:

Remember to find points of contact, solve of tangent and circle:

We solve $3x - 4y + 20 = 0$

And $x^2 + y^2 = 16\ for\ points\ of\ contact.$

Now $x^2 + y^2 = 16 \to (ii)$

$$3x - 4y + 20 = 0$$

⇨ $x = \frac{4y-20}{3}\ put\ in\ )ii)$

⇨ $\left(\frac{4y-20}{3}\right)^2 + y^2 = 16$

⇨ $\frac{16y^2+400-160y}{9} + y^2 = 16$

⇨ $16y^2 + 400 - 160y + 9y62 = 144$

⇨ $25y^2 - 8y - 80y + 256 = 0$

⇨ $5y(5y-16) - 16(5y-16) = 0$

⇨ $(5y-16)(5y-16) = 0$

⇨ $(5y-16)^2 = 0$

⇨ $5y - 16 = 0$

⇨ $5y = 16$

⇨ $y = \frac{16}{5}$

Now $x = \frac{4\left(\frac{16}{5}\right)-20}{3}$

$$= \frac{64-100}{15} = -\frac{36}{15}$$

$$x = -\frac{12}{5}$$

So point of contact is $\left(-\frac{12}{5}, \frac{16}{5}\right)$

Now we slove $3x + 4y - 20 = 0$

And $x^2 + y^2 = 16$

$$3x + 4y - 20 = 0$$

⇨ $x = \frac{20-4y}{3}n\ put\ in\ (ii)$

$$\left(\frac{20-4y}{3}\right)^2 + y^2 = 16$$

$$\frac{400 + 16y^2 - 160y}{9} + y^2 = 16$$

⇨ $400 + 16y^2 - 160y + 9y^2 = 144$

⇨ $25y^2 - 160y + 256 = 0$

⇨ $(5y-16)^2 = 0$

⇨ $5y - 16 = 0$

⇨ $y = \frac{16}{5}$

So $x = \frac{20-4\left(\frac{16}{5}\right)}{3} = \frac{20-\frac{64}{5}}{3}$

$$x = \frac{100-64}{15} = \frac{36}{15} = \frac{12}{5}$$

Also $\left(\frac{12}{5}, \frac{16}{5}\right)$ $is\ point\ of\ contact.$

(ii)

$$(-1, 2)\ to\ x^2 + y^2 + 4x + 2y = 0$$

Also find points of contact.

Solution:

$$x^2 + y^2 + 4x + 2y = 0 \to (i)$$

Compare with

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$2g = 4$$

$$g = 2$$

$$2f = 2 \Rightarrow f = 1, c = 0$$

$$r = a = \sqrt{g^2 + f^2 - c} = \sqrt{(2)^2 + (1)^2 - 0}$$

$$\Rightarrow r = a = \sqrt{5}$$

Centre $(-g, -f) = (-2, -1)$

Let

$m\ be\ slope\ of\ tangent\ drawn\ form\ point\ (-1,2)$
$to\ given\ circle\ so\ eq.of\ tangent\ os$

$$y - y_1 = m(x - x_1)$$

⇨ $y - 2 = m(x+1)$

⇨ $y - 2 = mx + m$

⇨ $mx - y + m + 2 = 0 \to (i)$

Distance of line from

Centre $(-2, -1) = r$

⇨ $\frac{|m(-2)-(-1)+m+2|}{\sqrt{m^2+1}} = \sqrt{5}$

⇨ $\frac{|-m+3|}{\sqrt{m^2+1}} = \sqrt{5}$

⇨ $\frac{m^2-6m+9}{m^2+1} = 5\quad (squaring)$

⇨ $5m^2 + 5 = m^2 - 6m + 9$

⇨ $5m^2 + 5 - m^2 + 6m - 9 = 0$

Or $4m^2 + 6m - 4 = 0$

$$4m^2 + 6m - 4 = 0$$
$$4m^2 + 8m - 2m - 4 = 0$$
$$4m(m+2) - 2(m+2) = 0$$
$$(m+2)(4m-2) = 0$$
$$m + 2 = 0 \quad or\ m = \frac{2}{4} = \frac{1}{2}$$

⇨ $m = -2\ or\ \ m = \frac{1}{2}$

When $m = -2\ \ so\ (i)$

$$-2x - y - 2 + 2 = 0$$

⇨ $-2x - y = 0$

⇨ $2x + y = 0$

When $m = \frac{1}{2}\ \ \ so\ (i)$

$$\frac{1}{2}x - y + \frac{1}{2} + 2 = 0$$

⇨ $x - 2y + 1 + 4 = 0$

Or$x - 2y + 5 = 0$

Hence req. tangent are

$$2x + y = 0\ and\ x - 2y + 5 = 0$$

Points of contact:

$$\boldsymbol{for}\ 2x + y = 0 \Rightarrow y = -2x\ put\ in\ (i)$$
$$x^2 + (-2x)^2 + 4x + 2(-2x) = 0$$
$$x^2 + 4x^2 + 4x - 4 = 0$$

⇨ $5x^2 = 0 \Rightarrow x = 0\ so\ y = -(0) \Rightarrow y = 0$

Thus $(0,0) is\ the\ point\ of\ contact.$

For $x - 2y + 5 = 0 \quad \Rightarrow x = 2y - 5\ put\ in\ (i)$

$$(2y-5)^2 + y^2 + 4(2y-5) + 2y = 0$$
$$4y^2 - 20y + 25 + y^2 + 8y - 20 + 2y = 0$$
$$5y^2 - 10y + 5 = 0$$

⇨ $y^2 - 2y + 1 = 0\ \Rightarrow (y-1)^2 = 0$

$$y - 1 = 0$$

⇨ $y - 1 = 0 \Rightarrow y = 1$

So $x = 2(1) - 5 \Rightarrow x = -3$

Thus $(-3,1) is\ point\ of\ contact.$

(iii) $\boldsymbol{(-7,-2) to\ (x+1)^2 + (y-2)^2 = 26\quad also}$

Find points of contact.

Solution:

$$(x+1)^2 + (y-2)^2 = 26 \rightarrow (i)$$

⇨ $\left(x - (-1)\right)^2 + (y-2)^2 = \left(\sqrt{26}\right)^2$

Compare with

$$(x-h)^2 + (y-k)^2 = r^2$$
$$centre\ (h,k) = (-1,2)$$
$$r = \sqrt{26}$$

Suppose $eq.of\ tangent$

$$y - y_1 = m(x - x_1)$$

⇨ $y + 2 = m(x+7) at\ (-7,-2)$

$$y + 2 = mx + 7m$$

Or $mx - y + 7m - 2 = 0 \rightarrow (ii)$

Distance of tangent from $(-1,2) = r$

⇨ $\frac{|m(-1)-2+7m-2|}{\sqrt{m^2+(-1)^2}} = \sqrt{26}$

⇨ $\frac{|6m-4|}{\sqrt{m^2+1}} = \sqrt{26}$

⇨ $\frac{26m^2+16-48m}{m^2+1} = 26$ (squaring)

⇨ $36m^2 + 16 - 48m = 26m^2 + 26$

⇨ $36m^2 - 26m^2 - 48m + 16 - 26 = 0$

Or $10m^2 - 48m - 10 = 0$

$$10m^2 - 50m + 2m - 10 = 0$$
$$10m(m-5) + 2(m-5) = 0$$
$$(10m+2)(m-5) = 0$$
$$m - 5 = 0 \quad 10m + 2 = 0$$
$$m = 5 \quad ; 10m = -2$$
$$\Rightarrow m = -\frac{2}{10} = -\frac{1}{5}$$

$when$ m=5 so (ii)

$$5x - y + 7(5) - 2 = 0$$

⇨ $5x - y + 35 - 2 = 0$

⇨ $5x - y + 33 = 0$

When $m = -\frac{1}{5}\quad so\ (ii)$

$$-\frac{1}{5}x - y + 7\left(-\frac{1}{5}\right) - 2 = 0$$
$$-\frac{1}{5}x - y - \frac{7}{5} - 2 = 0$$
$$\Rightarrow -x - 5y - 7 - 10 = 0$$
$$\Rightarrow -x - 5y - 17 = 0$$
$$or\ \ x + 5y + 17 = 0$$

$hence\ 5x - y + 33 = 0\ and\ x + 5y + 17 = 0$

are required tangents.

Points of contact:

For $x + 5y + 17 = 0$

$$\Rightarrow x = -5y - 17\ put\ in\ (i)$$
$$(-5y - 17 + 1)^2 + (y-2)^2 = 26$$
$$\Rightarrow (-5y - 16y)^2 + (y-2)^2 = 26$$
$$25y^2 + 160y + 256 + y^2 + 4 - 4y - 26 = 0$$
$$26y^2 + 156y + 234 = 0$$

⇨ $y^2 + 6y + 9 = 0\ \ (\div by\ 26)$

Or$(y+3)^2 = 0$

⇨ $y + 3 = 0\ \ \Rightarrow y = -33$

So $x = -5(-3) - 17$

$$x = 15 - 17 = -2$$

So $(-2,-3)\ is\ point\ of\ contact$

For $5x - y + 33 = 0$

⇨ $y = 5x + 33\ put\ in\ (i)$

$$(x+1)^2 + (5x + 33 - 2)^2 = 26$$
$$x^2 + 1 + 2x + (5x + 31)^2 = 26$$
$$x^2 + 1 + 2x + 25x^2 + 310x + 961 - 26 = 0$$
$$26x^2 + 312x + 936 = 0$$

⇨ $x^2 + 12x + 36 = 0\ \ (\div by\ 26)$

Or $(x+6)^2 = 0 \Rightarrow x+6 = 0$

$$x = -6 \quad so\ y = 5(-6) + 33$$

$$\Rightarrow y = -30 + 33 = 3$$

$$so\ (-6,3) is\ point\ of\ contact.$$

Q#9) Find an equation of the chord of contact of the tangent drawn from $(4,5)$ to the circle $2x^2 + 2y^2 - 8x + 12y + 21 = 0$.

Sol:

Circle: $2x^2 + 2y^2 - 8x + 12y + 21 = 0$

$\Rightarrow x^2 + y^2 - 4x + 6y + \frac{21}{2} = 0.$

Comparing it with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow 2g = -4 \Rightarrow g = -2$ also $\Rightarrow 2f = 6 \Rightarrow f = 3$

and $c = \frac{21}{2}$

Let $A(x_1, y_1)$ *and* $B(x_2, y_2)$ be the points of contact of two tangents. Equation of the tangent at $A(x_1, y_1)$

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

$\Rightarrow\ xx_1 + yy_1 - 2(x + x_1) + 3(y + y_1) + \frac{21}{2} = 0$

Since it passes through $(4,5)$

$\Rightarrow\ 4x_1 + 5y_1 - 2(4 + x_1) + 3(5 + y_1) + \frac{21}{2} = 0$

$\Rightarrow\ 4x_1 + 5y_1 - 8 - 2x_1 + 15 + 3y_1 + \frac{21}{2} = 0$

$\Rightarrow\ 2x_1 + 8y_1 + 7 + \frac{21}{2} = 0$

$\Rightarrow\ 4x_1 + 16y_1 + 14 + 21 = 0$

$\Rightarrow\ 4x_1 + 16y_1 + 14 + 21 = 0$

$\Rightarrow\ 4x_1 + 16y_1 + 35 = 0 \rightarrow 1)$

In similar fashion, we equation of the tangent at $B(x_2, y_2)$

$\Rightarrow\ 4x_2 + 16y_2 + 35 = 0 \rightarrow (2)$

Equation (1) and (2) represent that $A(x_1, y_1)$ *and* $B(x_2, y_2)$ lie on

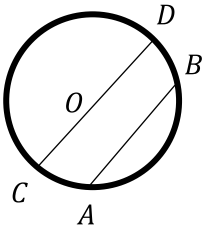
$\Rightarrow\ 4x + 16y + 35 = 0$

Which is the required equation of line.

Analytic Proofs of Important Properties of a circle:

"Chord of a Circle:

A line segment where end points lie on a circle is called chord of a circle. In figure AB is chord.



Diameter of a circle:

A chord passing through Centre of a circle. In figure , if O is Centre the CD is diameter of circle.

Theorem 1

Length of a diameter of the circle $x^2 + y^2 = a^2$ is $2a$

Proof:

Given equation of circle is $x^2 + y^2 = a^2$

Centre $= O(0,0)$

Radius $= r = a$

Let

$A(a,0) and\ B(-a,o) be\ ends\ of\ diameter\ AB\ then$

Length of diameter

$$= |AB| = \sqrt{(a-(-a)^2 + (0-0)^2}$$

$$|AB| = \sqrt{(2a)^2}$$

$$= 2a$$

Hence Proved.

Theorem:

Perpendicular dropped from the Centre of a circle on a chord bisects the chord.

Proof:

Consider $eq. of\ circle\quad x^2 + y^2 = a^2 \rightarrow (i)$

Let

$A(x_1, y_1) and\ B(x_2, y_2) be\ ends\ of\ chord\ AB.$

$\because A(x_1, y_1) and\ B(x_2, y_2) lie\ on\ the\ circle\ (i) so$

$$x_1^2 + y_1^2 = a^2 \rightarrow (ii) and$$

$$x_2^2 + y_2^2 = a^2 \rightarrow (iii)$$

Let we draw a $\perp$

$ar\ OC\ from\ centre\ O\ on\ the\ chord\ AB$

$$Slope\ of\ chord\ AB = \frac{y_2 - y_1}{x_2 - x_1}$$

$$slope\ of\ OC = -\left(\frac{x_2 - x_1}{y_2 - y_1}\right)$$

$$\because \overline{OC} \perp \overline{AB}$$

$\because line\ passing\ through\ (x_1, y_1) and\ having\ slope\ m$

Eq. of OC is

$$y - 0 = -\left(\frac{x_2 - x_1}{y_2 - y_1}\right)(x - 0)$$

$\Rightarrow\ y(y_2 - y_1) = -x(x_2 - x_1)$

$$or\ (x_2 - x_1)x + (y_2 - y_1)y = 0 \rightarrow (iv)$$

Midpoint of chord AB is

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ Now OC will bisect the chord AB

$if\ \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) satisfies. its\ equation$

$$i.e; put\ x = \frac{x_1 + x_2}{2}, y = \frac{y_1 + y_2}{2}\ in\ (iv)$$

We have

$$(x_2 - x_1)\left(\frac{x_2 + x_1}{2}\right) + (y_2 - y_1)\left(\frac{y_1 + y_2}{2}\right) = 0$$

⇨ $(x_2-x_1)(x_2+x_1)+(y_2-y_1)(y_2+y_1)=0$

⇨ $x_2^2-x_1^2+y_2^2-y_1^2=0$

⇨ $x_2^2+y_2^2=x_1^2+y_1^2$

⇨ $a^2=a^2$ *by (ii) and (iii) Hence ⊥ ar dropped from centre of a circle on a chord bisect the chord*

Theorem3.

The perpendicular bisector of any chord of a circle passes through the Centre of the circle.

Proof:

Consider equation of circle is

$$x^2+y^2=a^2 \rightarrow (i)$$

Let

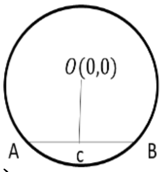
$A(x_1,y_1)$ *and* $B(x_2,y_2)$ *be ends of chord AB.*

∵ *the points A and B lie on the circle (i) then*

$$x_1^2+y_1^2=a^2 \rightarrow (ii) \quad and$$

$$x_2^2+y_2^2=a^2 \rightarrow (iii)$$

let C *be midpoint of chord AB so*



Coordinates of C *are* $C\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

Slope of chord AB=$\frac{y_2-y_1}{x_2-x_1}$

Slope of ⊥ *ar bisector AB* $=-\left(\frac{x_2-x_1}{y_2-y_1}\right)$

Using $y-y_1=m(x-x_1)$

⇨ $y-\left(\frac{y_1+y_2}{2}\right)=-\left(\frac{x_2-x_1}{y_2-y_1}\right)\left(x-\frac{x_1+x_2}{2}\right)\rightarrow (iv)$

This perpendicular bisector will pass through Centre of (0,0) satisfies eq. (iv) for this

$$put\ x=0\ ,y=o\ in\ (iv)$$

$0-\left(\frac{y_1+y_2}{2}\right)=-\left(\frac{x_2-x_1}{y_2-y_1}\right)\left(0-\frac{x_1+x_2}{2}\right)$

⇨ $-\left(\frac{y_1+y_2}{2}\right)=\left(\frac{x_2-x_1}{y_2-y_1}\right)\left(\frac{x_1+x_2}{2}\right)$

⇨ $-(y_2+y_1)(y_2-y_1)=(x_2-x_1)(x_2+x_1)$

⇨ $-(y_2^2-y_1^2)=x_2^2-x_1^2$

⇨ $x_1^2+y_1^2=x_2^2+y_2^2$

$$a^2=a^2 \quad (by\ (ii) and\ (iii)$$

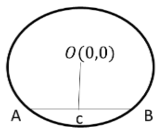
thus

⊥ *ar bisector of any chord of a circle passes* Through Centre of the circle.

Theorem:4

The line joining the Centre of a circle to the mid-point of a chord is perpendicular to the chord.

Proof:



Consider eq. of circle.

$$\boldsymbol{x^2+y^2=a^2\rightarrow (i)}$$

Let

$A(x_1,y_1)$ *and* $B(x_2,y_2)$ *be ends of chord AB.*

∵ *the points A and B lie on the circle (i) then*

$$x_1^2+y_1^2=a^2 \rightarrow (ii) \quad and$$

$$x_2^2+y_2^2=a^2 \rightarrow (iii)$$

let C *be midpoint of chord AB so*

Coordinates of C *are* $C\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

Slope of chord AB=$\frac{y_2-y_1}{x_2-x_1}$

Here OC is line joining Centre O of circle to midpoint C of chord AB then

Slope of OC=$\frac{\frac{y_1+y_2}{2}-o}{\frac{x_1+x_2}{2}-0}=\frac{y_1+y_2}{x_1+x_2}$

$$(slope\ of\ OC)(slope\ of\ chord\ AB)$$

$$=\left(\frac{y_1+y_2}{x_1+x_2}\right)\left(\frac{y_2-y_1}{x_2-x_1}\right)$$

$$=\frac{(y_2+y_1)(y_2-y_1)}{(x_2+x_1)(x_2-x_1)}$$

$$=\frac{y_2^2-y_1^2}{x_2^2-x_1^2}$$

By $(iii)-(ii)$

$$x_2^2-x_1^2+y_2^2-y_1^2=a^2-a^2$$

⇨ $y_2^2-y_1^2=-(x_2^2-x_1^2)$

Now

$$(slope\ of\ OC)(slope\ of\ chord\ AB)$$

$$=-\frac{(x_2^2-x_1^2)}{x_2^2-x_1^2}=-1$$

⇨ $OC\perp AB$

Hence line joining the Centre of a circle to the midpoint of chord is perpendicular to chord.

Congruent Chords:

“two chords are congruent if they are equal in length.”

Theorem:5

Congruent chords of a circle are equidistance from the Centre.

Proof:

Consider eq. of circle

Consider eq. of circle $x^2+y^2=a^2\rightarrow (i)$

Let AB and CD be two congruent chords.

Let coordinates of the points be

$A(x_1,y_1), B(x_2,y_2)$,

$C(x_3,y_3)$ *and D. Sine these points lie on the circle (i) so*

$$x_1^2+y_1^2=a^2\rightarrow (ii) \quad x_2^2+y_2^2=a^2\rightarrow (iii)$$

$$x_3^2+y_3^2=a^2\rightarrow (iv) \quad x_4^2+y_4^2=a^2\rightarrow (v)$$

For *congruent chords*

So $|AB|=|CD|$

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(x_4-x_3)^2+(y_4-y_3)^2}$$

Squaring both sides

$\Rightarrow\ (x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_4 - x_3)^2 + (y_4 - y_3)^2$

$$x_2^2 + x_1^2 - 2x_1x_2 + y_2^2 + y_1^2 - 2y_1y_2$$
$$= x_4^2 + x_3^2 - 2x_3^2 - 2x_3x_4 + y_4^2 + y_3^2 - 2y_3y_4$$

from $(ii), (iii), (iv)$, and (v)

$\Rightarrow\ a^2 + a^2 - 2(x_1x_2 + y_1y_2) = a^2 + a^2 - 2(x_3x_4 + y_3y_4)$

$\Rightarrow\ -2(x_1x_2 + y_1y_2) = -2(x_3x_4 + y_3y_4)$

$$or\ x_1x_2 + y_1y_2 = x_3x_4 + y_3y_4 \rightarrow (vi)$$

Let us draw ⊥ *ars OM and ON from centre O on the chords AB and CD resp. Now by property of circle, M and N will be midpoints of chords AB And CD respectively Now*

Coordinates of M are $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Also coordinates of N are $N\left(\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}\right)$

We are to prove that

$$|OM| = |ON|$$

$\Rightarrow\ |OM|^2 = |ON|^2$

$$|OM|^2 = \left(\frac{x_1 + x_2}{2}\right)^2 + \left(\frac{y_1 + y_2}{2}\right)^2$$
$$= \frac{x_1^2 + x_2^2 + 2x_1x_2}{4} + \frac{y_1^2 + y_2^2 + 2y_1y_2}{4}$$
$$= \frac{x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2(x_1x_2 + y_1y_2)}{4}$$
$$= \frac{a^2 + a^2 + 2(x_1x_2 + y_1y_2)}{4}\ by\ (ii) and\ (iii)$$
$$|OM|^2 = \frac{2a^2 + 2(x_1x_2 + y_1y_2)}{4}$$

Also

$$|ON|^2 = \left(\frac{x_3 + x_4}{2}\right)^2 + \left(\frac{y_3 + y_4}{2}\right)^2$$
$$= \frac{x_3^2 + x_4^2 + 2x_3x_4}{4} + \frac{y_3^2 + y_4^2 + 2y_3y_4}{4}$$
$$= \frac{x_3^2 + x_4^2 + y_3^2 + y_4^2 + 2(x_3x_4 + y_3y_4)}{4}$$
$$= \frac{a^2 + a^2 + 2(x_3x_4 + y_3y_4)}{4}$$
$$|ON|^2 = \frac{2a^2 + 2(x_1x_2 + y_1y_2)}{4}$$

By $eq\ (vi)$

$$|ON|^2 = \frac{2a^2 + 2(x_1x_2 + y_1y_2)}{4}$$

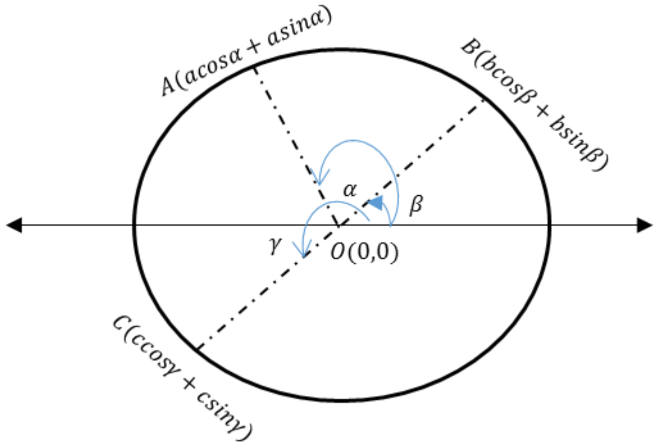
Hence we conclude that $|OM|^2 = |ON|^2$

Thus congruent chords of a circle are equidistant from Centre.

Remember from fig.

Ab is minor arc ACB is major are O(o,o) is Centre of circle OA,OB,OC are radii,

$x^2 + y^2 = a^2 \quad and\ |OA| = |OB| = |OC| = a$



From fig $|OA| = |OB| = |OC| = a$

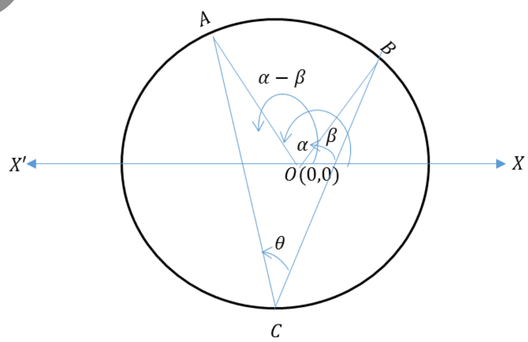
Let $m\angle XOA = \alpha, m\angle XOB = B, m\angle XOC = r$

Coordinates of points are $A(acos\alpha, asin\alpha)$, $B(acos\beta, asin\beta)$ and $C(acos\gamma, asin\gamma)$

Theorem 6:

Show that the measure of the central angle of a minor arc is double the measure of the angle in the corresponding major arc.

Proof:



Consider eq. of circle

$$x^2 + y^2 = a^2 \rightarrow (i)$$
$$where\ a\ is\ radius$$

Let AB be minor arc and ACB be major arc. Let $m\angle XOA = \alpha, m\angle XOB = \beta, and\ m\angle XOC = \gamma$

Then $m\angle BOA = \alpha - \beta\ (angle\ subtended\ by\ minor\ arc\ AB)$

Let $m\angle BCA = \theta = (angle\ subtended\ by\ major\ arc\ ACB)$

We are to prove that $\alpha - \beta = 2\theta\ now$

$$A(acos\alpha, asin\alpha)$$
$$B(acos\beta, asin\beta)$$
$$C(acos\gamma, asin\gamma)$$

Let $m_1 = slope\ of\ AC$

$m_2 = slope\ of\ BC$

$$\because m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_1 = slope\ of\ AC = \frac{asin\gamma - asin\alpha}{acos\gamma - acos\alpha}$$

$$m_1 = \frac{a(sin\gamma - sin\alpha)}{a(cos\gamma - cos\alpha)}$$

$$m_1 = \frac{2\cos\frac{\gamma+\alpha}{2}\sin\frac{\gamma-\alpha}{2}}{-2\sin\frac{\gamma+\alpha}{2}\sin\frac{\gamma-\alpha}{2}}$$

$$m_1 = -\cot\left(\frac{\gamma+\alpha}{2}\right)$$

$$\because \tan\left(\frac{\pi}{2}+\theta\right) = -cot\theta$$

$$m_1 = Tan\left(\frac{\pi}{2}+\frac{\gamma+\alpha}{2}\right)$$

$$m_1 = Tan\left(\frac{\pi+\gamma+\alpha}{2}\right)$$

Similarly,

$$m_2 = slope\ of\ BC = Tan\left(\frac{\pi+\gamma+\beta}{2}\right)$$

$\because\ \theta\ is\ angle\ from\ BC\ to\ AC\ so,$

$$Tan\theta = \frac{Tan\left(\frac{\pi+\gamma+\alpha}{2}\right) - Tan\left(\frac{\pi+\gamma+\beta}{2}\right)}{1+Tan\left(\frac{\pi+\gamma+\alpha}{2}\right)Tan\left(\frac{\pi+\gamma+\beta}{2}\right)}$$

$$\because \frac{Tan\alpha - tan\beta}{1+tan\alpha Tan\beta} = Tan(\alpha-\beta)$$

⇨ $Tan\theta = Tan\left(\frac{\pi+\gamma+\alpha}{2} - \frac{\pi+\gamma+\beta}{2}\right)$

⇨ $Tan\left(\frac{\pi+\gamma+\alpha-\pi-\gamma-\beta}{2}\right)$

⇨ $Tan\theta = Tan\left(\frac{\alpha-\beta}{2}\right)$

Or $\theta = \frac{\alpha-\beta}{2}$

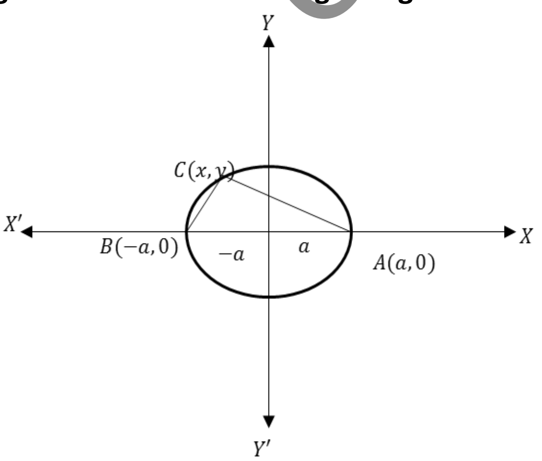
⇨ $\alpha - \beta = 2\theta$

Hence prove.

Theorem 7.

An angle in a semi-circle is a right angle.

Proof:



Consider eq. of circle $x^2 + y^2 = a^2$ $with\ centre\ O(0,0) and\ radius\ a. let\ A(a,0) and\ B(-a,0) be\ ends\ of\ its\ diameter. let\ C(x,y) be\ any$ Point on the circle. We are to prove that

$$AC \perp BC$$

i.e;

$$(slope\ of\ AC)(slope\ of BC) = -1$$

Now

$$slope\ of\ AC = \frac{y-o}{x-a} = \frac{y}{x-a}$$

$$slope\ of\ BC = \frac{y-0}{x-(-a)} = \frac{y}{x+a}$$

So

$$(slope\ of\ AC)(Slope\ of\ BC) = \left(\frac{y}{x-a}\right)\left(\frac{y}{x+a}\right)$$

$$= \frac{y^2}{x^2-a^2}$$

$$= -\frac{y^2}{y^2} = -1$$

$hence\ proved$

Theorem 8:

The tangent to a circle at any point of the circle is perpendicular to the radial segment at that point.

Proof:

Consider eq. of circle

$x^2 + y2 = a^2 \to (i)$

Let the point $P(x_1, y_1) be\ point\ of\ tangency.$

$\because eq. of\ tangent\ to\ the\ circle\ (i) at\ P(x_1, y_1) is$

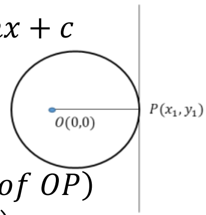
$$xx_1 + yy_1 = a^2$$

⇨ $yy_1 = -xx_1 + a^2$

⇨ $y = \left(-\frac{x_1}{y_1}\right)x + \frac{a^2}{y_1}$

$compare\ with\ y = mx + c$

Slope of tangent line= $-\frac{x_1}{y_1}$



Slope of radial segment $OP = \frac{y_1}{x_1}$

$$(slope\ of\ tangent)(slope\ of\ OP)$$

$$= \left(-\frac{x_1}{y_!}\right)\left(\frac{y_1}{x_1}\right) = -1$$

This show that tangent line to a circle at any point of the circle is perpendicular to the radial segment at that point.

Theorem 9:

The perpendicular at the outer end of a radial segment is tangent to the circle.

Proof:

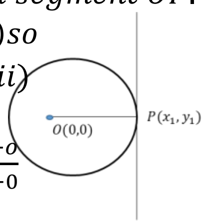
Consider eq. of circle $x^2 + y^2 = a^2 \to (i)$

Let

$P(x_1, y_1) be\ outer\ end\ of\ radial\ segment\ OP.$

As $P(x_1, y_1) lies\ on\ the\ circle\ (i) so$

$$x_1^2 + y_1^2 = a^2 \to (ii)$$



Slope of radial segment $OP = \frac{y_1 - 0}{x_1 - 0}$

$$= \frac{y_1}{x_1}$$

Slope of line ⊥ *ar to radial segment and passing through* $P(x_1, y_1)$ *is*

$$y - y_1 = \frac{-x_1}{y_1}(x - x_1) \quad (\because y - y_1 = m(x - x_1))$$

⇨ $yy_1 - y_1^2 = -xx_1 + x_1^2$

⇨ $xx_1 + yy_1 = x_1^2 + y_1^2$

⇨ $xx_1 + yy_1 = a^2$ *by (ii)*

Which represent eq. of tangent to the circle at point $P(x_1, y_1)$

Exercise 6.3

Q#1) Prove that normal lines of a circle pass through the Centre of circle.

Sol: Consider a circle $x^2 + y^2 = r^2 \rightarrow (1)$

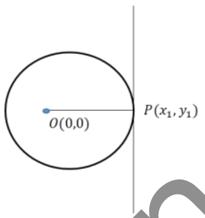
Centre$=O(0,\ 0)$

Radius$=r$

Let $P(x_1, y_2)$ be any point on the circle

$\Rightarrow x_1^2 + y_2^2 = \text{r}^2$

Differentiate (1) w.r.t x



$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(r^2)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$\Rightarrow 2y\frac{dy}{dx} = -2x$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

Slope of tangent at $P = \frac{dy}{dx}|_P = -\frac{x_1}{y_1}$

Slope of Normal at $P = m = \frac{y_1}{x_1}$

Equation of Normal through $P(x_1, y_1)$ is

$(y - y_1) = m(x - x_1)$

$\Rightarrow (y - y_1) = \frac{y_1}{x_1}(x - x_1)$

$\Rightarrow x_1y - x_1y_1 = xy_1 - x_1y_1$

$\Rightarrow x_1y = xy_1$

At $O(0,\ 0)$

Put $x = 0$ *and* $y = 0$

$\Rightarrow x_1y = xy_1 \Rightarrow x_1(0) = (0)y_1$

$\Rightarrow 0 = 0$

Hence normal line pass through Centre.

Q#2) Prove that the straight line drawn from the Centre of a circle perpendicular to a tangent passes through the point of tangency.

Sol: Consider a circle $x^2 + y^2 = \text{r}^2 \rightarrow (1)$

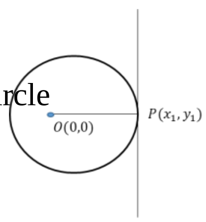
Centre$=O(0,\ 0)$

Radius$=r$

Let $P(x_1, y_2)$ be any point on the circle

$\Rightarrow x_1^2 + y_2^2 = \text{r}^2$

Differentiate (1) w.r.t x



$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(r^2)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$\Rightarrow 2y\frac{dy}{dx} = -2x$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

Slope of tangent at $P = \frac{dy}{dx}|_P = -\frac{x_1}{y_1}$

Let l be the line through "O" and ⊥ to the tangent at P, so

Slope of line $l = m = \frac{y_1}{x_1}$

Equation of l $O(0,\ 0)$ is

$(y - y_1) = m(x - x_1)$

$\Rightarrow (y - 0) = \frac{y_1}{x_1}(x - 0)$

$\Rightarrow x_1y = xy_1$

At $P(x_1, y_1)$

Put $x = x_1$ *and* $y = y_1$

$\Rightarrow x_1y = xy_1 \Rightarrow x_1(y_1) = (x_1)y_1$

$\Rightarrow x_1y_1 = x_1y_1$

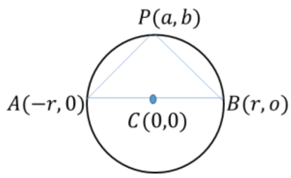
Thus, the ⊥ *ar* line "l" passes through the point of tangency $P(x_1, y_1)$.

Q#3) Prove that the midpoint of the hypotenuse of a right triangle is the circumcenter of the triangle.

Sol:

$$let\ x^2 + y^2 = r^2 \rightarrow (i)$$

Is eq. of circle with Centre $O(0,0)$ *and* $P(a, b)$ *any* Point lies on (i)



So,

$$a^2 + b^2 = r^2 \rightarrow (ii)$$

$|OA| = \sqrt{(-r - 0)^2 + (0 - 0)^2} = r$

$|OB| = \sqrt{(r - o)^2 + (0 - 0)^2} = r$

$|OP| = \sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$

$= \sqrt{r^2} = r$ *by (ii)*

⇨ $|OA| = |OB| =$ $|OP|$ *and O is circumcentre*

Hence Proved.

Q#4) Prove that the perpendicular dropped from a point of a circle on a diameter is a mean proportional between the segments into which it divides the diameter.

Sol: Consider a circle $x^2 + y^2 = \text{r}^2$

$P(a, b)$ *lies on it. so* $a^2 + b^2 = r^2$

⇨ $b^2 = r^2 - a^2 \rightarrow (i)$

Now

$$|PQ| = \sqrt{(a - a)^2 + (b - 0)^2}$$

$$= b$$
$$|AQ| = |AO| + |OQ| = r + a$$
$$|BQ| = |OB| - |OQ| = r - a$$
$$|AQ|.|QB| = (r+a)(r-a) = r^2 - a^2$$
$$|AQ|.|QB| = b^2 \quad by\ (i)$$

$$|AQ|.|QB| = |PQ|^2$$

⇨ $|AQ|.|QB| = |PQ||PQ|$

Hence proved.

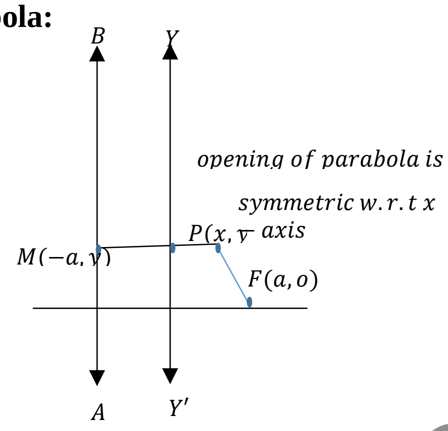
Parabola:

Set of all points which are equidistance from a fixed point and a fixed line.

- Fixed point is called focus of parabola.
- Fixed line is called Directrix of parabola.

Equation of parabola:

$y^2 = 4ac$



Let AB be a fixed line called directrix drawn parallel to y- axis such that its equation is $x = -a\ or\ x + a = 0$

Let $P(x, y)$ be any point on the parabola. Let $M(-a, y) be\ point\ on\ the\ direcrrix\ AB.$

By def.

$$\frac{|FP|}{|PM|} = 1 \Rightarrow |FP| = |PM|$$

⇨ $\sqrt{(x-a)^2 + (y-0)^2} = \sqrt{x-(-a)^2 + (y-y)^2}$

⇨ $(x-a)^2 + y^2 = (x+a)^2 \quad (by\ squaring)$

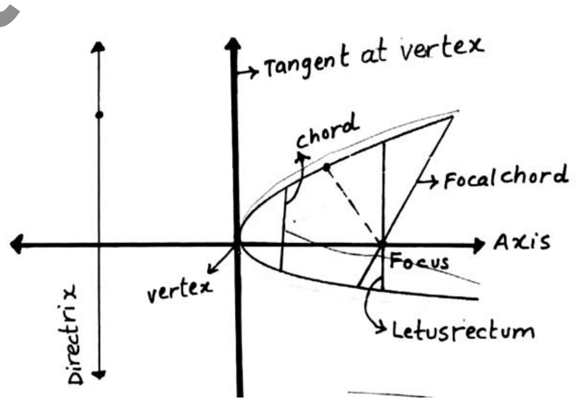
⇨ $x^2 + a^2 - 2ax + y^2 = x^2 + a^2 + 2ax$

Or $y^2 = 4ax$

This is req. eq. of parabola.

Definitions:

1) The line through the focus and perpendicular to the *directrix* is called **axis of parabola.**
2) The point where the axis meets the parabola is called **vertex** of parabola.
3) In parabola, fixed point not on the *directrix* is called **focus of parabola.**
4) In parabola, fixed line is called ***directrix* of parabola.**
5) A line passing through vertex and perpendicular to axis of parabola **is called tangent at vertex of parabola.**
6) A line whose points lie on parabola is called chord of parabola.
7) A chord passing through focus of parabola is called **focal chord.**
8) A focal chord perpendicular to axis of parabola is called **Let rectum.**
9) If $|FP|\ is\ distance\ between$ focus F and point P on parabola and $|PM| is \perp ar\ dis.$ of point p from $directrix$ of parabola then $\frac{|FP|}{|PM|}$ Is called **eccentricity. Denoted by e. for parabola $e = 1$**
10) If the point $(at^2, 2at) lies$ on the parabola $y^2 = 4ax\ for\ any\ t \in R\ then\ x = at^2,$ $y = 2at\ are\ called$ ***parametric equation of parabola.***



General form of equation of parabola.

$let\ P(x, y) be\ any\ point\ on\ the\ parabola\ having\ F(h, k) as\ focus\ and\ M\ be\ point\ on\ the\ directrix$ $lx + my + c = 0\ by\ def; eq\ of\ parabola\ is$

$$\sqrt{(x-h)^2 + (y-k)^2} = \frac{|lx + my + n|}{\sqrt{l^2 + m^2}}$$

$$or\ \ (x-h)^2 + (y-k)^2 = \frac{(lx + my + n)^2}{l^2 + m^2}$$

Remember that

Every second degree equation of the form

$$ax^2 + by^2 + 2gx + 2fy + c = 0$$

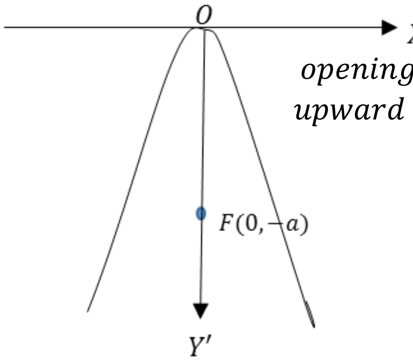
Will represented parabola if either $a = 0\ or\ b - o\ but$

Both $are\ not\ zero\ simultaneously.$

Other standard parabolas:

1. If the focus lies on the $y-axis\ with$ $coordinates\ F(0,a) and\ directrix\ of\ the\ parabola$ is $y=-a\ then\ eq. of\ parabola\ is\ x^2=4ay$.

The graph is shown as;

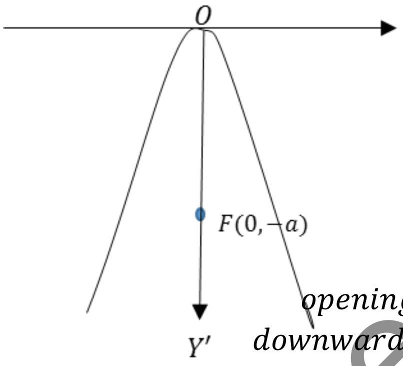


$opening\ of\ parabola\ is\ upward\ and\ symmetric\ w.r.t\ y-axis.$

2. If the focus is $F(o,-a)\ and\ directrix\ is\ the$

Line $y=a\ the\ eq. of\ parabola\ is\ x^2=-4ay$

The graph is shown as,



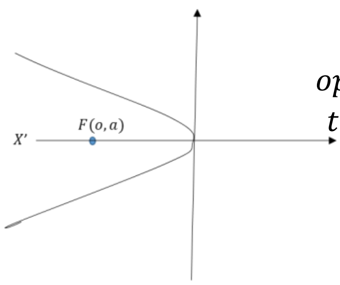
$opening\ of\ parabola\ is\ downward\ and\ symmetric\ w.r.t\ y-axis.$

3.

If the focus of parabola is $F(-a,o) and\ is\ directrix$

Is the line $x=a, then\ eq. of\ parabola\ is\ y^2=-4ax$

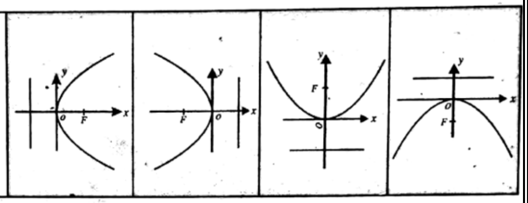
The graph is shown as;



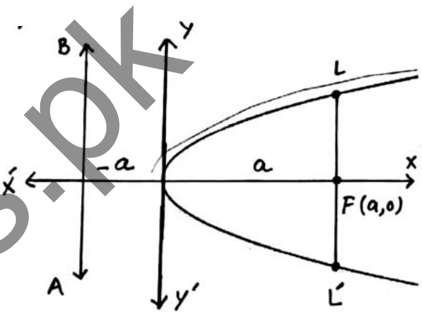
$opening\ of\ parabola\ is\ to\ left\ and\ symmetric\ w.r.t\ x-axis.$

Summary of Standard Parabolas

Sr.No.	1	2	3	4
Equation	$y^2=4ax$	$y^2=-4ax$	$x^2=4ay$	$x^2=-4ay$
Focus	$(a,0)$	$(-a,0)$	$(0,a)$	$(0,-a)$
Directrix	$x=-a$	$x=a$	$y=-a$	$y=a$
Vertex	(0,0)	(0,0)	(0,0)	(0,0)
Axis	$y=0$	$y=0$	$x=0$	$x=0$
Latusrectum	$x=a$	$x=-a$	$y=a$	$y=-a$
Graph				



Prove that length of $letusrectum$ of parabola is 4a.



Consider eq. of parabola is $y^2=4ax\rightarrow(i)$

Let $LL' be\ focal\ chord\ \perp$ $to\ axis\ of\ parabola\ is\ call\ ed\ letusrecum. let\ x=a\ be\ eq. of\ letusretum. LL'$

Put $x=a\ in\ (i)$

⇨ $y^2=4a(a)\Rightarrow y^2=4a^2$

$or\ y=\pm 2a\ \Rightarrow y=2a\ or\ y=-2a$

$so\ coordinates\ of\ L\ and\ L' are\ L(a,2a) and\ L'(a,-2a)$

$so\ coordinates\ of\ L\ and\ L' are\ L(a,2a) and\ L'(a,-2a)$

Now,

Length $of\ letusretum=|LL'|=$

$\sqrt{(a-a)^2+(-2a-2a)^2}=\sqrt{(-4a)^2}=\sqrt{16a^2}$

$=4a$

Hence proved.

Theorem:

The point of a parabola which is close to the focus is the vertex

Proof:

Consider eq. of parabola $y^2=4ax\rightarrow(i)\ ,a>0$

Let

$F(a,o) be\ focus\ and\ P(x,y) be\ any\ point\ on\ the$ Parabola, Now

$$|PF|=\sqrt{(x-a)^2+(y-0)^2}$$

$$=\sqrt{x^2+a^2-2ax+y^2}$$

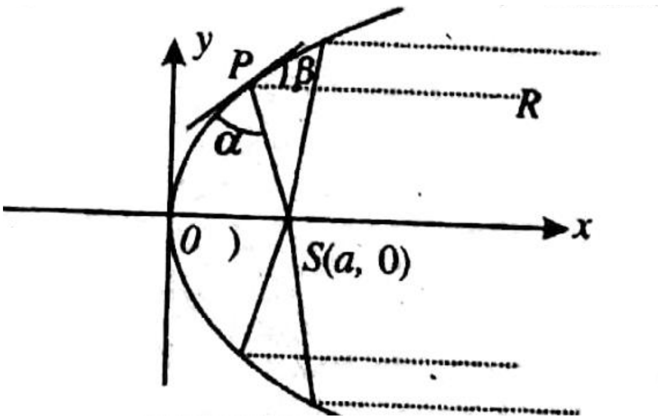
$$\sqrt{x^2+a^2-2ax+4ax} \quad \because y^2=4ax$$
$$=\sqrt{x^2+a^2+2ax}$$
$$=\sqrt{(x+a)^2}$$
$$|PF|=x+a \quad ,a>0,x>0$$

If $V(0,0) is\ vertex\ then$

$$|VF|=\sqrt{(a,o)^2+(0,0)^2}=\sqrt{a^2}=a$$
$$|VF|<|FP|$$
$$\Rightarrow distance\ b\backslash w\ focus, Vertex$$
$$< Distance\ b\backslash w\ Focus, point$$

hence the point of parabola which is closest to the focus is vertex of parabola.

Reflecting property of parabola:



When a light source is placed at the focus of a parabolic reflecting surface, then light ray travelling from focus S to a point P on parabola will be reflected in the direction PR which is parabola to the axis of the parabola shown in figure.

Exercise 6.4

Question.1.

Find the focus, vertex and directrix of the parabola sketch its graph

(i). $y^2=8x$

Solution.

Given Parabola

$$y^2=8x$$
$$Here \quad 4a=8$$
$$a=2$$
$$Vertex: V(0,0)$$

The axis of the parabola is along $x-axis$ and opening of parabola is to the right side.

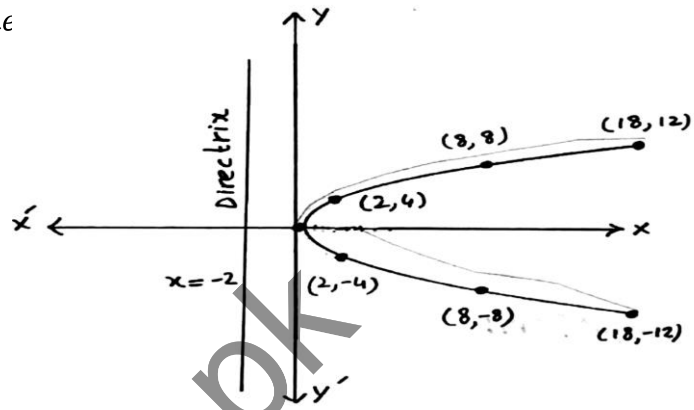
$$Focus: F(a,0)=F(2,0)$$
$$Directrix: \quad x+a=0$$
$$x+2=0 \Rightarrow x=-2$$

Sketch:

$$\because y^2=4ax$$

$$\Rightarrow \quad y=\pm\sqrt{8y}, D_f=[0,\infty)$$

x	$y=\pm\sqrt{8x}$	Points
0	$y=\pm\sqrt{8(0)}=0$	(0,0)
2	$y=\pm\sqrt{8(2)}=\pm 4$	(2,4), (2,−4)
8	$y=\pm\sqrt{8(8)}=\pm 8$	(8,8), (8,−8)
18	$y=\pm\sqrt{8(18)}=\pm\sqrt{144}=\pm 12$	(18,12), (18,−12)



(16y

Solution.

Given Parabola

$$x^2=-16y$$
$$Here \quad 4a=16$$
$$a=4$$
$$Vertex: V(0,0)$$

The axis of the parabola is along $y-axis$ and opening of parabola is to the downward side.

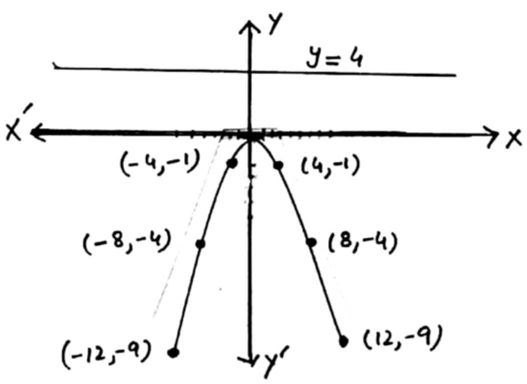
$$Focus: F(0,-a)=F(0,-4)$$
$$Directrix: \quad y-a=0$$
$$y-4=0 \Rightarrow y=4$$

Sketch:

$$\because x^2=-16y$$

$$\Rightarrow \quad x=\pm\sqrt{-16y}, D_f=(-\infty,-1]$$

y	$x=\pm\sqrt{-16y}$	Points
−1	$x=\pm\sqrt{-16(-1)}=\pm\sqrt{16}=\pm 4$	(4,−1), (−4,−1)
0	$x=\pm\sqrt{-16(0)}=0$	(0,0)
−4	$x=\pm\sqrt{-16(-4)}=\pm\sqrt{64}=\pm 8$	(8,−4), (−8,−4)
−9	$x=\pm\sqrt{-16(-9)}=\pm\sqrt{144}=\pm 12$	(12,−9), (−12,−9)



(iii). $x^2 = 5y$

Solution.

Given Parabola

$$x^2 = 5y$$
$$Here \quad 4a = 5$$
$$a = \frac{5}{4}$$
$$Vertex: V(0,0)$$

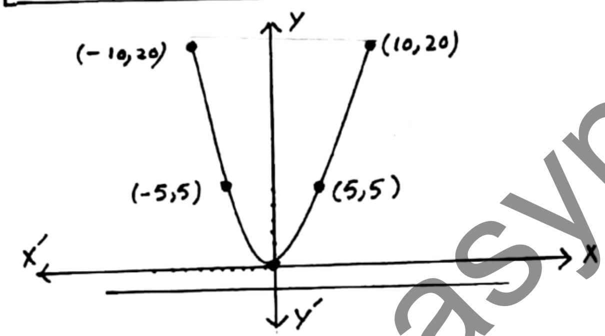
The axis of the parabola is along $y - axis$ and opening of parabola is to the upward side.

$$Focus : F(0\,,a) = F\left(0\,,\frac{5}{4}\right)$$
$$Directrix : \quad x + a = 0$$
$$x + \frac{5}{4} = 0.$$

Sketch:

$$x^2 = 5y \Rightarrow x = \pm\sqrt{5y}$$
$$D_f = [o, +\infty)$$

y	$x = \pm\sqrt{5y}$	Points
0	$x = \pm\sqrt{5(0)} = 0$	$(0,0)$
5	$x = \pm\sqrt{5(5)} = 5 \ \pm 5$	$(5,5), (-5,5)$
20	$x = \pm\sqrt{5(20)} = \pm\sqrt{100} = \pm 10$	$(10,20), (-10,20)$



(iv). $\boldsymbol{y^2 = -12x}$

Solution.

Given Parabola

$$y^2 = -12x$$
$$Here \quad 4a = 12$$
$$a = 3$$
$$Vertex: V(0,0)$$

The axis of the parabola is along $x - axis$ and opening of parabola is to the left side.

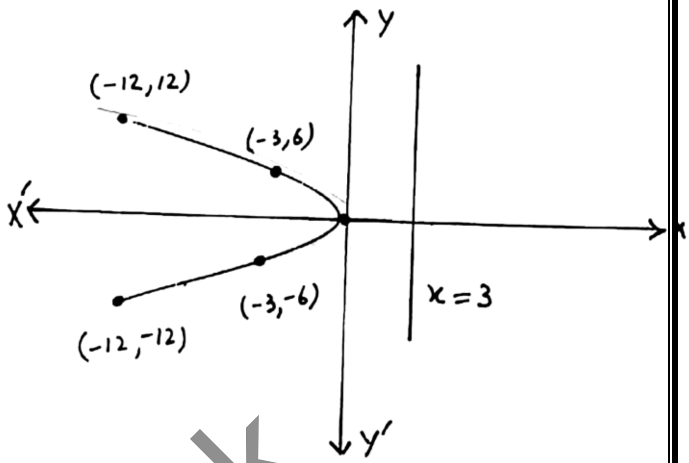
$$Focus : F(-a\,,0) = F(-3\,,0)$$
$$Directrix : \quad x - a = 0$$
$$x - 3 = 0.$$

Sketch:

$$\because y^2 = -12x$$

⇨ $y = \pm\sqrt{-12x}$, $D_f = (-\infty, o)$

x	$y = \pm\sqrt{-12x}$	Points
0	$y = \pm\sqrt{-12(0)} = 0$	$(0,0)$
−3	$y = \pm\sqrt{-12(-3)} = \pm\sqrt{36} = \pm 6$	$(-3,6), (-3,-6)$
−12	$y = \pm\sqrt{(-12)(-12)} = \pm\sqrt{144} = \pm 12$	$(-12,12)(-12,-12)$



(v). $\boldsymbol{x^2 = 4(y-1)}$

Solution.

Given Parabola

$$x^2 = 4(y-1) \rightarrow (i)$$

Put $X = x \quad and \quad Y = y - 1\,, we\ have$

$$X^2 = 4Y \rightarrow (ii)$$
$$Here \quad 4a = 4$$
$$a = 1$$

The vertex of (ii) is $O(0,0)$ with axis of the parabola is along Y-axis and open upward.

$$Vextex: V(0,0)$$
$$\Rightarrow \quad X = 0 \quad and\ Y = 0$$
$$\Rightarrow \quad x = 0 \quad and \quad y - 1 = 0$$
$$\Rightarrow \quad x = 0 \quad and \quad y = 1$$

Hence Vertex of (i) parabola is $(0,1)$.

Now $Focus : (0\,,a) = (0\,,1)$

$$X = 0 \quad and \quad Y = 1$$
$$x = 0 \quad and \quad y - 1 = 1$$
$$x = 0 \quad and \quad y = 2$$

Hence Focus of the parabola (i) is $F(0,2)$.

Directrix of the parabola (ii) is

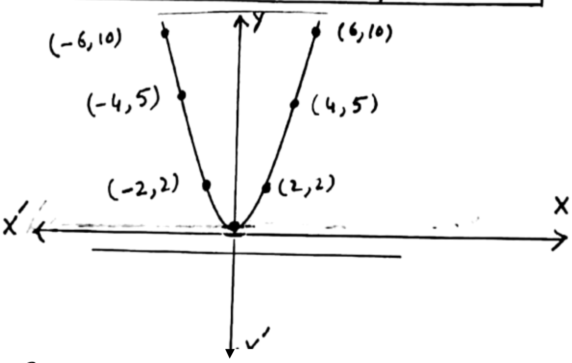
$$Y + a = 0$$
$$Y + 1 = 0$$
$$y - 1 + 1 = 0$$
$$y = 0 \ \ is\ the\ directrix\ of\ the\ parabola\ (i).$$

Sketch:

$$\because x^2 = 4(y-1)$$

⇨ $x = \pm 2\sqrt{y-1} \quad D_f = [1, \infty)$

y	$x = \pm 2\sqrt{y-1}$	Points
1	$x = \pm 2\sqrt{1-1} = 0$	$(0,0)$
2	$x = \pm 2\sqrt{2-1} = \pm 2$	$(2,2),(-2,2)$
5	$x = \pm 2\sqrt{5-1} = \pm 2(2) = \pm 4$	$(4,5),(-4,5)$
10	$x = \pm 2\sqrt{10-1} = \pm 2(3) = \pm 6$	$(6,10),(-6,10)$



vi) $y^2 = ` - 8(x-3)$

⇨ $(y-0)^2 = -8(x-3)$

Compare with $y^2 = -4ax$

$$Y = y - 0 = Y, \quad X = x - 3$$

$$4a = 8 \Rightarrow a = 2$$

Focus: $F(X = -a, Y = 0)$

$$\because X = x - 3 \Rightarrow -a = -2 + 3$$

$$or\ x = 1 \quad (\because a = 2)$$

$$Y = 0 \Rightarrow y = 0 \quad (\because Y = y)$$

So $F(1,0)$

Vertex; $(X = 0, Y = 0)$

$$\because X = x - 3 \Rightarrow 0 = x - 3 \quad (\because X = 0)$$

Or $x = 3$

Also $Y = y \Rightarrow 0 = y \quad (\because Y = 0)$

Or $y = 0$

So vertex $= V(3,0)$

Directrix

$$X = a$$

$$\Rightarrow x - 3 = a \Rightarrow x - 3 = 2$$

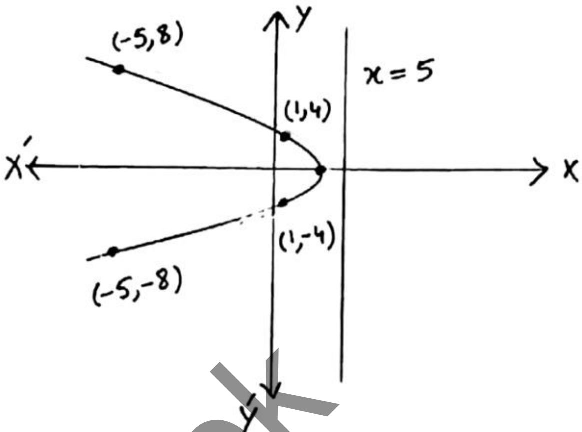
$$or\ x = 2 + 3 \Rightarrow x = 5$$

Sketch:

$$\because y^2 = -8(x-3)$$

⇨ $y = \pm\sqrt{-8(x-3)} \quad , D_f = (-\infty, 3)$

x	$y = \pm\sqrt{-8(x-3)}$	Points
3	$y = \pm\sqrt{-8(3-3)} = 0$	$(0,0)$
1	$y = \pm\sqrt{-8(1-3)} = \pm\sqrt{16} = \pm 4$	$(1,4),(1,-4)$
−5	$y = \pm\sqrt{-8(-5-3)} = \pm\sqrt{64} = \pm 8$	$(-5,8),(-5,-8)$



vii) $\boldsymbol{x - 1)^\wedge 2 = 8(y + 2)}$

Compare with $X^2 = 4aY$

$$X = x - 1, \ Y = y + 2, \ \ 4a = 8 \Rightarrow a = 2$$

Focus; $F(X = 0, Y = a)$

$$X = x - 1 \Rightarrow 0 = x - 1 \Rightarrow x = 1$$

$$Y = y + 2 \Rightarrow a = y + 2 \Rightarrow 2 = y + 2$$

$$\Rightarrow y = 0$$

So $F(1,0)$

Vertex:

$$V(X = 0, Y = 0)$$

$$X = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$$

$$Y = o \Rightarrow y + 2 = 0 \Rightarrow y = -2$$

$$V(1,-2)$$

Directrix:

$$Y = -a$$

⇨ $y + 2 = -a \Rightarrow y + 2 = -2$

⇨ $y = -2 - 2 \Rightarrow y = -4$

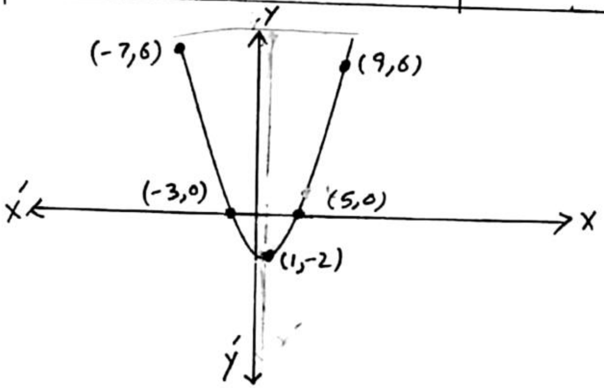
Sketch:

$$(x-1)^2 = 8(y+2)$$

⇨ $x - 1 = \pm\sqrt{8(y+2)}$

⇨ $x = 1 \pm \sqrt{8(y+2)} \quad , D_f = [-2, +\infty)$

y	$x = 1 \pm \sqrt{8(y+2)}$	Points
-2	$x = 1 \pm \sqrt{8(-2+2)} = 1 \pm 0 = 1$	(1, -2)
0	$x = 1 \pm \sqrt{8(0+2)} = 1 \pm 4 = 5, -3$	(5,0), (-3,0)
6	$x = 1 \pm \sqrt{8(6+2)} = 1 \pm 8 = 9, -7$	(9,6), (-7,6)



viii) $y = 6x^2 - 1$

⇨ $6x^2 = y + 1$

Or $(x - 0)^2 = \frac{1}{6}(y + 1)$

Compare with $X^2 = 4aY$

$$X = x - o = x, \qquad Y = y + 1$$

$$4a = \frac{1}{6} \Rightarrow a = \frac{1}{24}$$

Focus; $F(X = 0,\ Y = a)$

$$X = 0 \Rightarrow y + 1 = \frac{1}{24} \qquad \left(\because a = \frac{1}{24}\right)$$

⇨ $y = \frac{1}{24} - 1 \Rightarrow y = -\frac{23}{24}$

$$so, \qquad F\left(0, -\frac{23}{24}\right)$$

Vertex:

$$V(X = 0, Y = 0)$$

⇨ $X = 0 \Rightarrow x = 0$

⇨ $Y = 0 \Rightarrow y + 1 = 0 \Rightarrow y = -1$

$$so\ V(0, -1)$$

Directrix:

$$Y = -a$$

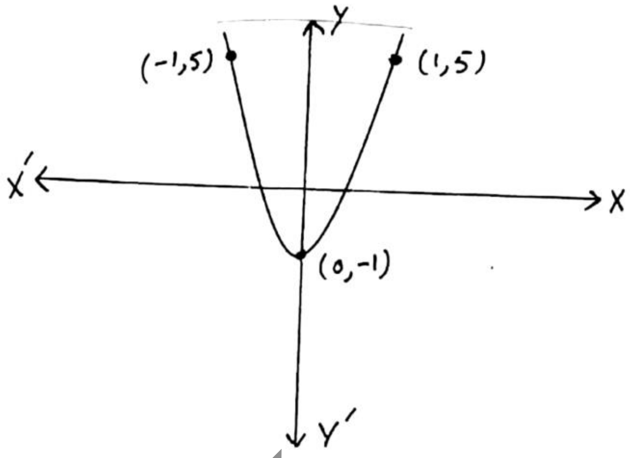
⇨ $y + 1 = -\frac{1}{24} \Rightarrow y + 1 + \frac{1}{24} = 0$

⇨ $y + \frac{25}{24} = 0$

Sketch:$\because x^2 = \frac{1}{6}(y + 1)$

⇨ $\pm\sqrt{\frac{y+1}{6}}, \qquad \boldsymbol{D_f} = [\mathbf{-1}, +\infty)$

y	$x = \pm\sqrt{\frac{y+1}{6}}$	Points
-1	$x = \pm\sqrt{\frac{-1+1}{6}} = 0$	(0, -1)
5	$x = \pm\sqrt{\frac{5+1}{6}} = \pm 1$	(1,5), (-1,5)



ix) $x + 8 - y^2 + 2y = 0$

⇨ $x + 8 = y^2 - 2y$

⇨ $x + 8 + 1 = y^2 - 2y + 1\ (adding\ 1\ both\ sides)$

⇨ $x + 9 = (y - 1)^2$

$$or\ \ (y - 1)^2 = 1(x + 9)$$

Compare with $Y^2 = 4aX$

⇨ $Y = y - 1, X = x + 9, 4a = 1 \Rightarrow a = \frac{1}{4}$

Focus:

$$F(X = a, Y = 0)$$

$$X = x + 9 \ \ \Rightarrow a = x + 9 \Rightarrow \frac{1}{4} = x + 9$$

$$x = \frac{1}{4} - 9 \Rightarrow x = -\frac{35}{4}$$

$$Y = 0 \Rightarrow y - 1 = 0 \Rightarrow y = 1$$

$$so, F\left(-\frac{35}{4}, 1\right)$$

Vertex:

$$V(X = 0, Y = 0)$$

$$X = 0 \Rightarrow x + 9 = 0 \Rightarrow x = -9$$

$$Y = 0 \Rightarrow y - 1 = 0 \Rightarrow y = 1$$

$$s0 \quad V(-9, 1)$$

Directrix:

$$X = -a$$

⇨ $X = -\frac{1}{4} \Rightarrow x + 9 + \frac{1}{4} = 0$

⇨ $x + \frac{37}{4} = 0$

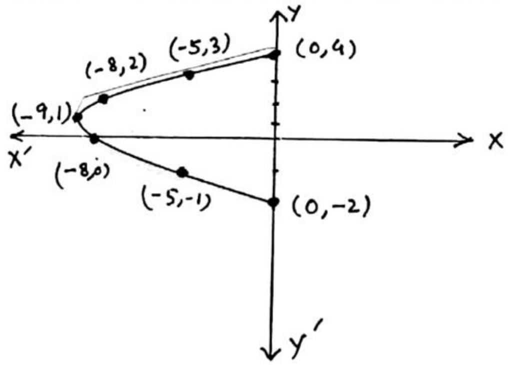
Sketch;

$$\because (y - 1)^2 = x + 9$$

$$\Rightarrow y - 1 = \pm\sqrt{x + 9}$$

$$or\ y = 1 \pm \sqrt{x + 9} \quad D_f = [-9, +\infty)$$

x	$y = 1 \pm \sqrt{x+9}$	Points
-9	$y = 1 \pm \sqrt{-9+9} = 1 \pm 0 = 1$	$(-9, 1)$
-8	$y = 1 \pm \sqrt{-8+9} = 1 \pm 1 = 2, 0$	$(-8, 0), (-8, 2)$
-5	$y = 1 \pm \sqrt{-5+9} = 1 \pm \sqrt{4} = 1 \pm 2 = 3, -1$	$(-5, 3), (-5, -1)$
0	$y = 1 \pm \sqrt{0+9} = 1 \pm 3 = 4, -2$	$(0, 4), (0, -2)$



x) $x^2 - 4x - 8y + 4 = 0$

$\Rightarrow x^2 - 4x + 4 = 8y$

$$\Rightarrow (x-2)^2 = 8(y-0)$$

Compare with $X^2 = 4aY$

$$X = x - 2, 4a = 8 \Rightarrow a = 2$$
$$Y = y - 0 = y$$

Focus:

$$F(X = 0, Y = a)$$
$$X = x - 2 \Rightarrow 0 = x - 2 \Rightarrow x = 2$$
$$Y = a \Rightarrow y = 2 \ (\because a = 2)$$
$$Y = 0 \Rightarrow y = 0$$

So $V(2,0)$

Directrix:

$$Y = -a$$

⇨ $Y = -2 \Rightarrow y = -2$

$$or\ y + 2 = 0$$

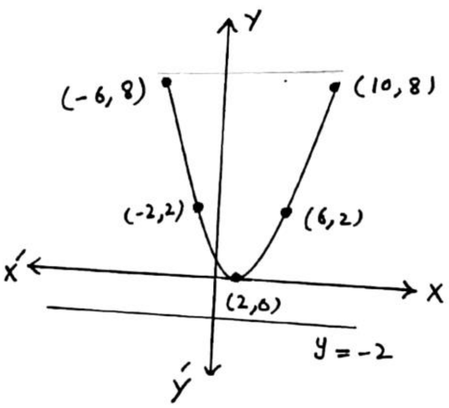
Sketch:

$$(x-2)^2 = 8y$$

⇨ $x - 2 = \pm\sqrt{8y}$

$$or\ x = 2 \pm \sqrt{8y} \quad , D_f = [0, +\infty)$$

y	$x = 2 \pm \sqrt{8y}$	Points
0	$x = 2 \pm \sqrt{8(0)} = 2$	$(2, 0)$
2	$x = 2 \pm \sqrt{8(2)} = 2 \pm \sqrt{16} = 6, -2$	$(6, 2), (-2, 2)$
8	$x = 2 \pm \sqrt{8(8)} = 2 \pm 8 = 10, -6$	$(10, 8), (-6, 8)$



Question. 2

Write an equation of the parabola with given elements

(i). $Focus\ (-3, 1)\ ; Directrix\ x = 3$

Solution.

$$Focus\ (-3, 1)$$
$$M; x - 3 = 0$$

Suppose $P(x, y)$ be any point on parabola then by definition

$$|PF|^2 = |PM|^2$$
$$\sqrt{(x+3)^2 + (y-1)^2} = \frac{|x-3|}{\sqrt{1^2 + 0^2}}$$
$$\sqrt{x^2 + 6x + 9 + y^2 - 2y + 1} = |x - 3|$$

On squaring both sides

$$x^2 + 6x + 9 + y^2 - 2y + 1 = x^2 - 6x + 9$$
$$x^2 + 6x + 9 + y^2 - 2y + 1 - x^2 + 6x - 9 = 0$$
$$y^2 + 12x - 2y + 1$$

Which is required equation of the parabola.

(ii). $Focus\ (2, 5)\ ; Directrix\ y = 1$

Solution.

$$Focus\ (2,5)$$
$$M; y - 1 = 0$$

P(x,y) be any point on parabola then by definition

$$|PF|^2 = |PM|^2\ (\ where\ |PM| is$$
$$\perp ar\ distance\ of\ F$$
$$from\ y - 1 = 0)$$

$$\sqrt{(x-2)^2 + (y-5)^2} = \frac{|y-1|}{\sqrt{0^2 + 1^2}}$$
$$\sqrt{x^2 - 2x + 4 + y^2 - 10y + 25} = |y - 1|$$

On squaring both sides

$$x^2 - 2x + 4 + y^2 - 10y + 25 = y^2 - 2y + 1$$
$$or\ x^2 - 4x + 29 - 10y - 1 + 2y = 0$$
$$or\ x^2 - 4x - 8y + 28 = 0$$

Which is required equation of the parabola.

(iii). $Focus\ (-3, 1)\ ; Directrix\ x - 2y - 3 = 0$

Solution.

$$Focus\ (-3, 1)$$
$$M; x - 2y - 3 = 0$$

take point $P(x, y)$ be any point on parabola then by definition

$$|PF|^2 = |PM|^2$$
$$\sqrt{(x+3)^2 + (y-1)^2} = \frac{|x - 2y - 3|}{\sqrt{1^2 + (-2)^2}}$$
$$\sqrt{x^2 + 6x + 9 + y^2 - 2y + 1} = \frac{|x-3|}{\sqrt{5}}$$

On squaring both sides

$$x^2 + 6x + 9 + y^2 - 2y + 1 = \frac{x^2 - 6x + 9}{5}$$
$$5(x^2 + 6x + 9 + y^2 - 2y + 1) = x^2 - 6x + 9$$

$$5x^2+30x+45+5y^2-10y+5-x^2+6x-9=0$$
$$4x^2+y^2+36x-22y+4xy+41$$
Which is required equation of the parabola.

Note:

"$\boldsymbol{When\ Focus\ and\ vertex\ have\ same\ abscissa}$ $(\boldsymbol{x-\ coordinates})$"

- $Ordiante\ vertex < ordinate\ of\ focus$

  If $ordinate\ of\ vertex > ordinate\ of\ focus$

  Then parabola opens downwards.
- **when Focus and vertex have same ordinate** $(\boldsymbol{y-coordinates})$"

  $Abscissa\ of\ Vertex < Abisssa\ of\ Fcocus$

Then parabola opens towards right

If

$$Abscissa\ of\ Vertex > Abscissa\ of\ Fpcus$$

Then parabola opens towards left.

(vi). $\boldsymbol{Focus}$ $(-3\,,1)$; $\boldsymbol{Vertex}$: $(3,2)$

Solution. Given that

$$Focus\ (-3\,,1)$$
$$Vertex\ (3,2)=V(h,k)$$

Focus and vertex have same ordinate and

$$Abscissa\ of\ Vertex > Abscissa\ of\ Fpcus$$

Then parabola opens towards left.

$$i.e\ (y-k)^2=-4a(x-h)$$
$$(y-2)^2=-4a(x-3)$$

Now $a=distance\ between\ focus\ and\ vertex$

$$a=|FV|=\sqrt{(3-1)^2+(2-2)^2}=\sqrt{4+0}=2$$

Putting in (i) , we have

$$(y-2)^2=-4(2)(x-3)$$
$$y^2-4y+4=-8(x-3)$$
$$y^2-4y+4=-8x+24$$
$$y^2+8x-4y-24+4=0$$
$$y^2+8x-4y-20=0$$

Which is required equation of the parabola.

(v). $\boldsymbol{Focus}$ $(2\,,5)$; $\boldsymbol{Vertex}$ $(-1\,,2)$

Solution.

$$Focus\ (-1\,,0)$$
$$Vertex\ (-1,2)=V(h,k)$$

Focus and vertex have same ordinate and

$$ordinate\ of\ vertex > ordinate\ of\ focus$$

Then parabola opens downwards.

$$\boldsymbol{i.e}\ (x-h)^2=-4a(y-k)$$

⇨ $(x+1)^2=-a(y-2)\to(i)$

$$a=|FV|=\sqrt{(2-0)^2+(-1+1)^2}=2$$

Put $a=2\ in\ (i)$

$$(x+1)^2=-a(2)(y-2)$$

⇨ $x^2+1+2x=-8y+16$

$$or\ x^2+2x+1+8y-15=0$$
$$(required\ parabola)$$

(vi). $\boldsymbol{Directrix\ x=-2}$ $\boldsymbol{Focus}$ $(2\,,2)$

Solution.

$$take\ any\ poit\ P(x,y)$$
$$F((2,2),M;x+2=0$$

By definition.

$$|PF|^2=|PM|^2$$

⇨ $\left(\sqrt{(x-2)^2+(y-2)^2}\right)^2=\left(\frac{|x+2|}{\sqrt{(1)^2+(0)^2}}\right)^2$

⇨ $x^2+4-4x+y^2+4-4y=\frac{x^2+4+4x}{1}$

⇨ $x^2-4x+y^2-4y+8-x^2-4-4x=0$

⇨ $y^2-4y+4-8x=0$

or$(y-2)^2=8x\ (req.parabola)$

(vii). $\boldsymbol{Vertex}$ $(2\,,2)$; $\boldsymbol{Directrix\ y=3}$

Solution.

$$Vertex\ (2\,,2)$$
$$y-3=0$$

Since axis of parabola is parallel to $Y-axis(because\ direcrix\ is\ parallel\ to\ X-axis)$

And opening is downward.

So equation of parabola with vertex $(h,k)=(2,2)$

$$(x-h)^2=-4a(y-k)^2$$
$$(x-2)^2=-4a(y-2)^2---(i)$$

Now $a=Distance\ of\ vertex\ (2,2)from\ directrix$

$$a=\frac{|2-3|}{\sqrt{(0)^2+(1)^2}}=\frac{|-1|}{\sqrt{0+1}}=1$$

Putting in equation (i), we have

$$(x-2)^2=-4(1)(y-2)$$
$$x^2-4x+4=-y+8$$
$$x^2-4x+4+y-8=0$$
$$x^2-4x+y-4=0$$

Which is required equation of the parabola.

(viii). $\boldsymbol{rectrix\ y=1}$, Length of latusrectum is 8, Opens Downward.

Solution.

$$Directrix\ y=1$$
$$Length\ of\ latusrectum\ is=4a=8=>a=2$$

Parabola is open downward.

Consider Vertex $=(h,-1)$.

Equation of the parabola is

$$(x-h)^2=-4a(y-k)$$
$$(x-h)^2=-4(2)(y+1)$$
$$x^2-2xh+h^2=-8(y+1)$$
$$x^2-2xh+h^2=-8y-8$$
$$x^2-2xh+h^2+8y+8=0$$
$$x^2-2xh+8y+h^2+8=0$$

Which is required equation of the parabola.

(ix). $\boldsymbol{Axis\ y=0\ through}$ $(2,1)$ $\boldsymbol{and}$ $(11,-2)$.

Solution.

Let vertex is (h,k).

Since it lies on $x-axis\ so\ k=0$.

Now equation of the parabola with vertex $(h,0)$.

$$(y-0)^2 = 4a(x-h)$$
$$y^2 = 4a(x-h) ---(i)$$
Since (2,1) lies on the parabola (i),we have
$$(1)^2 = 4a(2-h)$$
$$1 = 4a(2-h) ---(ii)$$
Also $(11,-2)$ lies on the parabola (i),we have
$$(-2)^2 = 4a(11-h)$$
$$4 = 4a(11-h) ---(iii)$$
Dividing (i) and (ii), we have
$$\frac{1}{4} = \frac{4a(2-h)}{4a(11-h)}$$
$$\frac{1}{1} = \frac{4(2-h)}{(11-h)}$$
$$11-h = 4(2-h)$$
$$11-h = 8-4h$$
$$-h+4h = 8-11$$
$$3h = -3$$
$$h = -1.$$
Putting in (ii), we have
$$1 = 4a(2-(-1))$$
$$1 = 4a(2+1)$$
$$1 = 4a(3)$$
$$1 = 12a$$
$$a = \frac{1}{12}$$
Putting in (i), we have
$$y^2 = 4\left(\frac{1}{12}\right)(x-(-1))$$
$$y^2 = \frac{1}{3}(x+1)$$
$$3y^2 = (x+1)$$
$$3y^2 - x - 1 = 0.$$
Which is required equation of the parabola.

(x). ***Axis paralel to Y − axis , the points (0,3), (3,4) and (4,11) lie on the graph.***

Solution.

Let eq. of parabola is
$$(x-h)^2 = 4a(y-k) \to (i)$$
Put $(0,3)$ *in* (i)

⇨ $h^2 = 4a(3-k)$
$$h^2 = 12a - 4ak \to (ii)$$
Put $(3,4)$ *in* (i)

⇨ $(3-h)^2 = 4a(4-k)$
$$9 + h^2 - 6h = 16a - 4ak \to (iii)$$
Put$(4,1)$*in* (i)

⇨ $(4-k)^2 = 4a(11-k)$
$$16 + h^2 - 8h = 44a - 4ak \to (iv)$$
By$(iv)-(iii)$
$$16 - 8h + h^2 = 44a - 4ak \to (iv)$$
By$(iv)-(iii)$
$$16 - 8h + h^2 = 44a - 4ak$$
$$\pm 9 \mp 6h \pm h^2 = \pm 12a \mp 4ak$$
$$7 - 2h = 28a \to (v)$$
By $(iii)-(ii)$
$$9 - 2h + h^2 = 16a - 4ak$$
$$\pm h^2 = \pm 12a \mp 4ak$$
$$9 - 6h = 4a \to (vi)$$
Multiplying $(v) by\ 3$
$$21 - 6h = 84a \to (vii)$$
By$(vii)-(vi)$
$$21 - 6h = 84a$$
$$\pm 9 \mp 6h = -4a$$
$$12 = 80a$$
⇨ $a = \frac{12}{80} = \frac{3}{20}$

⇨ $Put\ a = \frac{3}{20}\ in\ (vi)$
$$9 - 6h = 4\left(\frac{3}{20}\right)$$
⇨ $9 - 6h = \frac{3}{5}$

⇨ $9 - \frac{3}{5} = 6h\ or\ 6h - \frac{42}{5}$

⇨ $h = \frac{7}{5}\ put\ in\ (ii)$

⇨ $\left(\frac{7}{5}\right)^2 = 12\left(\frac{3}{20}\right) - 4\left(\frac{3}{20}\right)k$

⇨ $\frac{49}{25} = \frac{18}{10} - \frac{49}{25}$

⇨ $\frac{3}{5}k = \frac{450-490}{250}$

⇨ $\frac{3}{5}k = \frac{-40}{250}$

⇨ $k = -\frac{4}{25} \times \frac{5}{3}$

⇨ $k = -4/15$

⇨ $Now\ (I)\ becomes\ as\ \left(x - \frac{7}{5}\right)^2 = 4\left(\frac{3}{20}\right)\left(y + \frac{4}{15}\right)$

⇨ $\left(x - \frac{7}{5}\right)^2 = \frac{3}{5}\left(y + \frac{4}{15}\right)$

Question.3.

Find an equation of the parabola having its focus at the origin and directrix parallel to the

$(i).\ x - axis$ $(ii).\ y - axis$

Solution.

(i). $Directrix; y = a, Focus = F(o,o)$

or $M; y - a = 0\ P(x,y) any\ point$

by definition of parabola
$$|PF|^2 = |PM|^2$$
$$\sqrt{(x-0)^2 + (y-0)^2} = \frac{|y-a|}{\sqrt{(0)^2 + (1)^2}}$$
$$\sqrt{x^2 + y^2} = \frac{|y-a|}{1}$$
On squaring both sides , we have
$$x^2 + y^2 = y^2 - 2ay + a^2$$
$$x^2 + y^2 - y^2 + 2ay - a^2 = 0.$$
$$x^2 + 2ay - a^2 = 0.$$

Which is required parabola.

CaseII:

$$Directrix\ ; y = -a\ (aslo\ ||to\ x - axis)$$

Or$M; y + a = 0\ \ , F(o, o), P(x, y)$

By definition.

$$|PF|^2 = |PM|^2$$

⇨ $\left(\sqrt{(x-0)^2 + (y-0)^2}\right)^2 = \left(\frac{|y+a|}{\sqrt{(1)^2+(0)^2}}\right)^2$

⇨ $x^2 + y^2 = y^2 + a^2 + 2ax$

⇨ $x^2 - a^2 - 2ax = 0$

(ii). $y - axis$:

CaseI: $\boldsymbol{Directrix; x = a\ , F(o, o), P(x, y)}$

$$or\ M; x - a = 0$$

$$by\ def; |PF|^2 = |PM|^2$$

$$\sqrt{(x-0)^2 + (y-0)^2} = \frac{|x-a|}{\sqrt{(1)^2 + (0)^2}}$$

$$\sqrt{x^2 + y^2} = \frac{|x-a|}{1}$$

On squaring both sides, we have

$$x^2 + y^2 = x^2 - 2ax + a^2$$

$$x^2 + y^2 - x^2 + 2ax - a^2 = 0.$$

$$y^2 + 2ax - a^2 = 0.$$

Which is required parabola.

Case II.

$$Directrix\ ; x = -a\ (also\ ||to\ y - axis)$$

$$orM, x + a = 0, \qquad F(0,0), P(x, y)$$

By definition.

⇨ $|PF|^2 = |PM|^2$

⇨ $\left(\sqrt{(x-0)^2(y-0)^2}\right)^2 = \left(\frac{|x+a|}{\sqrt{(1)^2+(0)^2}}\right)^2$

⇨ $x^2 + y^2 = x^2 + a^2 + 2ax$

Or $y^2 - a^2 - 2ax = 0$

Question.4

Show that an equation of the parabola with focus at $(\boldsymbol{aCos\alpha\ , a\ Sin\alpha})$ $\boldsymbol{and\ directrix}$

$$\boldsymbol{xCos\alpha + ySin\alpha + a = 0\ is\ (xSin\alpha - yCos\alpha)^2 = 4a(xCos\alpha + ySin\alpha)}$$

Solution.

$$Focus\ (aCos\alpha\ , a\ Sin\alpha)$$

$$M;\ \ xCos\alpha + ySin\alpha + a = 0$$

P(x,y) be any point on parabola then by definition

$$|PF|^2 = |PM|^2$$

$$\sqrt{(x - aCos\alpha)^2 + (y - aSin\alpha)^2} = \frac{|xCos\alpha + ySin\alpha + a|}{\sqrt{Cos^2\alpha + Sin^2\alpha}}$$

$$\sqrt{(x - aCos\alpha)^2 + (y - aSin\alpha)^2} = \frac{|xCos\alpha + ySin\alpha + a|}{1}$$

On squaring both sides

$$(x - aCos\alpha)^2 + (y - aSin\alpha)^2 = |xCos\alpha + ySin\alpha + a|^2$$

$$x^2 + a^2Cos^2\alpha - 2axCos\alpha + y^2 + a^2Sin^2\alpha - 2aySin\alpha = x^2Cos^2\alpha + y^2Sin^2\alpha + a^2 + 2axCos\alpha + 2aySin\alpha + 2xySin\alpha Cos\alpha$$

$$x^2 - 2axCos\alpha + y^2 - 2aySin\alpha + a^2(Cos^2\alpha + Sin^2\alpha) = x^2Cos^2\alpha + y^2Sin^2\alpha + a^2 + 2axCos\alpha + 2aySin\alpha + 2xySin\alpha Cos\alpha$$

$$x^2 - 2axCos\alpha + y^2 - 2aySin\alpha + a^2 = x^2Cos^2\alpha + y^2Sin^2\alpha + a^2 + 2axCos\alpha + 2aySin\alpha + 2xySin\alpha Cos\alpha$$

$$x^2(1 - Cos^2\alpha) + y^2(1 - Sin^2\alpha) + a^2 - a^2 - 2xySin\alpha Cos\alpha = 4axCos\alpha + 4aySin\alpha$$

$$x^2Sin^2\alpha + y^2Cos^2\alpha - 2xySin\alpha Cos\alpha = 4a(xCos\alpha + ySin\alpha)$$

$$(xsin\alpha - ycos\alpha)^2 = 4a(xCos\alpha + ySin\alpha)$$

Which is required equation of the parabola.

Hence Proved.

Question.5.

Show that ordinate at any point P of the parabola is a mean proportional between the length of the latus rectum and the abscissa of P.

Solution.

Consider the equation of the parabola is

$$y^2 = 4ax$$

$$y.y = 4a.x$$

$$\frac{4a}{y} = \frac{y}{x}$$

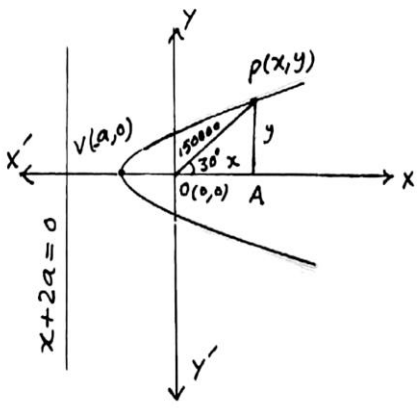
$$\frac{Latus\ ractum}{odinate} = \frac{ordinate}{abscissa}$$

Hence ordinate is mean proportional between latus rectum and abscissa.

Qustion.6.

A comet has a parabolic orbit with the earth at the focus. When the comet is 150,000km from the earth, the line joining the comet and the earth makes an 30$\boldsymbol{with\ the\ axis\ of\ the\ parabola}$. How close will the comet come to the earth.

Solution.



Suppose earth be at focus is origin and $V(-a,o)$ *be* vertex of parabola. Then directrix of parabola is

$x = 2a$

⇨ $M; x + 2a = 0$

Let comet be at point $P(x,y)$ *then by def.*

$$|PF|^2 = |PM|^2$$

⇨ $\left(\sqrt{(x-0)^2+(y-0)^2}\right)^2 = \left(\frac{|x+2a|}{\sqrt{(1)^2+(0)^2}}\right)^2$

⇨ $x^2 + y^2 = a^2 \rightarrow (i)$

InΔOAP *, by pathgoras theorem*

$$|OA|^2 + |AP|^2 = |OP|^2$$

⇨ $x^2 + y^2 = (150000)^2 \rightarrow (ii)$

by (i)and (ii)

$$(x+2a)^2 = \pm 150000$$

⇨ $(x+2a)^2 = \pm 150000 \rightarrow (iii)$

From right ΔOAP

$$cos30^0 = \frac{|OA|}{|AP|} \Rightarrow \frac{\sqrt{3}}{2} = \frac{x}{150000}$$

⇨ $x = \frac{\sqrt{3}}{2}(150000)$

By (iii)

$$\frac{\sqrt{3}}{2}(150000) + 2a = \pm 150000$$

⇨ $2a = \pm 150000 - \frac{\sqrt{3}}{2}(150000)$

⇨ $2a = \pm 150000 - \sqrt{3}(75000)$

⇨ $2a = 75000(\pm 2 - \sqrt{3})$

⇨ $a = 37500(\pm 2 - \sqrt{3})$

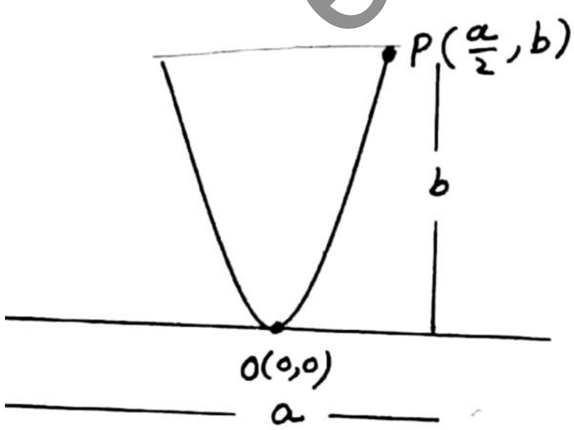
$\because$ *a is short distance and can'tbe − ve so*

$$a = 37500(2-\sqrt{3})km$$

Question.7.

Find the equation of the parabola formed by the cables of a suspension bridge whose span is a m and the vertical height of the supporting towards is b m.

Solution.



Consider the equation of the parabola with vertex O(0,0)

$$x^2 = 4a'y \rightarrow (i)$$

Since $P\left(\frac{a}{2}, b\right)$ *lies on the parabola*

$$\left(\frac{a}{2}\right)^2 = 4a'(b)$$

$$a' = \frac{a^2}{16b}$$

Putting this value in (i), we have

$$x^2 = 4\left(\frac{a^2}{16b}\right)y$$

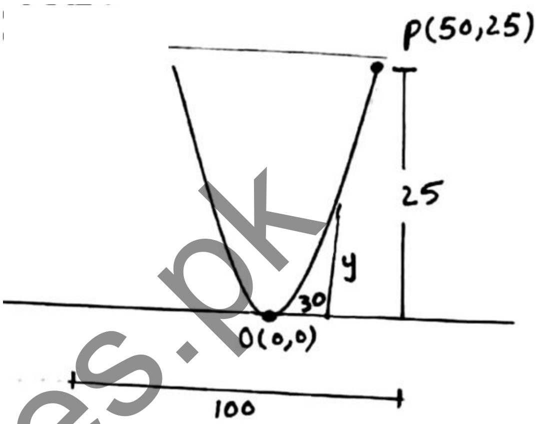
$$x^2 = \frac{a^2}{4b}y$$

Which is required equation.

Question.8.

A parabolic arch has a 100m base and height 25m. Find the height of the arch at the point 30m from the center of the base.

Solution.



Consider the equation of the parabola with vertex O(0,0)

$$x^2 = 4ay ----(i)$$

Since $P(50, 25)$ *lies on the parabola*

$$(50)^2 = 4a(25)$$

$$2500 = 100a$$

$$a = 25$$

Putting in (i), we have

$$x^2 = 4(25)y = 100y$$

When $x = 30$ *then*

$$(30)^2 = 100y$$

$$y = \frac{900}{100} = 9$$

Hence the required height is 9m.

Question.9.

Show that tangent at any point P of a parabola marks equal angles which the line PF and line through P and parallel to $x - axis$.

Solution:

Let *eq. of parabola is* $y^2 = 4ax \rightarrow (i)$

$$m_1 = 0(\because line \ ||to\ x - axis)$$

For m_2; *take derivative of* (i)

$$2y\frac{dy}{dx}\Big|_{x_1,y_1} = \frac{2a}{y}$$

⇨ $m_3 = \frac{y_1 - 0}{x_1 - a} = \frac{y_1}{x_1 - a} = \frac{y_1}{x_1 - a}$

$$\text{Tan}\theta_1 = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{\frac{2a}{y_1} - 0}{1 + \left(\frac{2a}{y_1}\right)(0)}$$

$$Tan\theta_1 = \frac{2a}{y_1} \Rightarrow \theta_1 = Tan^{-1}\left(\frac{2a}{y_1}\right) \to (i)$$

$$Tan\theta_2 = \frac{m_3 - m_2}{1 + m_3 m_2}$$

$$= \frac{\left(\frac{v_1}{x_1 - a}\right) - \left(\frac{2a}{y_1}\right)}{1 + \left(\frac{y_1}{x_1 - a}\right)\left(\frac{2a}{y_1}\right)}$$

$$Tan\theta_2 = \frac{\frac{y_1^2 - 2ax_1 + 2a^2}{(x_1 - a)(y_1)}}{\frac{y_1x_1 - ay_1 + 2ay_1}{y_1(x_1 - a)}}$$

$$put\ y_1^2 = 4ax_1 \quad (\because \ P\ is\ a\ parabola)$$

⇨ $$Tan\theta_2 = \frac{\frac{4ax_1 - 2ax_1 + 2a^2}{y_1(x_1 - a)}}{\frac{y_1x_1 - ay_1 + 2ay_1}{y_1(x_1 - a)}}$$

⇨ $Tan\theta_2 = \frac{2a(x_1 + a)}{y_1(x_1 + a)}$

⇨ $\theta_2 = Tan^{-1}\left(\frac{2a}{y_1}\right) \to (ii)$

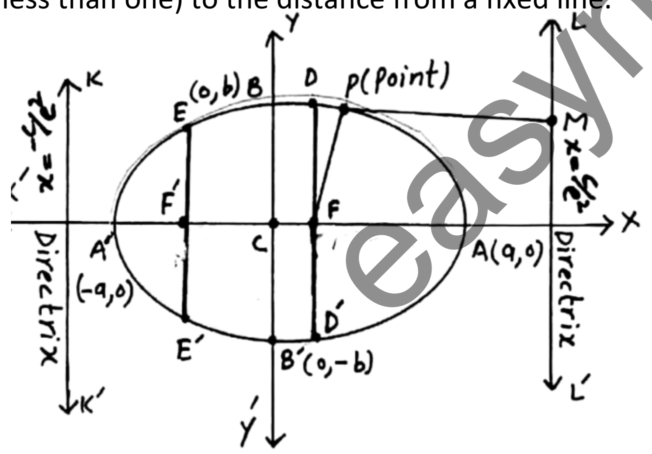
By $(i) and\ (ii)$

$\theta_1 = \theta_2\ \ Hence\ proved.$

Ellipse and its Elements

Ellipse:

Set of all points in a plane, such that distance of each from a fixed point bears a constant ratio (less than one) to the distance from a fixed line.



Note:

- Fixed point is called Focus. Here F is Focus
- Fixed line LL' is directrix.
- Constant ratio is called eccentricity, denoted by $e\ in\ Fig. \frac{|FP|}{|PM|} = e$

⇨ $|FP| = e|PM| \quad where\ |FP| < |PM|$

So in ellipse $e < 1$

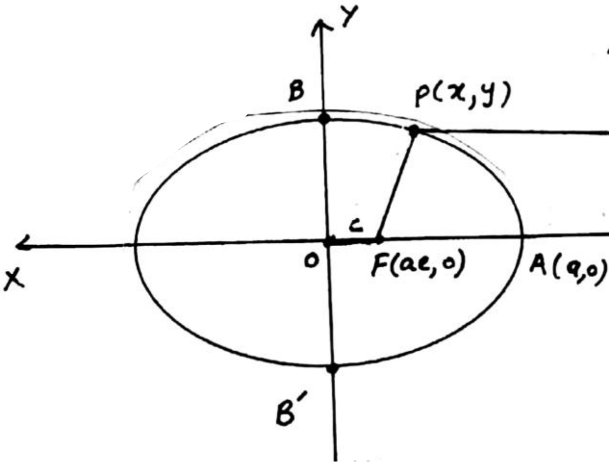
Definition:

1) An ellipse has two diameters, called axes of symmetry. In fig. $AA' and\ BB'$ are diameter of ellipse.
2) The largest diameter is called major axis. In fig. AA' is major axis and length of major axis $|AA'|\ is$ 2a.
3) The smallest diameter is called minor axis. In fig. $BB' is$ Minor axis. And length of minor axis and length of minor axis $BB' is\ 2b$.
4) An ellipse has two directrix, called **directrices. In fig**. $LL' and\ kk' are\ directrices$.
5) Two fixed points are called foci (plural of focus) in fig.
   $F\ and\ F' are\ foci. Foci\ always\ lie\ on\ major\ axis.$
6) The end points of major axis called vertices. In fig.
   $A(a, o) and\ A'(-a, 0) are\ vertices\ of\ ellipse.$
7) The end points of minor axis are called co-vertices. In fig $B(o, b) and\ B'(o, -b) are\ co-vertices$ of ellipse.
8) A straight line joining two points of ellipse is called chord of ellipse.
9) A straight and passing through focus is called focal chord of ellipse.
10) Focal chord perpendicular to mojor axis is called letusrectum. or focal chord parallel to minor axis is called letusrectum. In fig. $DD'\ and\ EE' are$ letusrectra (plural of letusrectum) length of letusrectum is $\frac{2b^2}{a}$
11) The point of intersection of major and minor axis or foci is called Centre.
    Or minor axis or foci is called Centre of ellipse.
    In figure C is Centre.
12) In fig. AC and CA' are semi-major asix.
13) In fig BC and CB' are semi minor axis.
14) If the point $acos\theta, bsin\theta)\ lies\ on\ ellipse \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ where\ \ \theta \in \mathbb{R}\ then\ x = acos\theta, y = bsin\theta$ $are\ called\ parametric\ equations\ of\ ellipse.$

Standard Equation of Ellipse:

$$\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

Proof:



Let

$P(x, y) be\ any\ point\ on\ the\ ellipse\ having\ focus$

Having focus $F(ae, o)$ and directed $x = \frac{a}{e}$

⇨ $ex - a = 0$

Let $|PM|$ be $\perp$

$ar\ distance\ of\ P(x, y) from\ directrix$

$$x = \frac{a}{e}$$

⇨ $ex - a = 0$

By def. $|FP|^2 = e|PM|^2$

$$\left(\sqrt{(x - ea)^2 + (y - o)^2}\right)^2$$

$$= e\left(\frac{|ex - a|}{\sqrt{(e)^2 + (0)^2}}\right)^2$$

⇨ $x^2 + e^2a^2 - 2eax + y^2 = e\frac{(ex-a)^2}{e}$

⇨ $x^2 + e^2a^2 - 2eax + y^2 = e^2x^2 + a^2 - 2eax$

⇨ $x^2 - e^2x^2 + y^2 = a^2 - e^2a^2$

Or $x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

$$\div\ by\ \ a^2(1 - e^2)$$

⇨ $\frac{x^2(1-e^2)}{a^2(1-e^2)} + \frac{y^2}{a^2(1-e^2)} = \frac{a^2(1-e^2)}{a^2(1-e^2)}$

⇨ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ where\ \ a^2(1 - e^2) = b^2$

Note:

$$\because b^2 = a^2(1 - e^2)$$

⇨ $b^2 = a^2 - a^2e^2$

⇨ $b^2 = a^2 - (ae)^2 \quad \because ae = c$

⇨ $b^2 = a^2 - c^2$

Or $c^2 = a^2 - b^2$

Prove that circle is special case of ellipse.

Proof:

$$\because \frac{\boldsymbol{x^2}}{\boldsymbol{a^2}} + \frac{\boldsymbol{y^2}}{\boldsymbol{b^2}}$$

$= \boldsymbol{1}\ \ \boldsymbol{represents\ an\ ellipse\ having}$

Centre at origin. Let $\boldsymbol{b = a}$

Then eq. of ellipse becomes as

$$\frac{\boldsymbol{x^2}}{\boldsymbol{a^2}} + \frac{\boldsymbol{y^2}}{\boldsymbol{a^2}} = \boldsymbol{1} \Rightarrow \boldsymbol{x^2 + y^2 = a^2}$$

Which represents eq. of Circle having Centre at origin.

⇨ **Circle is a special case of an ellipse.**

Note:

In an ellipse

$$\boldsymbol{b^2 = a^2(1 - e^2)}$$

$$\boldsymbol{for\ a\ circle\ b = a}$$

⇨ $\boldsymbol{a^2 = a^2(1 - e^2)}$

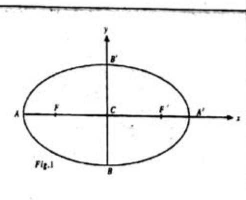
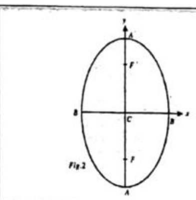
⇨ $\boldsymbol{1 = 1 - e^2}$

⇨ $\boldsymbol{1 = 1 - e^2 \Rightarrow e^2 = 1 - 1}$

$$\boldsymbol{or\ e^2 = 0 \Rightarrow e = 0}$$

So $\boldsymbol{for\ a\ circle, eccentricity\ is\ e = o}$

Summary of standard Ellipses

Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$ $c^2 = a^2 - b^2$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b$ $c^2 = a^2 - b^2$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Directrices	$x = \pm\frac{c}{e^2}$	$y = \pm\frac{c}{e^2}$
Major axis	$y = 0$	$x = 0$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Covertices	$(0, \pm b)$	$(\pm b, 0)$
Centre	$(0, 0)$	$(0, 0)$
Eccentricity	$e = \frac{c}{a} < 1$	$e = \frac{c}{a} < 1$
Graph		

Note: In each ellipse

Length of major axis $= 2a$, Length of minor axis $= 2b$

Length of Latusrectum $\approx \frac{2b^2}{a}$, Foci lie on the major axis

Exercise No.6.5

Q1. Find an equation of the ellipse with given data and sketch its graph.

(i) Foci

$(\pm\boldsymbol{3}, \boldsymbol{0})\ \boldsymbol{and\ minor\ axis\ of\ length}\ \boldsymbol{10}$.

Solution:

Length of minor axis= 10

$$c = 3\ and\ 2b = 10 \Rightarrow b = 5$$

Now $c^2 = a^2 - b^2 \Rightarrow a^2 = c^2 + b^2$

⇨ $a^2 = 9 + 25 \Rightarrow a^2 = 34$

Now $eq. of\ ellipse\ is$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{34} + \frac{y^2}{25} = 1$$

Sketch:

$$Centre = \ mid\ point\ of\ foci$$

$F(3,0) and\ F'(-3,0)$

Centre $= \left(\frac{3-3}{2}, \frac{0+0}{2}\right) = (0,0)$

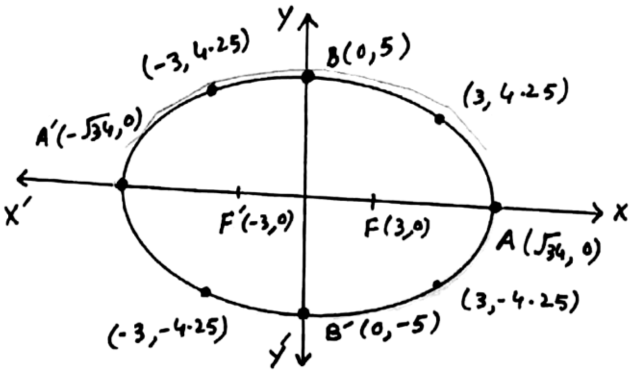
Vertices $= (\pm a, 0) = (\pm\sqrt{34}, o)$

Co$-vertices$ $(0, \pm b) = (0, \pm 5)$

$i.e\ B(0,5), B'(0,-5)$

Length of letusrect:

$$\frac{2b^2}{a} = \frac{2(25)}{\sqrt{34}} = \frac{50}{\sqrt{34}}$$



(ii)

$Foci(0,-1) and\ (0,-5), major\ axis\ of\ length\ 6.$

Solution:

$$centre = midpoint\ of\ foci$$

$$(h,k) = \left(\frac{0+0}{2}, \frac{-1-5}{2}\right) = (0,-3)$$

$$\because 2C = Distance\ b\backslash w\ foci$$

$$2C = \sqrt{(0-0) + (-5+1)^2} = \sqrt{16} = 4$$

⇨ $2C = 4 \Rightarrow C = 2 \Rightarrow C^2 = 4$

$$\because length\ of\ major\ axis = 2a = 6$$

$$\Rightarrow a = 3 \Rightarrow a^2 = 9$$

$$\because c^2 = a^2 - b^2 \Rightarrow b^2 = a^2 - c^2 = 9 - 4 = 5$$

$\because foci\ have\ same\ abscissa, so\ major\ axis\ is\ along$

$y - axis\ is\ along\ y - axis$

Now eq. of ellipse is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \to (i)$$

$$\because h = 0, k = -3, a^2 = a, b^2 = 5$$

$$\Rightarrow \frac{x^2}{5} + \frac{(y+3)^2}{a^2} = 1 \to (i)$$

$$\because h = 0, k = -3, a^2 = a, b^2 = 5$$

$$\Rightarrow \frac{x^2}{5} + \frac{(y+3)^2}{9} = 1$$

Sketch:

Vertices$(X = 0, Y = \pm a)$

⇨ $X = x \Rightarrow x = 0, y = \pm a$

⇨ $y + 3 = \pm 3 \Rightarrow y = -3 \pm 3 = 0, -6$

So vertex are $A(0.0)$ $A'(0,-6)$

Co-vertices$(X = \pm b, Y = 0)$

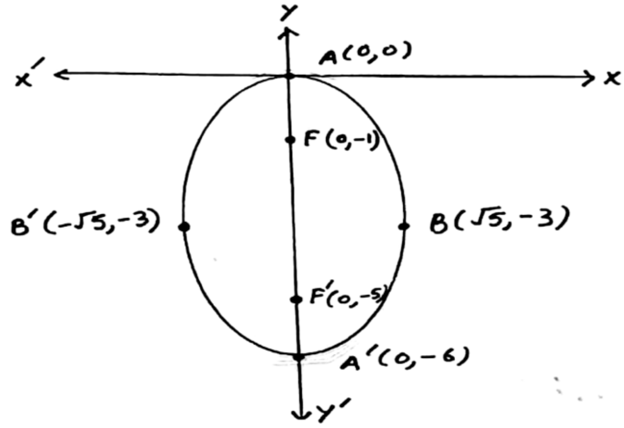
$$\because X = x \to x = \pm b \Rightarrow x = \pm\sqrt{5}$$

$$Y = 0 \Rightarrow y + 3 = 0 \Rightarrow y = -3$$

So $co - vertices\ are$ $(\sqrt{5}, -3), (-\sqrt{5}, -3)$

Length of $letusrectum$:

$$\frac{2b^2}{a} = \frac{2(5)}{3} = \frac{10}{3}$$



(iii) $Foci$ $(\pm 3\sqrt{3}, 0)$ $and\ Vertices$ $(\pm 6, 0)$

Solution:

$$Foci(\pm c, o) = (\pm 3\sqrt{3}, 0)$$

⇨ $C = 3\sqrt{3} \Rightarrow C^2 = 27$

⇨ Centre $= Midpoint\ of\ foci$

$$= \left(\frac{3\sqrt{3} - 3\sqrt{3}}{2}, \frac{0+0}{2}\right) = (0,0)$$

$$vertices = (\pm a, 0) = (\pm 6, 0)$$

⇨ $a = 6 \Rightarrow a^2 = 36$

$$\because Foci = (\pm 3\sqrt{3}, 0))$$

⇨ $same\ ordinates, so\ major\ axis\ is\ along\ x - axis$

So,

$$ellipse\ is \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

⇨ $\frac{x^2}{36} + \frac{y^2}{9} = 1$

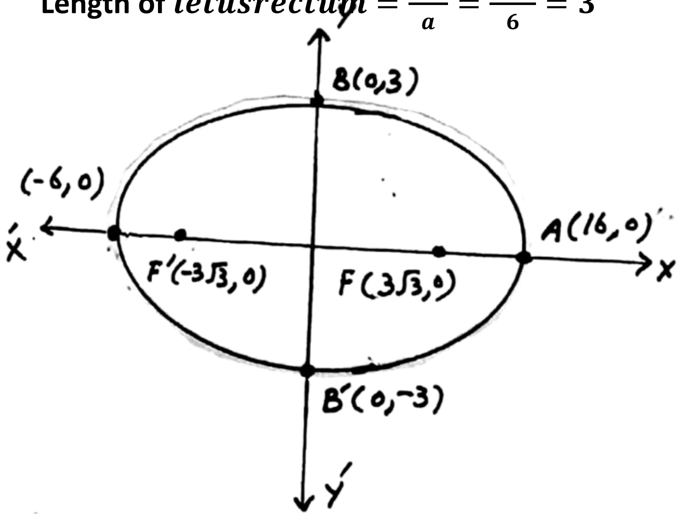
Sketch:

Vertices $= (\pm 6, 0) i.e\ A(6,0) A'(-6,0)$

Co-vertices $= (0, \pm b) = (0, \pm 3)$

$i.e\ B(o,3), B'(0,-3)$

Length of $letusrectum$ $= \frac{2b^2}{a} = \frac{2(9)}{6} = 3$



(iv) ***Vertices*** $(-1,1)$ ***and*** $(5,1)$; ***Foci*** $(4,1)$ ***and*** $(0,1)$

Solution:

$$\because Foci = F(4,1), F'(0,1)$$

Centre= $midpoint\ of\ foci$

$$(h,k) = \left(\frac{4+0}{2}, \frac{1+!}{2}\right) = (2,1)$$

2C= $Dist.\ b\backslash w\ vertics$

$$2a = \sqrt{(5+1)^2 + (1-1)^2} = 6$$

⇨ $2a = 6 \Rightarrow a = 3\ \ or\ a^2 = 9$

$$\because c^2 = a^2 - b^2 \Rightarrow b^2 = a^2 - c^2$$

$$\Rightarrow b^2 = 9 - 4 \Rightarrow b^2 = 5$$

$\because$ *foci have same ordinate so major axis is along* $x - axis.$

$Eq.$ Of ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

$$\because h = 2, k = 1, a^2 = 9, b^2 = 5\ \ so$$

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{5} = 1$$

Sketch:

Vertices: $A(5,1), A'(-1,1)$

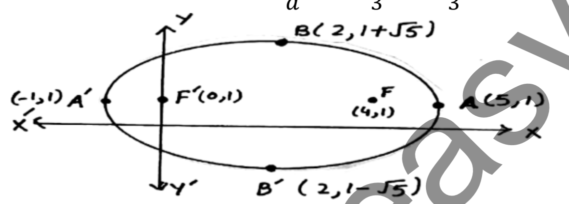
Co-vertices$(X = 0, Y = \pm b)$

$$\because X = x - 2 \Rightarrow x - 2 = 0 \Rightarrow x = 2$$

$$Y = y - 1 \Rightarrow y - 1 = \pm b \Rightarrow y = 1 \pm \sqrt{5}$$

Covertices are $B(2,1+\sqrt{5}), B'(2,1-\sqrt{5})$

Length of letusrectum $\frac{2b^2}{a} = \frac{2(5)}{3} = \frac{10}{3}$



(v)

Foci $(\pm\sqrt{5}, o)$ *and passing through point* $\left(\frac{3}{2}, \sqrt{3}\right)$

Solution:

foci= $(\pm c, o) = (\pm\sqrt{5}, 0)$

⇨ $c = \sqrt{5} \Rightarrow c^2 = 5$

As $Foci$: $F(\sqrt{5}, 0), F'(-\sqrt{5}, 0)$

Centre $=$ $midpoint\ of\ foci$

$$= \left(\frac{\sqrt{5} - \sqrt{5}}{2}, \frac{0+0}{2}\right) = (0,0)$$

$$\because c^2 = a^2 - b^2 \Rightarrow a^2 - b^2 = 5$$

$$\Rightarrow a^2 = b^2 + 5 \rightarrow (i)$$

$$\because\ \ ellipse\ (ii) passes\ through\ \left(\frac{3}{2}, \sqrt{3}\right)$$

$$so\ (ii) \Rightarrow \frac{\left(\frac{3}{2}\right)^2}{a^2} + \frac{(\sqrt{3})^2}{b^2} = 1$$

⇨ $\frac{9}{4a^2} + \frac{3}{b^2} = 1$

⇨ $9b^2 + 12a^2 = 4a^2b^2\ \ \ (\times\ by\ 4a^2b^2)$

⇨ $9b^2 + 12(b^2 + 5) = 4(b^2 + 5)b^2\ \ \ (\because a^2 = b^2 + 5)$

⇨ $9b^2 + 12b^2 + 60 = 4b^4 + 20b^2$

⇨ $21b^2 - 20b^2 + 60 = 4b^2$

Or$4b^4 - b^2 - 60 = 0$

$$4b^2 - 16b^2 + 15b^2 - 60 = 0$$

$$4b^2(b^2 - 4) + 15(b^2 + 15) = 0$$

$$b^2 - 4, 4b^2 + 15 = 0$$

⇨ $b^2 = 4, b^2 = -\frac{15}{4} (impossible)$

$$\because b^2 = 4 \Rightarrow a^2 = 4 + 5\ \ (by (i)$$

⇨ $a^2 = 9$

So(ii) $becomes$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Sketch:

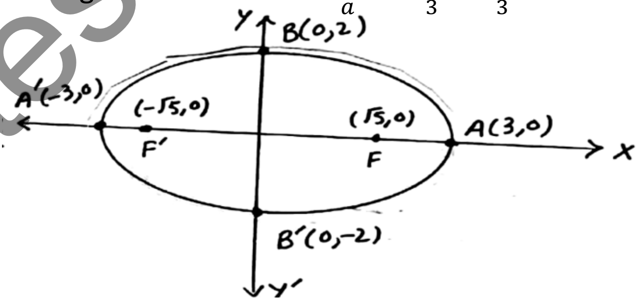
Vertices $= (\pm a, 0) = (\pm 3, o)$

⇨ $A(3,0), A'(-3,0)$

Co-vertices $= (0, \pm b) = (0, \pm 2)$

⇨ $B(0,2), B'(o, -2)$

Length of leturectum : $\frac{2b^2}{a} = \frac{2(4)}{3} = \frac{8}{3}$



vi) ***vertics*** $(o, \pm 5)$***and eccentricity*** $\left(\frac{3}{5}\right)$

Solution:

$$Vertices = (0, \pm a) = (0, \pm 5)$$

⇨ $a = 5, a^2 = 25$

$$centre = midpont\ of\ vertices$$

$$= \left(\frac{0+0}{2}, \frac{5-5}{2}\right) = (0,0)$$

$$\because eccebtricity = l = \frac{3}{5}$$

$$\because c = ae \Rightarrow c = 5\left(\frac{3}{5}\right) = 3$$

$$\Rightarrow c^2 = 9$$

$$\because c^2 = a^2 - b^2 \Rightarrow b^2 = a^2 - c^2 = 25 - 9$$

$$\Rightarrow b^2 = 16$$

$\because$ *vertices have same abscissa so major axis is y − axis. and eq. of ellipse is*

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

⇨ $\frac{x^2}{16} + \frac{y^2}{25} = 1$

Sketch:

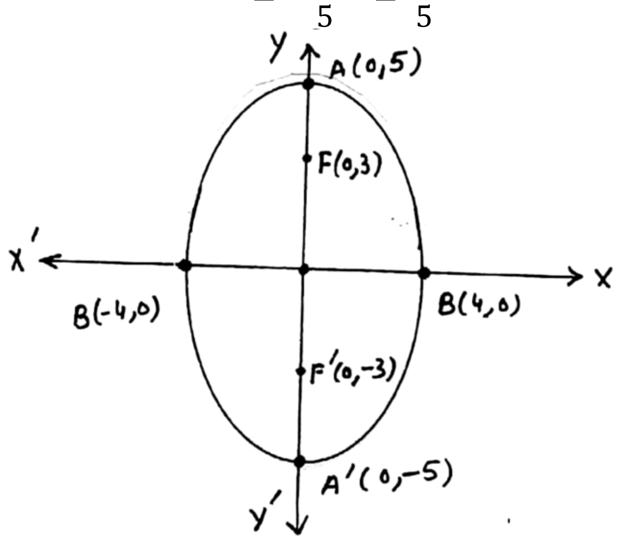
Vertices:$(o,\pm 5) \Rightarrow A(0,5)\ A'(0,-5)$

Covertices;$(\pm b, o) = (\pm 4,0)$

⇨ $B(4,0), B'(-4,0)$

Length $of\ letusrectum$: $\frac{2b^2}{a}$

$$= \frac{2(16)}{5} = \frac{32}{5}$$



(vii) Centre (o,o), $focus$ $(0,-3)$ $and\ Vertex$ $(0,4)$

Solution:

Centre $= (0,0), Focus = f'(0,-3)$

So other focus $F(0,3)$

⇨ $foci = (0,\pm c) = (0,\pm 3)$

⇨ $c = 3 \Rightarrow c^2 = 9$

Also vertex $= (0,4)$

Other vertex $= (0,-4)$

⇨ $vertices = (0,\pm a) = (o,\pm 4)$

⇨ $a = 4\ or\ a^2 = 16$

⇨ $c^2 = a^2 - b^2 \Rightarrow b^2 = a^2 - c^2$

$or\ b^2 = 16 - 9 = 7$

$\because foci\ have\ same\ abscissa\ so\ eq.of\ ellipse\ is$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad (\because major\ is\ y - axis)$$

⇨ $\frac{x^2}{7} + \frac{y^2}{16} = 1$

Sketch:

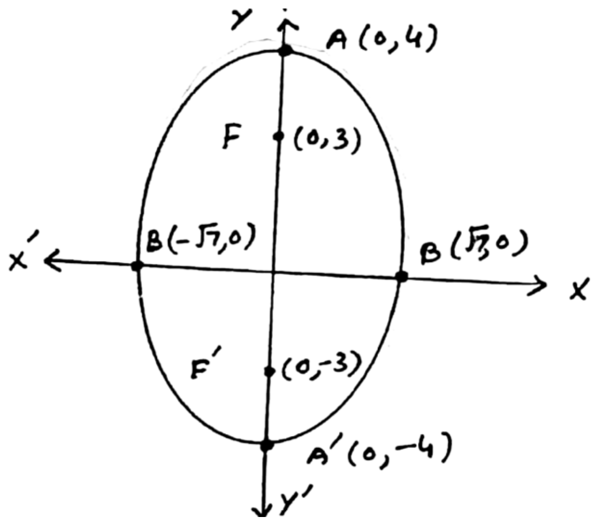
Vertices $= (0,\pm a) = (0,\pm 4)$

⇨ $A(0,4), A'(0,-4)$

Co-vertices= $(\pm b, o) = (\pm\sqrt{7}, 0)$

⇨ $B(\sqrt{7},0), B'(-\sqrt{7},o)$

Length of $letusrectum = \frac{2b^2}{a} = \frac{2(7)}{4} = \frac{7}{2}$



(viii) Centre $(2,2)$ major axis parallel to $y - axis$ and length 8 units, minor axis parallel to $x - axis$ and of length 6 units.

Solution:

Centre $= (h,k) = (2,2)$

Length of major axis $= 2a = 8$

⇨ $a = 4 \Rightarrow a^2 = 16$

Length of minor axis $= 2b = 6$

⇨ $b = 3 \Rightarrow b^2 = 9$

$\because c^2 = a^2 - b^2 = 16 - 9 = 7$

⇨ $c = \sqrt{7}$

Equation of Ellipse is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$(\because major\ axis\ is\ ||to\ y - axis.)$

$Put\ h = 2, k = 2, b^2 = 9, a^2 = 16$

⇨ $\frac{(x-2)^2}{9} + \frac{(y-2)^2}{16} = 1$

Sketch:

Vertices: $(X = 0, Y = \pm a)$

$\because X = x - 2 \Rightarrow x - 2 \Rightarrow x = 2$

$Y = y - 2 \Rightarrow y - 2 = \pm 2$

$\Rightarrow y - 2 = \pm 4 \Rightarrow y = 2 \pm 4$

$or\ y = 6, -2$

$So vertices\ area\ A(2,6), A'(2,-2)$

Covertices: $(X = \pm b, Y = 0)$

$\because X = x - 2 \Rightarrow x - 2 = \pm b$

$\Rightarrow x - 2 = \pm 3 \Rightarrow x = 2 \pm 3$

$\Rightarrow x = 5, -1$

$Y = y - 2 \Rightarrow y - 2 = 0 \Rightarrow y = 2$

$so\ converties\ are\ B(5,2), B'(-1,2)$

Foci: $(X = 0, Y = \pm c)$

$X = x - 2 \Rightarrow x - 2 = 0 \Rightarrow x = 2$

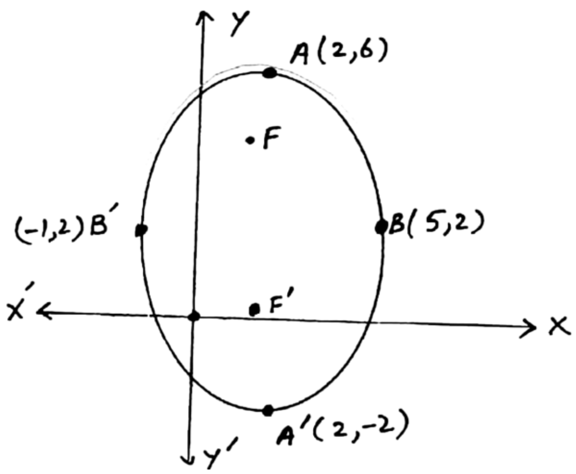
$Y = y - 2 \Rightarrow y - 2 = \pm c$

⇨ $y - 2 = \pm\sqrt{7} \Rightarrow y = 2 \pm \sqrt{7}$

$So Foci\ are : F(2,2+\sqrt{7})$

$$F'\left(2,2-\sqrt{7}\right)$$

Length of *letusrectum*: $\frac{2b^2}{a}=\frac{2(9)}{4}=\frac{9}{2}$



(ix) Centre (0,0) ,symmetric with respect to both the axis and passing through the points $(2,3)$ *and* $(6,1)$

Solution:

Centre $=(0,0)$, *symmetric w.r.t both axis*

Eq. of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\because (i)$ *passses through* $(2,3)$ *so*

$$\frac{(2)^2}{a^2}+\frac{(3)^2}{b^2}=1\Rightarrow\frac{4}{a^2}+\frac{9}{b^2}=1$$

⇨ $4b^2+9a^2=a^2b^2\to(ii)$

Also (i) *passes through* $(6,1)$ *so*

$$\frac{(6)^2}{a^2}+\frac{(1)^2}{b^2}=1\Rightarrow\frac{36}{a^2}+\frac{1}{b^2}=1$$

⇨ $36b^2+a^2=a^2b^2\to(iii$

⇨ By (ii) *and* (iii)

$$36b^2+a^2=4b^2+9a^2$$
$$\Rightarrow 32b^2=8a^2$$
$$\Rightarrow a^2=4b^2\to(iv)\ put\ in\ (ii)$$
$$4b^2+9(4b^2)=(4b^2)b^2$$
$$\Rightarrow 40b^2=4b^4$$
$$or\ 10=b^2(\div\ by\ b^2)$$
$$\Rightarrow b=\sqrt{10}$$
$$(iv)\Rightarrow a^2=4(10)\Rightarrow a^2=40$$
$$or\ a=\sqrt{40}=2\sqrt{10}$$

req. ellipse (i) *becomes as*

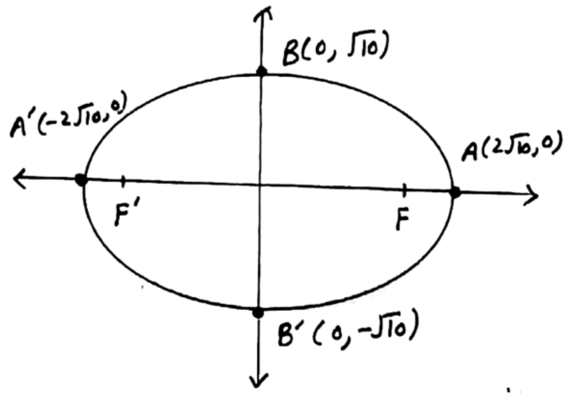
$$\frac{x^2}{40}+\frac{y^2}{10}=1$$

Sketch:

$$vertices:(\pm a,0)=\left(\pm 2\sqrt{10},0\right)$$
$$A\left(2\sqrt{10},o\right),A'\left(-2\sqrt{10},0\right)$$
$$covertices:(0,\pm b)=\left(0,\pm\sqrt{10}\right)$$
$$B\left(0,\sqrt{10}\right),B'\left(o,-\sqrt{10}\right)$$

Length of *letusrectum* $=\frac{2b^2}{a}$

$$=\frac{2(10)}{2\sqrt{10}}=\frac{10}{\sqrt{10}}=\sqrt{10}$$



(x) Centre (0,0) Major axis horizontal, the points $(3,1)$ *and* $(4,0)$ *lie on the graph*.

Solution:

Centre $=(0,0)$

Major*axis horizonal* $(i.e\ x-$ *axis*$)$ *Eq. of ellipse is*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\to(i)$$

$\because (i)$ *passes through* $(3,1)$ *so* $\frac{(3)^2}{a^2}+\frac{(1)^2}{b^2}=1$

⇨ $\frac{9}{a^2}+\frac{1}{b^2}=1$

⇨ $9b^2+a^2=a^2b^2\to(ii)$

Also (i) *passes through* $(4,0)$ *so*

$$\frac{(4)^2}{a^2}+\frac{(0)^2}{b^2}=\frac{9}{a^2}+\frac{1}{b^2}=1$$

⇨ $9b^2+a^2=a^2b^2\to(ii)$

Also (i) *passes through* $(4,0)$ *so*

$$\frac{(4)^2}{a^2}+\frac{(0)^2}{b^2}=1\Rightarrow\frac{16}{a^2}=1$$

⇨ $a^2=16\Rightarrow a=4$

Now $(ii)\Rightarrow 9b^2+16=16b^2$

⇨ $16=16b^2-9b^2$

⇨ $16=7b^2\Rightarrow b^2=\frac{16}{7}$

⇨ $or\ b=\frac{4}{\sqrt{7}}$

$$so\ req.\ ellipse\ (i)becomes\ \frac{x^2}{16}+\frac{y^2}{\frac{16}{7}}=1$$

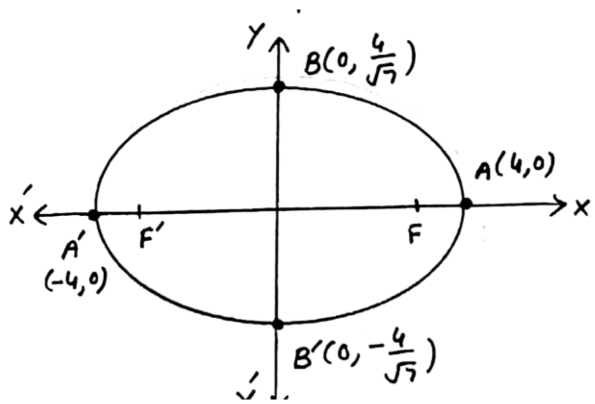
Sketch:

$$Vertices: A(4,0),A'(-4,0)$$

Convertices:

$$B\left(0,\frac{4}{\sqrt{7}}\right),B'\left(0,-\frac{4}{\sqrt{7}}\right)$$

$$Length\ of\ latusrectum=\frac{2b^2}{a}=\frac{2\left(\frac{16}{7}\right)}{4}=\frac{8}{7}$$



Question No.2

Find the Centre, foci, eccentricity, vertices and directrices of the ellipse whose equation is given:

(i) $\boldsymbol{x^2+4y^2=16}$

Solution:

$$x^2+4y^2=16$$

⇨ $\frac{x^2}{16}+\frac{y^2}{4}=1\ (\div\ by\ 16)$

$$compare\ with\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

⇨ $a^2=16\Rightarrow a=4$

$$b^2=4\Rightarrow b=2$$

$$\because c^2=a^2-b^2=16-4=12$$

⇨ $c=\sqrt{12}\Rightarrow c=2\sqrt{3}$

Centre: (0,0)

Foci= $(\pm c,0)=(\pm 2\sqrt{3},0)$

Eccentricity: e=$\frac{c}{a}=\frac{2\sqrt{3}}{4}\ \Rightarrow e=\frac{\sqrt{3}}{2}$

$\boldsymbol{Vertices}:(\pm a,0)=(\pm 4,0)$

$\boldsymbol{Directrices}: x=\pm\frac{a}{e}$

$$\Rightarrow x=\pm\frac{4}{\frac{\sqrt{3}}{2}}\Rightarrow x=\pm\frac{8}{\sqrt{3}}$$

$\boldsymbol{(ii)\ 9x^2+y^2=18}$

$\boldsymbol{solution}$:

$$9x^2+y^2=18$$

⇨ $\frac{x^2}{2}+\frac{y^2}{18}=1\ (\div\ by\ 18)$

$$compare\ with\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

⇨ $a^2=18\Rightarrow a=\sqrt{18}=3\sqrt{2}$

$$b^2=2\Rightarrow b=\sqrt{2}$$

$$\because c^2=a^2-b^2=18-2=16$$

⇨ $c=4$

Centre: (0,0)

Foci= $(o,\pm c)=(o,\pm 4)$

Eccentricity: e=$\frac{c}{a}=\frac{4}{3\sqrt{2}}\ \Rightarrow e=\frac{2\times 2}{3\sqrt{2}}=\frac{2\sqrt{2}}{3}$

$\boldsymbol{Vertices}:(0,\pm a)=(0,\pm 3\sqrt{2})$

$\boldsymbol{Directrices}: y=\pm\frac{a}{e}$

$$\Rightarrow y=\pm\frac{\frac{3\sqrt{2}}{2\sqrt{2}}}{\frac{2\sqrt{2}}{3}}\Rightarrow y=\pm\frac{9}{2}$$

$\boldsymbol{(iii)\ 25x^2+9y^2=225}$

Solution:

$$25x^2+9y^2=225$$

⇨ $\frac{x^2}{9}+\frac{y^2}{25}=1\ (\div\ by\ 225)$

$$compare\ with\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

⇨ $a^2=5\Rightarrow a=\sqrt{5}$

$$b^2=9\Rightarrow b=3$$

$$\because c^2=a^2-b^2=25-9=16$$

⇨ $c=4$

Centre: (0,0)

Foci= $(o,\pm c)=(o,\pm 4)$

Eccentricity: e=$\frac{c}{a}=\frac{4}{5}$

$\boldsymbol{Vertices}:(0,\pm a)=(0,\pm 5)$

$\boldsymbol{Directrices}:\ y=\pm\frac{a}{e}$

$$\Rightarrow y=\pm\frac{5}{\frac{4}{5}}\Rightarrow y=\pm\frac{25}{4}$$

$\boldsymbol{(iv)}\quad \boldsymbol{\frac{(2x-1)^2}{16}+\frac{(y+2)^2}{16}=1}$

Solution:

$$\frac{(2x-1)^2}{16}+\frac{(y+2)^2}{16}=1$$

⇨ $\frac{\left[2\left(x-\frac{1}{2}\right)\right]^2}{16}+\frac{(y+2)^2}{16}=1\rightarrow(i)$

$$compare\ with\ \frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$$

⇨ $X=x-\frac{1}{2}\ ,Y=y+2$

⇨ $a^2=16\Rightarrow a=4$

$$b^2=4\Rightarrow b=2$$

$$\because c^2=a^2-b^2=16-2=12$$

⇨ $c=\sqrt{12}=2\sqrt{3}$

Centre: $(X=0,Y=0)$

$$\because\ X=x-\frac{1}{2}\Rightarrow x-\frac{1}{2}=0\Rightarrow x=\frac{1}{2}$$

⇨ $so\ centre\ \left(\frac{1}{2},-2\right)$

Foci= $Major\ axis\ is\ along\ y-axis$

Foci $(\boldsymbol{X=0,Y=\pm c})$

$$\because\ X=x-\frac{1}{2}\Rightarrow x-\frac{1}{2}=0\Rightarrow x=\frac{1}{2}$$

$$Y=y+2\ \Rightarrow y+2\Rightarrow y+2=\pm c$$

⇨ $y+2=\pm 2\sqrt{3}$

⇨ $y=-2\pm 2\sqrt{3}$

Hence $foci\ \left(\frac{1}{2}, -2 \pm 2\sqrt{3}\right)$

Eccentricity: $e = \frac{c}{a} = \frac{2\sqrt{3}}{4} \Rightarrow e = \frac{3}{\sqrt{2}}$

Vertices: $(X = 0, Y = \pm a)$

$$\because\ X = x - \frac{1}{2} \Rightarrow x - \frac{1}{2} = 0 \Rightarrow x = \frac{1}{2}$$

$$Y = y + 2 \Rightarrow y + 2 = \pm a \Rightarrow y + 2 = \pm 4$$

⇨ $Y = y + 2 \Rightarrow y + 2 = \pm a \Rightarrow y + 2 = \pm 4$

⇨ $y = -2 \pm 4 \Rightarrow y = 2, -6$

Hence vertices are $\left(\frac{1}{2}, 2\right), \left(\frac{1}{2}, -6\right)$

$$\boldsymbol{Directrices}: y = \pm\frac{a}{e} \Rightarrow y + 2 = \pm\frac{4}{\frac{\sqrt{3}}{2}}$$

$$\boldsymbol{y + 2} = \pm\frac{8}{\sqrt{3}} \Rightarrow y = -2 \pm \frac{8}{\sqrt{3}}$$

$\boldsymbol{(ii) x^2 + 16x + 4y + 76 = 0}$

Solution:

$$\boldsymbol{x^2 + 16x + 4y^2 - 16y + 76 = 0}$$

⇨ $\boldsymbol{x^2 + 16x + 4(y^2 - 4y) = -76}$

⇨ $x^2 + 2(8)(x) + (8)^2 + 4(y^2 - 4y + 4 - 4) = -76 + (8)^2$

⇨ $(x + 8)^2 + 4(y - 2)^2 - 16 = -12$

⇨ $(x + 8)^2 + 4((y - 2)^2 - 4) = -76 + 64$

⇨ $(x + 8)^2 + 4(y - 2)^2 - 16 = -12$

⇨ $(x + 8)^2 + 4(y - 2)^2 = 4$

⇨ $\frac{(x+8)^2}{4} + \frac{(y-2)^2}{1} = 1$

⇨ $compare\ with \frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$

⇨ $X = x + 8,\ \ Y = y - 2 \qquad a^2 = 4$

⇨ $a = 2 \because c^2 = a^2 - b^2 \Rightarrow c^2 = 4 - 1 = 3$

⇨ $c = \sqrt{3}$

⇨ $b^2 = 1 \Rightarrow b = 1$

Centre: $(X = 0, Y = 0)$

$$\because\ X = x + 8 \Rightarrow x + 8 = 0 \Rightarrow x = -8$$

$$Y = y - 2 \Rightarrow y - 2 = 0 \Rightarrow y = 2$$

⇨ $so\ centre\ (-8,2)$

Foci= $Major\ axis\ is\ along\ x - axis$

Foci $(\boldsymbol{X = \pm c, Y = 0})$

$$\because\ X = x + 8 \Rightarrow x + 8 = \pm c$$

$$x + 8 = \pm\sqrt{3} \Rightarrow x = -8 \pm \sqrt{3}$$

$$Y = y - 2 \Rightarrow y - 2 \Rightarrow y = 2$$

Hence $foci\ \left(-8 \pm \sqrt{3}, 2\right)$

Eccentricity: $e = \frac{c}{a} \Rightarrow e = \frac{\sqrt{3}}{2}$

Vertices: $(X = \pm a, Y = 0)$

$$\because\ X = x + 8 \Rightarrow x + 8 = \pm a$$

$$\Rightarrow x + 8 = \pm 2 \Rightarrow x = -8 \pm 2$$

$$\Rightarrow x = -6, -10$$

$$Y = y - 2 \Rightarrow y - 2 \Rightarrow y = 2$$

Hence vertices are $(-6,2), (-10,2)$

$Directrices: X = \pm\frac{a}{e}$

$$X = x + 8 \Rightarrow x + 8 = \pm\frac{a}{e}$$

⇨ $x + 8 = \pm\frac{2}{\frac{\sqrt{3}}{2}}$

⇨ $x = -8 \pm \frac{4}{\sqrt{3}}$

$$\boldsymbol{(vi)\ 25x^2 + 4y^2 - 250x - 16y + 541 = 0}$$

Solution:

$$25x^2 + 4y^2 - 250x - 16y + 541 = 0$$

$$25(x^2 - 10x) + 4(y^2 - 4y) = -541$$

$$25(x^2 - 10x + 25 - 25) + 4(y^2 - 4y + 4 - 4) = -541$$

$$25[(x - 5)^2 - 25] + 4[(y - 2) - 4] = -541$$

$$25(x - 5)^2 - 625 + 4(y - 2)^2 - 16 = -541$$

$$25(x - 5)^2 + 4(y - 2)^2 = -451 + 625 + 16$$

$$\Rightarrow 25(x - 5)^2 + 4(y - 2)^2 = 100$$

$$or\ \frac{(x - 5)^2}{4} + \frac{(y - 2)^2}{25} = 1\ \ (\div\ by\ 100)$$

⇨ $compare\ with \frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$

⇨ $X = x - 5,\ \ Y = y - 2 \qquad a^2 = 25$

⇨ $a = 5 \because c^2 = a^2 - b^2 \Rightarrow c^2 = 25 - 4 = 21$

⇨ $c = \sqrt{21}$

⇨ $b^2 = 4 \Rightarrow b = 2$

Centre: $(X = 0, Y = 0)$

$$\because\ X = x - 5 \Rightarrow x - 5 = 0 \Rightarrow x = 5$$

$$Y = y - 2 \Rightarrow y - 2 = 0 \Rightarrow y = 2$$

⇨ $so\ centre\ (5,2)$

Foci= $Major\ axis\ is\ along\ y - axis$

Foci $(\boldsymbol{X = 0, Y = \pm c})$

$$\because\ X = x - 5 \Rightarrow x - 5 = x = 5$$

$$Y = y - 2 \Rightarrow y - 2 \Rightarrow y = 2$$

Hence $foci\ \left(5, 2 \pm \sqrt{21}\right)$

Eccentricity: $e = \frac{c}{a} \Rightarrow e = \frac{\sqrt{21}}{5}$

Vertices: $(X = \pm 0, Y = a)$

$$\because\ X = x + 5 \Rightarrow x + 5$$

$$x = -5$$

$$Y = y - 2 \Rightarrow y - 2 = a \Rightarrow y - 2 = \pm 5$$

$$\Rightarrow y = 2 \pm 5 \Rightarrow y = -3, 7$$

Hence vertices are $(5,7), (7,-3)$

$Directrices: Y = \pm\frac{a}{e}$

$$Y = y - 2 \Rightarrow y - 2 = \pm\frac{a}{e}$$

⇨ $y - 2 = \pm\frac{5}{\frac{\sqrt{21}}{5}} \qquad \begin{pmatrix} \because a = 5 \\ e = \frac{\sqrt{21}}{5} \end{pmatrix}$

⇨ $y = 2 \pm 25/\sqrt{21}$

Question No.3 let "a" be a positive number and $0 < c < a$. let $F(c,o)$ and $F'(-c,0)$ be two given points. prove that the locus of points $P(x,y)$ such that $|PF| + |PF'| = 2a$ is an Ellipse.

Solution:

Given that

$$|PF'| + |PF| = 2a \quad F'(-c,o)\ F(c,o), P(x,y)$$

⇨ $\sqrt{(x+c)^2 + (y-o)^2} = \sqrt{(x-c)^2 + (y-o)^2} = 2a$

$\Rightarrow \sqrt{(x+c)^2 + (y)^2} = 2a - \sqrt{(x-c)^2 + (y)^2}$

Squaring both sides by

⇨ $x^2 + c^2 + 2cx + y^2 = 4a^2 + x^2 + c^2 - 2cx + y^2 - 4a\sqrt{(x-c)^2 + (y)^2}$

⇨ $2cx + 2cx - 4a^2 = -4a\sqrt{x^2 + c^2 - 2cx + y^2}$

⇨ $4(cx - a^2) = -4a\sqrt{x^2 + y^2 + c^2 - 2cx}$

⇨ $cx - a^2 = -a\sqrt{x^2 + y^2 + c^2 - 2cx}$

⇨ $a^2 - cx = a\sqrt{x^2 + y^2 + c^2 - 2cx}$

Again squaring

$$a^2 + c^2x^2 - 2a^2cx = a^2(x^2 + y^2 + c^2 - 2cx)$$

$$a^4 + c^2x^2 - 2a^2cx = a^2x^2 + a^2y^2 + a62c^2 - 2a^2cx$$

⇨ $c^2a^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

$$\div \text{ by } a^2(a^2 - c^2)$$

⇨ $\frac{x^2}{a^2(a^2-c^2)} + \frac{y^2}{a^2-c^2} = 1$

⇨ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\because a^2 - b^2 = c^2)$

$$\Rightarrow (a^2 - c^2 = b^2)$$

Which is an ellipse. Hence proved.

Question No.4 let a *be a positive number and $0 < c < a$. let $F(0,0)$ and $F'(1,1)$ be two given points $P(x,y)$ is an ellipse, such that $|PF| + |PF'| = 2$ find equation of ellipse.*

Solution:

$$0 < c < a$$

$$P(x,y), F(0,0), F'(1,1)$$

Given that

$$|PF| + |PF'| = 2$$

⇨ $\sqrt{(x-0)^2 + (y-o)^2} = \sqrt{(x-1)^2 + (y-1)^2}$

⇨ $\sqrt{x^2 + y^2} + \sqrt{x^2 + 1 - 2x + y^2 + 1 - 2y} = 2$

⇨ $\sqrt{x^2 + y^2} + \sqrt{x^2 + y^2 - 2x - 2y + 2} = 2$

⇨ $\sqrt{x^2 + y^2 - 2x - 2y + 2} = 2 - \sqrt{x^2 + y^2}$

squring both sides

$$x^2 + y^2 - 2x - 2y + 2 = 4 + x^2 + y^2 - 4\sqrt{x^2 + y^2}$$

$$-2x - 2y + 2 = 4 - 4\sqrt{x^2 + y^2}$$

$$\Rightarrow 4\sqrt{x^2 + y^2} = 2 + 2x + 2y$$

$$4(x^2 + y^2) = x^2 + y^2 + 1 + 2xy + 2y + 2x = 0$$

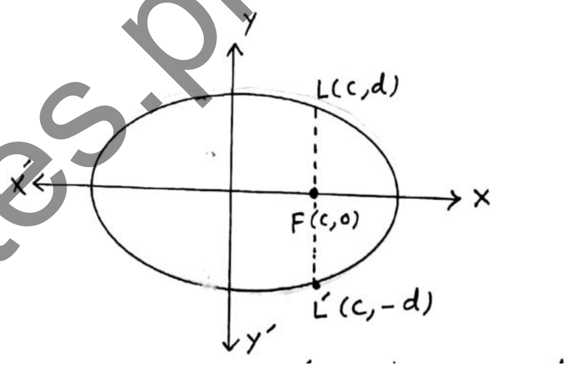
⇨ $4x^2 + 4y^2 - x^2 - y^2 - 2xy - 2x - 2y - 1 = 0$

$$or\ \ 3x^2 + 3y^2 - 2x - 2y - 2xy - 1 = 0$$

Which is ellipse.

Question No.5 prove that letusrectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$

Proof:



Let

$L(c,d)$ and $L'(c,d)$ be points of Letusrectum LL' of

Given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow (i)$

$$\because Length\ of\ letusrectum = |LL'| = 2d$$

$$\because L(c,d) lies\ on\ (i) so$$

$\frac{c^2}{a^2} + \frac{d^2}{b^2} = 1$

⇨ $c^2b^2 + a^2d^2 = a^2b^2$

⇨ $a^2d^2 = a^2b^2 - b^2c^2$

⇨ $a^2d^2 = b^2(a^2 - c^2)$

⇨ $a^2d^2 = b^2(b^2)$

$$\because\ c^2 = a^2 - b^2$$

$$\Rightarrow b^2 = a^2 - c^2$$

$$or\ d^2 = \frac{b^4}{a^2} \Rightarrow d = \sqrt{\frac{b^4}{a^2}}$$

$$\Rightarrow d = \frac{b^2}{a}\ \ so$$

$$Length\ of\ letusrectum = 2d = \frac{2b^2}{a}$$

Hence proved.

Question No.6 the major axis of an ellipse in standard form lies along the x-axis and has length of the minor axis. Write an equation of ellipse.

Solution:

Given $2a = 4\sqrt{2}$

⇨ $a = 2\sqrt{2}$

⇨ $a^2 = 8$

$$also\ given\ that\ \ 2c = 2b \Rightarrow c = b$$

$$Now\ using\ c^2 = a^2 - b^2$$

$$b^2 = a^2 - b^2$$

$$\Rightarrow 2b^2 = a^2$$

$$\Rightarrow 2b^2 = 8 \Rightarrow b^2 = 4$$

$$Thus\ eq.of\ ellipse\ is \frac{x^2}{8} + \frac{y^2}{4} = 1$$

Question No.7 an asteroid has elliptic orbit with the sun at one focus, its distance from the sun ranges 17milions to 183 million miles.

Write an equation of the orbit of the asteroid.

Solution:

$$a - c = 17 \rightarrow (i)$$

$$a + c = 183 \rightarrow (ii)$$

By$(i) + (ii)$

$$2a = 200 \Rightarrow a = 100$$

$$(ii) \Rightarrow 100 + c = 183 \Rightarrow c = 83$$

Using $c^2 = a^2 - b^2$

$$b^2 = a^2 - c^2$$

$$b^2 = (100)^2 - (83)^3$$

$$b^2 = 10000 - 6889$$

$$b^2 = 3111$$

$$\because the\ eqution\ of\ elliptic\ orbits\ is$$

$$\frac{x^2}{10{,}000} + \frac{y^2}{3111} = 1$$

Question No.8 An arch in the shape of a semi –ellipse is 90m wide at the base and 30m high at the Centre. At what distance from the Centre is the arch $20\sqrt{2}\boldsymbol{m\ high}$?

Solution:

Here $2a = 90 \Rightarrow a = 45\ \ and\ b = 30$

$\because Eq.of\ ellipse\ is$

$$\frac{x^2}{(45)^2} + \frac{y^2}{(30)^2} = 1 \rightarrow (i)$$

At the high

$20\sqrt{2}m\ let\ x_1\ be\ the\ distance\ from$ $the\ Centre\ then\ the\ point\ \ (x_1, 20\sqrt{2})lies\ on$ Ellipse (i)

$$\frac{x_1^2}{(45)^2} + \frac{(20\sqrt{2})^2}{30} = 1$$

⇨ $\frac{x_1^2}{2025} + \frac{800}{900} = 1$

⇨ $\frac{x_1^2}{2025} = 1 - \frac{800}{900}$

⇨ $\frac{x_1^2}{2025} = 1 - \frac{8}{9}$

⇨ $\frac{x_1^2}{2025} = \frac{1}{9} \Rightarrow x_1^2 = \frac{2025}{9}$

⇨ $x_1^2 = 225 \Rightarrow x_1 = \pm 15$

⇨ $x_1 = 15m\ (neglect - ve\ value\ of\ x_1)$

$$\because Req\ distance\ from\ centre = 15m$$

Question No9. The moon orbits the earth in an elliptic path with the earth at one focus. The major and minor axes of the orbit are $\boldsymbol{7, 68, 806\ km\ and}$ $\boldsymbol{7, 6, 7, 746km\ respectively, . Find\ \ the\ greatest}$ And least distance.

Solution:

Let the earth be at F

Given that $2a = 768{,}806$

$$\Rightarrow a = 384403\ km$$

$$2b = 767{,}746$$

⇨ $b = 383873km$

Using $c^2 = a^2 - b^2$

⇨ $c^2 = (a - b)(a + b)$

⇨ $c^2 = (530)(768276)$

⇨ $c^2 = 407186280$

⇨ $c = 20178.86$

Now Required greatest distance

$$= a + c = 404582\ km(Approx)$$

$$and\ least\ distance = a - c$$

$$= 364224km(Approx)$$

Hyperbola

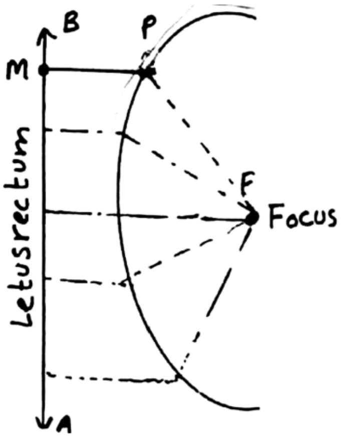
“A set of all points in a plane such that distance of each point from a fixed point bars a constant ratio (greater than one) to a distance from a fixed line.”

Note:

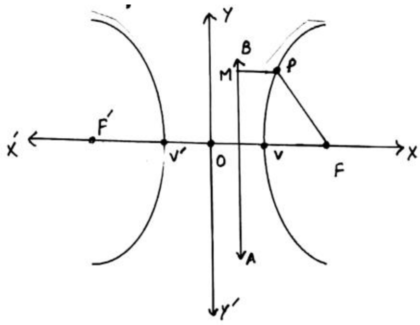
- Fixed point is called focus.
- Fixed line is called directrix.
- Constant ratio is called eccentricity, dented by e.
- In fig. $\frac{|FP|}{|PM|} = e$
- $|FP| = e|PM|\ where\ |FP| > |PM|$

⇨ $\frac{|FP|}{|PM|} > 1$

$$so\ for\ hyperbola\ e > 1$$

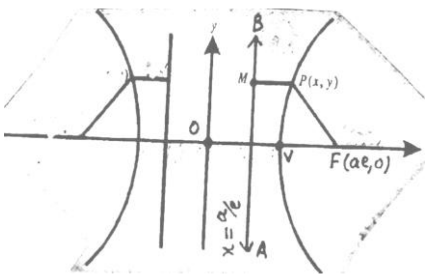


Definition:



1) The midpoint of line segment joining the foci is called Centre of hyperbola. In fig. F and F' are foci and O is Centre.
2) The line passing through the foci of hyperbola is called focal axis or Transverse axis.
3) The line passing through Centre of hyperbola and perpendicular to transverse axis is called conjugate axis. In fig. y-axis is conjugate axis.
4) The points where the hyperbola meets its transverse axis are called vertices of hyperbola.in fig. V and V' are $vertices$ of $hyperbola$.
5) Length of $leturectum$ is $\frac{2b^2}{a}$
6) If the point $asec\theta, bTan\theta)$ lies on hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $where$ $\theta \in R$ $then$ $x = asec\theta$ $y = atan\theta$ are $parametric$ $equations$ of $hyperbola$.

Standard Equation of hyperbola:



Let $F(ae, o)$ be $focus$ and AB be $directed$ $with$ $equation$ be $directrix$ $with$ $equation$ $x = \frac{a}{e} \Rightarrow ex - a = 0$ of $req.hyperbola$.

Let $|PM|$ be $\perp$ ar $distance$ of any $point$ $P(x, y)on$ Hyperbola to the $directrix$ $ex - a = 0$

By $def.of$ $hyperbola$

$$\frac{|FP|}{|PM|} = e \Rightarrow |FP| = e|PM|$$

⇨ $\sqrt{(x-ae)^2+(y-0)^2} = e\frac{|ex-a|}{\sqrt{(e)^2+(o)^2}}$

⇨ $\sqrt{x^2+a^2e^2-2aex+y^2} = \frac{e|ex-a|}{e}$

⇨ or $x^2+a^2e^2-2aex+y^2 = |ex-a|^2$

⇨ $x^2+a^2e^2-2aex+y^2 = e^2x^2+a^2-2aex$

⇨ $a^2e^2-a^2+y^2 = e^2x^2-x^2$

⇨ $a^2(e^2-1) = (e^2-1)x^2-y^2$

⇨ $(e^2-1)x^2 - \frac{y^2}{a^2(e^2-1)} = 1$ $\div$ by $a^2(e^2-1)$

Or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ let $a^2(e^2-1) = b^2$

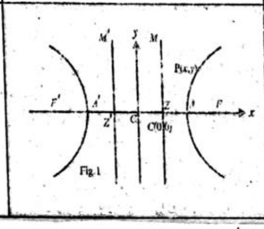
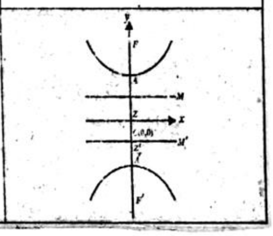
Note:

$\because b^2 = a^2(e^2-1)$

$\Rightarrow b^2 = a^2e^2-a^2 = (ae)^2-a^2$

$\Rightarrow b^2 = c^2-a^2$ $\because ae = c$ or $c^2 = a^2+b^2$

Summary of Standard Hyperbolas

Equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Directrices	$x=\pm\frac{a}{e}$ or $x=\pm\frac{c}{e^2}$	$y=\pm\frac{a}{e}$ or $y=\pm\frac{c}{e^2}$
Transverse axis	$y=0$	$x=0$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Eccentricity	$e=\frac{c}{a}>1$	$e=\frac{c}{a}>1$
Centre	(0, 0)	(0, 0)
Graph		

Exercise 6.6

Question No.1 find an equation of hyperbola with the given data, and sketch its graph

(i) ***Centre*** $(0,0)$, ***focus*** $(6,0)$ ***Vertex*** $(4,0)$

Solution:

Centre

$(0,0), Focus\ (6,0)\ so\ other\ focus\ (-6,0)$

Foci $(\pm c, 0) = (\pm 6, 0) \Rightarrow c = 6$

⇨ $c^2 = 36$

Vertex= $(4,0), other\ vertex(-4,0)$

Vertices= $(\pm a, 0) = (\pm 4, 0) \Rightarrow a = 4$

⇨ $a = 16$

$$\because c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2$$

$$\Rightarrow b^2 = 36 - 16 \Rightarrow b^2 = 20 \Rightarrow b = \sqrt{20}$$

clearly from foci and vertces x − axis is transvrerse axis

So, eq.is

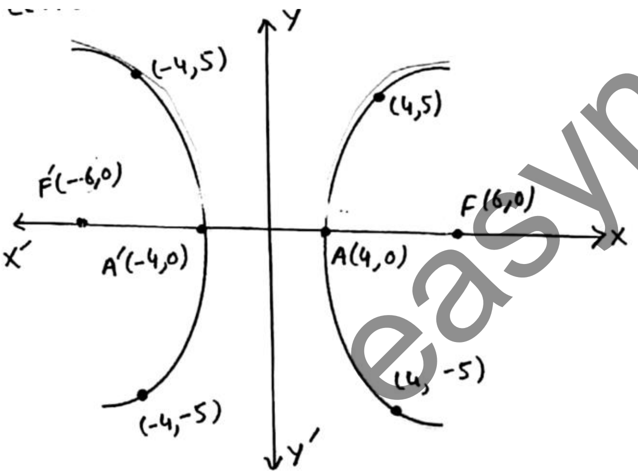
$$\frac{x^2}{a2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{16} - \frac{y^2}{20} = 1$$

Sketch:

$A(4,0), A'(-4,0), F(6,0), F'(-6,0)$

Length of $llletusrectum = \frac{2b^2}{a} = \frac{2(20)}{4} = 10$

$$Centre = (0,0)$$



(ii)***Foci*** $(\pm 5, 0)$, ***Vertex***$(3,0)$

Solution:

$Foci\ (\pm c, 0) = (\pm 5, 0) i.e\ F(5,0), F'(-5,0)$

⇨ $c = 5\ , c^2 = 25$

Centre $=$ ***midpoint of foci***

$$= \left(\frac{5-5}{2}, \frac{0+0}{2}\right) = (0,0)$$

$$vertex = (3,0), other\ vertex = (-3,0)$$

$$vertex\ (\pm a, o) = (\pm 3, 0) \Rightarrow a = 3$$

$$\boldsymbol{a^2} = 9 \quad \because c^2 = a^2 + b^2$$

⇨ $b^2 = c^2 - a^2 = 25 - 9 = 16$

Clearly from foci and vertices $x - axis\ is$

Transverse axis. So eq. is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

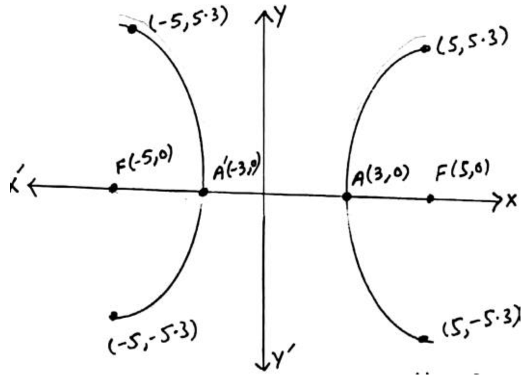
⇨ $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Sketch:

Vertices$F\ (5,0), F'(-5,0)$

Length of $letusrectum$: $\frac{2b^2}{a} = \frac{2(16)}{3} = \frac{32}{3}$

$$Centre = (0,0)$$



(iii)

Foci $\left(\boldsymbol{2 \pm 5\sqrt{2}, -7}\right)$, ***length of transverse axis*** $\mathbf{10}$

Solution:

$$Foci\ \left(2 \pm 5\sqrt{2}, -7\right) \Rightarrow F\left(2 + 5\sqrt{2}, -7\right)$$

$$F'\left(2 - 5\sqrt{2}, -7\right)$$

Centre $= midpoint\ of\ foci$

$$(h, k) = \frac{2 + 5\sqrt{2}, +2 - 5\sqrt{2}}{2}, \frac{-7-7}{2} = (2, -7)$$

Length of transverse axis $F\left(2 + 5\sqrt{2}, -7\right)$

And $Centre\ (2, -7) is$

$$c = \sqrt{\left(2 + 5\sqrt{2} - 2\right)^2 + (-7+7)^2} = 5\sqrt{2}$$

⇨ $c^2 = \left(5\sqrt{2}\right)^2 = 25(2) \Rightarrow c^2 = 50$

$$\because \ c^2 = a^2 + b^2 \ \Rightarrow b^2 = c^2 - a^2$$

$$= 50 - 25$$

$$\Rightarrow b^2 = 25$$

foci have same ordinate so eq. of hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

⇨ $\frac{(x-2)^2}{25} - \frac{(y+7)^2}{25} = 1$

$$compare\ with\ \frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$$

$$X = x - 2, Y = y + 7, \qquad a^2 = 25, b^2 = 25$$

⇨ $a = 5, b = 5$

Sketch:

$$vertics\ (X = \pm a, Y = 0)$$

$$X = x - 2 \Rightarrow x - 2 = \pm 5 \Rightarrow x = 2 \pm 5$$

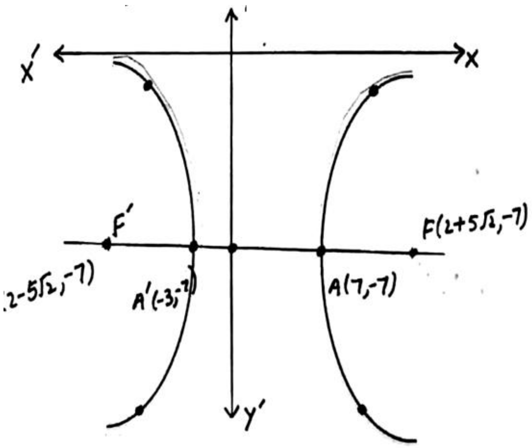
$$x = 7, -3$$

$$Y = y + 7 \Rightarrow y + 7 = 0 \Rightarrow y = -7$$

$$so\ vertices\ (7, -7), (-3, -7)$$

$$foci\ (X = \pm c, Y = 0)$$

$$X = x - 2 \Rightarrow x - 2 = \pm x$$
$$x = 2 \pm 5\sqrt{2}$$
$$y = 0 \Rightarrow y + 7 = 0y = -7$$
$$so\ foci\ \left(2 + 5\sqrt{2}, -7\right), F'\left(2 - 5\sqrt{2}, -7\right)$$
$$length\ of\ letusretum : \frac{2b^2}{a} = \frac{2(25)}{5} = 10$$



(*iv*) *Foci* (0, ±6), *e* = 2

Solution:

$$foci = (0, \pm c) = (0, \pm 6)$$

⇨ $c = 6 \Rightarrow c^2 = 36$

Centre $= midpoint\ of\ foci$

$$= \left(\frac{0 - 0}{2}, \frac{6 - 6}{2}\right) = (0,0)$$
$$\because e = 2 \Rightarrow e = \frac{c}{a}\ or\ a = \frac{c}{e}$$
$$\Rightarrow a = \frac{c}{2} = 3 \quad \Rightarrow a^2 = 9$$
$$\because c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2$$
$$\Rightarrow b^2 = 36 - 9 = 27$$
$$\because e\ foci\ have\ same\ abscissa.$$

So $y -$ $axis\ is\ trasverse\ axis. so\ eq. of\ hyperbola$

Is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \Rightarrow \frac{y^2}{9} - \frac{x^2}{27} = 1$

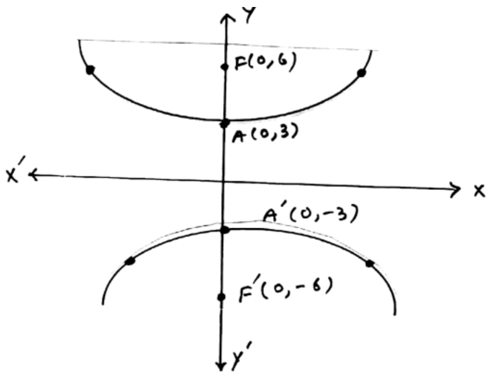
Sketch:

Vertices: $A(0,3), A'(0,-3)$

Foci: $F(0,0), \quad F'(0,-6)$

Length of $letusrectum = \frac{2b^2}{a} = \frac{2(37)}{3}$

$= 18\ and\ centre\ (0,0)$



(*v*) $Foci\ (0, \pm 9), directed\ y = \pm 4$

Solution:

$$foci = (0, \pm c) = (0, \pm 9)$$

⇨ $c = 9 \Rightarrow c^2 = 81$

Centre $= midpoint\ of\ foci$

$$= \left(\frac{0 + 0}{2}, \frac{9 - 9}{2}\right) = (0,0)$$
$$Directrices: y = \pm 4$$
$$\because \quad y = \pm \frac{a}{3}$$
$$\Rightarrow \frac{a}{e} = 4 \Rightarrow a = 4e \rightarrow (i) also$$

Also $\quad c = ea$

$$ae = 9 \ (\because c = 9)$$

⇨ $e = \frac{9}{a}\ put\ \ in\ (i)$

⇨ $a = 4\left(\frac{9}{a}\right) \Rightarrow a^2 = 36 \Rightarrow a = 6$

⇨ $c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2$

⇨ $b^2 = 81 - 36 = 45$

$\because foci\ have\ same\ abscissa\ so\ y - axis\ is\ transverse\ axis.$

Thus eq. of req. hyperbola is

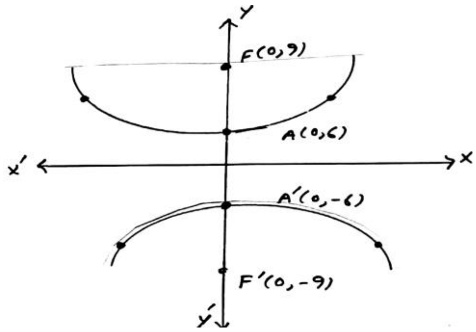
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \Rightarrow \frac{y^2}{36} - \frac{x^2}{45} = 1$$

Sketch:

Vertices $A(0,6), A'(0,-6)\ \ Fcoi: F(0,9)$

Length of $letusrectum : \frac{2b^2}{a} = \frac{2(45)}{6}$

$= 15 \quad centre (0,0)$



(*vi*) ***Centre* (2, 2), *horizontal transverse axis of***

Length 6 and eccentricity *e* = 2

Solution:

Centre $= (h, k) = (2,3)$

$$length\ of\ transerse\ axis = 2a = 6$$

⇨ $e = 2 \quad \because c = ea \ \Rightarrow c = (3)(2) = 6$

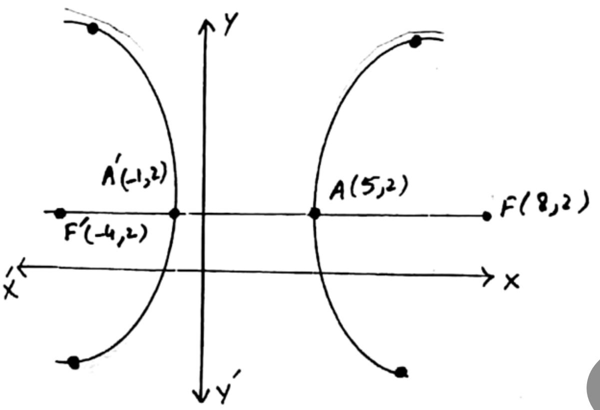
$$or\ c^2 = 36 \ \because c^2 = a^2 + b^2$$
$$\Rightarrow b^2 = c^2 - a^2 = 36 - 9 = 27$$

$\because$

$transverse\ axis\ is\ horizonatal\ axis\ i. e$
$x - axis\ so\ eq. of\ required\ hyperbola\ is$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$
$$\Rightarrow \frac{(x-2)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$sketch:$$
$$vertices: (X = \pm a, Y = 0)$$
$$X = x - 2 \Rightarrow x = 2 = \pm a$$
$$\Rightarrow x - 2 = \pm 3 \Rightarrow x = 2 \pm 3$$
$$or\ x = 5, -1$$
$$Y = y - 2 \Rightarrow y - 2 = 0 \Rightarrow y = 2$$
$$so\ vertices\ A(5,2), A'(-1,2)$$
$$Foci: (X = \pm c, Y = 0)$$
$$X = x - 2 \Rightarrow x - 2 = \pm 6$$
$$\Rightarrow x = 2 \pm 6 \Rightarrow x = 8, -4$$
$$Y = 0 \Rightarrow y - 2 \ \Rightarrow y = 0$$
$$so\ foci\ are\ F(8,2). F'(-4,2)$$
$$Centre: (2,2)$$
$$length\ of\ letusrectum = \frac{2b^2}{a} = \frac{2(27)}{3}$$
$$= 18$$



(vii) ***Vertices*** $(\mathbf{2}, \pm\mathbf{3}), (\mathbf{0}, \mathbf{5})$ ***lies on the curve.***

solution:

$$vertices = (2, \pm 3) \Rightarrow A(2,3), A'(2,-3)$$

Centre: $midpoint\ of\ vertices$

$$(h,k) = \left(\frac{2+2}{2}, \frac{3-3}{2}\right) = (2,0)$$
$$\because\ Distance\ b\backslash w\ vertices = 2a$$
$$\Rightarrow 2a = \sqrt{(2-2)^2 + (-3-3)^2} = \sqrt{36} = 6$$
$$\Rightarrow a = 3 \Rightarrow a^2 = 9$$

$\because$ *vertices have same abscssa so transverse axis is along* $y-$ *axis. so eq. of hyperbola is*

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

⇨ (i) *passes throuh* $(0,5)$ *so*

$$(i) \quad \frac{(5)^2}{9} - \frac{(0-2)^2}{b^2} = 1$$

⇨ $\frac{25}{9} - \frac{4}{b^2} = 1 \Rightarrow \frac{4}{b^2} = \frac{25}{9} - 1$

⇨ $\frac{4}{b^2} = \frac{16}{9} \Rightarrow 36 = 16b^2$

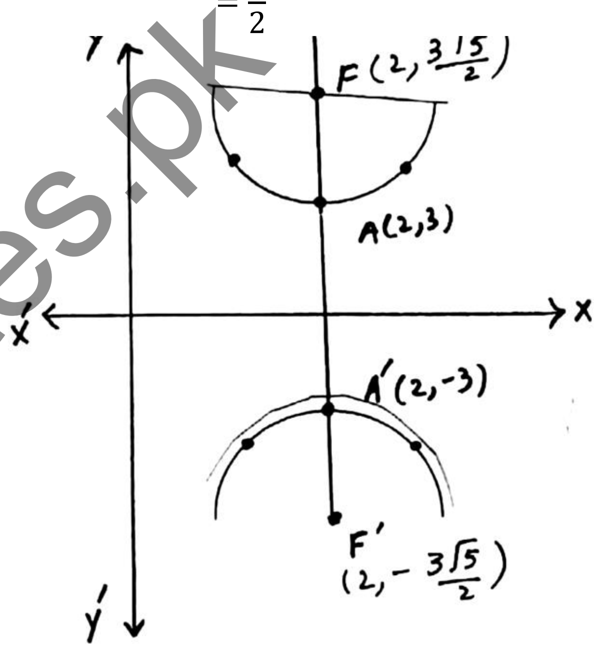
⇨ $b^2 = \frac{36}{16} = \frac{9}{4}$

$$\because \quad c^2 = a^2 + b^2 \Rightarrow c^2 = 9 + \frac{9}{4}$$

$$\Rightarrow c^2 = \frac{45}{4} \Rightarrow c = \sqrt{\frac{45}{4}} = \frac{3\sqrt{2}}{2}$$
$$so\ (i) \Rightarrow \frac{y^2}{9} - \frac{(x-2)^2}{\frac{9}{4}} = 1$$

Sketch:

$$vertices: \ (X = 0, Y = \pm a)$$
$$X = x - 2 \Rightarrow x - 2 = 0 \Rightarrow x = 2$$
$$\Rightarrow Y = y \Rightarrow y = \pm c \Rightarrow y = \pm\frac{3\sqrt{5}}{2}$$
$$\boldsymbol{so}\ foci\ F\left(2, \frac{3\sqrt{5}}{2}\right), F'\left(2, -\frac{3\sqrt{5}}{2}\right)$$
$$\boldsymbol{lehgth}\ of\ letusrectum: \frac{2b^2}{a} = \frac{2\left(\frac{9}{4}\right)}{2} = \frac{6}{4}$$
$$= \frac{3}{2}$$



$(viii)$ *Foci* $(5,-2), (5,4)$ *and one vertex* $(5,3)$

$$Foci\ : F'(5,-2), F(5,4)$$

Centre= $(h,k) = midpoit\ of\ foci$

$$= \left(\frac{5+5}{2}, \frac{4-2}{2}\right) = (5,1)$$
$$\because\ one\ vertex = (5,3)$$
$$\because\ Distance\ b\backslash w\ foci = 2c$$
$$2c = \sqrt{(5-5)^2 + (4+2)^2} = 6$$
$$c = 3 \Rightarrow c^2 = 9$$
$$\because c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2$$
$$b^2 = 9 - 4 = 5$$

as foci have same abscissa so transverse axis is along $y-$ *axis. so eq is*

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$
$$\frac{(y-1)^2}{4} - \frac{(x-5)^2}{5} = 1$$

Sketch:

$$Vertices: (X = 0, Y = \pm a)$$

$$X = x - 5 \Rightarrow x - 5 = 0 \Rightarrow x = 5$$

$$Y = y - 1 \Rightarrow y - 1 = \pm a \Rightarrow y - 1 = \pm 2$$

$$y = 1 \pm 2 \Rightarrow y - 1 = \pm a \Rightarrow y - 1 = \pm 2$$

$$y = 1 \pm 2 \Rightarrow y = 3, -1$$

So vertices $A(5,3), A'(5,-1)$

Foci$(X = 0, Y = \pm c)$

$$X = x - 5 \Rightarrow x - 5 = 0 \Rightarrow x = 5$$

$$Y = y - 1 \Rightarrow y - 1 = \pm c \Rightarrow y - 1 = \pm 3$$

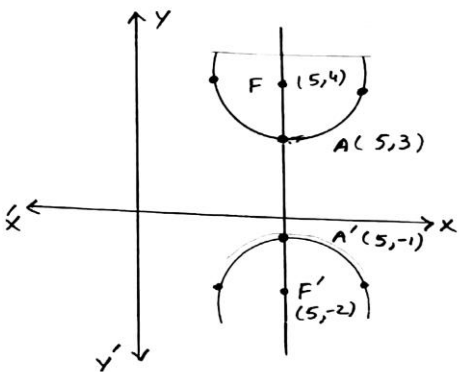
$$y = 1 \pm 3, \quad y = 4, -2$$

$$so\ foci\ (5,4), F'(5,-2)$$

Centre $(5,1)$

Length of $latusrectum$: $\frac{2b^2}{a}$

$$= \frac{2(5)}{2} = 5$$



Question No.2 find Centre, foci eccentricity, vertices and equation of *directrices* of

(*i*) $x^2 - y^2 = 9$

Solution:

$$x^2 - y^2 = 9 \Rightarrow \frac{x^2}{9} - \frac{y^2}{9} = 1$$

Here $a^2 = 9 \Rightarrow a = 3, b^2 = 9 \Rightarrow b = 3$

$$\because c^2 = a^2 + b^2 = 9 + 9 = 18$$

⇨ $c = \sqrt{18} = 3\sqrt{2}$

$$Centre: (0,0)$$

$$Eccentricity: e = \frac{c}{a}$$

⇨ $e = \frac{3\sqrt{2}}{3} = \sqrt{2}$

$$Foci: (\pm c, 0) = (\pm 3\sqrt{2}, 0)$$

$$Directrices: x = \pm\frac{a}{e} = \pm\frac{3}{\sqrt{2}}$$

$$Vertices: (\pm a, 0) = (\pm 3,0)$$

(*ii*) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

Solution:

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

Here $a^2 = 4 \Rightarrow a = 2, b^2 = 9 \Rightarrow b = 3$

$$\because c^2 = a^2 + b^2 = 4 + 9 = 13$$

⇨ $c = \sqrt{13}$

Centre: $(0,0)$

Eccentricty:

$$e = \frac{c}{a} \Rightarrow c = \frac{\sqrt{13}}{2}$$

Foci: $(\pm c, 0) = (\pm\sqrt{13}, 0)$

$$Directrices: x = \pm\frac{a}{e}$$

⇨ $x = \pm\frac{2}{\frac{\sqrt{13}}{2}} = \pm\frac{4}{\sqrt{13}}$

$Vertices: (\pm a, 0) = (\pm 2,0)$

(iii)

$\frac{y^2}{16} - \frac{x^2}{9} = 1$

$solution$:

$$\frac{y^2}{16} - \frac{x^2}{9} = 1 \quad here\ a^2 = 16 \Rightarrow a = 4$$

$$b^2 = 9 \Rightarrow b = 3$$

$$\because c^2 = a^2 + b^2$$

$$c^2 = 16 + 9 = 25 \Rightarrow c = 5$$

Centre: $(\mathbf{0}, \mathbf{0})$

$$eccentricity: e = \frac{c}{a} = \frac{5}{4}$$

$$foci: (o, \pm c) = (0, \pm 5)$$

$Directricity: y = \pm\frac{a}{e}$

$$\Rightarrow y = \pm\frac{4}{\frac{5}{4}} = \pm\frac{16}{5}$$

$Vertices: (0, \pm a) = (0, \pm 4)$

(*iv*) $\frac{y^2}{4} - x^2 = 1$

Solution:

$$\frac{y^2}{4} - x^2 = 1 \quad \Rightarrow \frac{y^2}{4} - \frac{x^2}{1} = 1$$

$$a^2 = 4 \Rightarrow a = 2, b^2 = 1 \Rightarrow b = 1$$

$$\because \quad c^2 = a^2 + b^2 \Rightarrow c^2 = 4 + 1 = 5$$

$$c = \sqrt{5}$$

Centre: $(0,0)$

$$Eccentricity: e = \frac{c}{e} = \frac{\sqrt{5}}{2}$$

Foci $= (0, \pm c) = (0, \pm\sqrt{5})$

$$Directrices: y = \pm\frac{a}{e}$$

⇨ $Diectricity: y = \pm\frac{a}{e}$

⇨ $y = \pm\frac{2}{\frac{\sqrt{5}}{2}} \Rightarrow y = \pm\frac{4}{\sqrt{5}}$

$Vertices: (0, \pm a) = (0, \pm 2)$

(v) $\frac{(x-1)^2}{2}-\frac{(y-1)^2}{9}=1$

Solution:

$$\frac{(x-1)^2}{2}-\frac{(y-1)^2}{9}=1$$

Compare with $\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$

$$X=x-1,\quad Y=y-1,\quad a^2=2,$$
$$b^2=9\ ,\quad a=\sqrt{2},\quad b=3$$
$$\because\quad c^2=a^2+b^2=2+9=11\Rightarrow c=\sqrt{11}$$

Centre:

$$(X=0,\quad Y=0)$$
$$X=x-1\Rightarrow x-1=0\Rightarrow x=1$$
$$Y=y-1\Rightarrow y-1=0\Rightarrow y=1$$

So Centre $(1,1)$

Eccentricity:

$$e=\frac{c}{a}\Rightarrow e=\frac{\sqrt{11}}{\sqrt{2}}=\sqrt{\frac{11}{2}}$$

Foci: $(X=\pm c, Y=0)$

$$\because X=x-1\Rightarrow x-1=\pm c\Rightarrow x=1\pm\sqrt{11}$$
$$Y=y-1\Rightarrow y-1=0\Rightarrow y=1$$

So $foci\left(1\pm\sqrt{11},1\right)$

$$\textit{Directrices}: X=\pm\frac{a}{e}$$
$$\Rightarrow x-1=\pm\frac{\sqrt{2}}{\frac{\sqrt{11}}{\sqrt{2}}}$$
$$x-1\pm\frac{2}{\sqrt{11}}$$

Vertices:

$$(X=\pm a,\quad Y=0)$$
$$\Rightarrow\ X=x-1\Rightarrow x-1=\pm a,\ \Rightarrow x=1\pm\sqrt{2}$$
$$Y=y-1\Rightarrow y-1=0\Rightarrow y=1$$
$$so\ Vertices\left(1\pm\sqrt{2},1\right)$$

(vi) $\frac{(y+2)^2}{9}-\frac{(x-2)^2}{16}=1$

Solution:

$$\frac{(y+2)^2}{9}-\frac{(x-2)^2}{16}=1$$

Compare with $\frac{Y^2}{a^2}-\frac{X^2}{b^2}=1$

$$X=x-2, Y=y+2, a^2=9\Rightarrow a=3$$
$$\because c^2=9+16=25$$
$$\Rightarrow c=5$$
$$b^2=16\Rightarrow b=4$$

Centre:

$$(X=0, Y=0)$$
$$X=x-2\Rightarrow x-2=0\Rightarrow x=2$$
$$Y=y+2\Rightarrow y+2=0\Rightarrow y=-2$$

So $Centre\ (2,-2)$

$$\textit{Eccentricity}:\ \because\ e=\frac{c}{a}\Rightarrow e=\frac{5}{3}$$

foci: $(X=0, Y=\pm c)$

$$X=x-2\Rightarrow x-2=0\Rightarrow x=2$$
$$Y=y+2\Rightarrow y+2=\pm c\Rightarrow y=-2\pm 5$$
$$\Rightarrow y=-7,3$$
$$Foci\ are\ (2,3),\quad (2,-7)$$

Directrices:

$$Y=\pm\frac{a}{e}$$
$$\Rightarrow\ y+2=\pm\frac{3}{\frac{5}{3}}\Rightarrow y=-2\pm\frac{9}{5}$$

Vertices: $(X=0\ , Y=\pm a)$

$$\Rightarrow X=x-2\Rightarrow x-2=0\Rightarrow x=2$$
$$\Rightarrow Y=y+2\Rightarrow y+2=\pm a$$
$$\Rightarrow y=-2\pm 3\ \Rightarrow y=-5,1$$

So vertices $(2,-5),(2,1)$

(vii) $9x^2-12x-y^2-2y+2=0$

Solution:

$$9x^2-12x-y62-2y+2=0$$
$$9\left(x^2-\frac{12}{9}x\right)-(y^2+2y)=-2$$
$$9\left(x^2-\frac{12}{9}x\right)-(y^2+2y)=-2$$
$$9\left\{x^2-2(x)\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^2\right\}$$
$$-\{y^2+2y+1-1\}=-2$$
$$9\left\{\left(x-\frac{2}{3}\right)^2-\frac{4}{9}\right\}-(y+1)^2+1=-2$$
$$9\left(x-\frac{2}{3}\right)^2-(y+1)^2=-2+4-1$$
$$\Rightarrow\ 9\left(x-\frac{2}{3}\right)^2-(y+1)^2=1$$
$$or\quad \frac{\left(x-\frac{2}{3}\right)^2}{\frac{1}{9}}-\frac{(y+1)^2}{1}=1\rightarrow(i)$$

$$compare\ with\ \frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$$
$$X=x-\frac{2}{3},\qquad Y=y+1,\ a^2=\frac{1}{9}\Rightarrow a=\frac{1}{3}$$
$$b^2=1\Rightarrow b=1$$
$$\because c^2=a^2+b^2=\frac{1}{9}+1=\frac{10}{9}$$
$$\Rightarrow c=\frac{\sqrt{10}}{3}$$

Centre:

$$X=0, Y=0$$

$$X = x - \frac{2}{3} \Rightarrow x - \frac{2}{3} = 0 \Rightarrow x = \frac{2}{3}$$
$$Y = y + 1 \Rightarrow y + 1 = 0 \Rightarrow y = -1$$

So $Centre\ \left(\frac{2}{3}, -1\right)$

Eccentricity:

$$e = \frac{c}{a} \Rightarrow e = \frac{\frac{\sqrt{10}}{3}}{\frac{1}{3}} = \sqrt{10}$$

Foci:

$$(X = \pm c, Y = 0)$$
$$X = x - \frac{2}{3} \Rightarrow x - \frac{2}{3} = \pm c \Rightarrow x = \frac{2}{3} \pm \frac{\sqrt{10}}{3}$$

⇨ $x = \frac{2 \pm \sqrt{10}}{3}$

$$Y = y + 1 \Rightarrow y + 1 = 0 \Rightarrow y = -1$$

So $foci\ \left(\frac{2 \pm \sqrt{10}}{3}, -1\right)$

***Directrices*:**

$$X = \pm \frac{a}{e}$$

⇨ $x - \frac{2}{3} = \pm \frac{\frac{1}{3}}{\sqrt{10}} \Rightarrow x = \frac{2}{3} \pm \frac{1}{3\sqrt{10}}$

Vertices:

$$(X = \pm a, Y = 0)$$
$$X = x - \frac{2}{3} \Rightarrow x - \frac{2}{3} = \pm a \Rightarrow x = \frac{2}{3} \pm \frac{1}{3}$$
$$\Rightarrow x = \frac{2}{3} + \frac{1}{3} \Rightarrow x = \frac{3}{3} = 1 \Rightarrow x = 1$$

$also\ x = \frac{2}{3} - \frac{1}{3} \Rightarrow x = \frac{1}{3}$

$hence\ vertices\ are\ (1, -1)\left(\frac{1}{3}, -1\right)$

(*viii*) $4y^2 + 12y - x^2 + 4x + 1 = 0$

Solution:

$$4y^2 + 12y - x^2 + 4x + 1 = 0$$
$$4(y^2 + 3y) - (x^2 - 4x) = -1$$
$$4\left\{y^2 + 2(y)\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right\} - \{x^2 - 4x + 4 - 4\} = -1$$
$$4\left(y + \frac{3}{2}\right)^2 - 9 - (x-2)^2 + 4 = -1$$
$$4\left(y + \frac{3}{2}\right)^2 - 9 - (x-2)^2 = -1 - 4 + 9$$
$$4\left(y + \frac{3}{2}\right)^2 - (x-2)^2 = 4$$
$$or\ \left(y + \frac{3}{2}\right)^2 - \frac{(x-2)^2}{4} = \frac{4}{4} = 1$$

Compare with $\frac{Y^2}{a^2} - \frac{X^2}{b^2} = 1$

$$X = x - 2, \qquad Y = y + \frac{3}{2},$$
$$a^2 = 1 \Rightarrow a = 1$$
$$b^2 = 4 \Rightarrow b = 2$$
$$\because\ c^2 = a^2 + b^2 = 1 + 4 \Rightarrow c^2 = 5$$
$$or\ \ c = \sqrt{5}$$

Centre:

$$(X = 0, Y = 0)$$
$$X = x - 2 \Rightarrow x - 2 = 0 \Rightarrow x = 2$$
$$Y = y + \frac{3}{2} \Rightarrow y + \frac{3}{2} = 0 \Rightarrow y = -\frac{3}{2}$$

So Centre $\left(2, -\frac{3}{2}\right)$

Eccentricity:

$$e = \frac{c}{a}$$

⇨ $e = \frac{\sqrt{5}}{1} = \sqrt{5}$

⇨

$Foci: (X = 0, Y = \pm c)$

$$X = x - 2 \Rightarrow x - 2 = 0 \Rightarrow x = 2$$
$$Y = y + \frac{3}{2} \Rightarrow y + \frac{3}{2} = \pm c \Rightarrow y = -\frac{3}{2} \pm \sqrt{5}$$

So foci $\left(2, -\frac{3}{2} \pm \sqrt{5}\right)$

***Directriecs*:**

$$Y = \pm \frac{a}{c}$$

⇨ $Y = y + \frac{3}{2} \Rightarrow y + \frac{3}{2} = \pm \frac{a}{c}$

⇨ $y = -\frac{3}{2} \pm \frac{1}{\sqrt{5}}$

Vertices:

$(X = 0, Y = \pm a)$

$X = x - 2 \Rightarrow x - 2 = 0, x = 2$

$Y = y + \frac{3}{2} \Rightarrow y + \frac{3}{2} = \pm a$

⇨ $y = -\frac{3}{2} \pm 1 \Rightarrow y = -\frac{1}{2}, -\frac{5}{2}$

$so\ vertices\ \left(2, \frac{1}{2}\right), \left(2, -\frac{5}{2}\right)$

(ix)

$$\boldsymbol{x^2 - y^2 + 8x - 2y - 10 = 0}$$

Solution:

$$x^2 + 8x - y^2 - 2y - 10 = 0$$

⇨ $x^2 + 2(x)(4) + (4)^2 - (4)^2 - (y^2 + 2y + 1 - 1) = 10$

⇨ $(x+4)^2 - 16 - (y+1)^2 + 1 = 10$

⇨ $(x+4)^2 - (y+1)^2 = 10 - 1 + 16$

⇨ $(x+4)^2 - (y+1)^2 = 25$

⇨ $\frac{(x+4)^2}{25} - \frac{(y+1)^2}{25} = 1$

$compare\ with$

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$$

$X = x + 4, Y = y + 1, a^2 = 25 \Rightarrow a = 5$

$$b^2 = 25 \Rightarrow c = 5$$
$$\because c^2 = a^2 + b^2 = 25 + 25 = 50$$
$$c = \sqrt{50} = 5\sqrt{2}$$

$Centre$:

$(X = 0, Y = 0)$

$$X = x + 4 \Rightarrow x + 4 = 0 \Rightarrow x = -4$$
$$Y = y + 1 \Rightarrow y + 1 = 0 \Rightarrow y = -1$$
$$so\ centre\ (-4, -1)$$
$$Eccentricity: e = \frac{c}{a}$$

⇨ $e = \frac{5\sqrt{2}}{5} = \sqrt{2}$

Foci:

$$(X = \pm c, Y = 0)$$
$$X = x + 4 \Rightarrow x + 4 = \pm c$$

⇨ $x = -4 \pm 5\sqrt{2}$

⇨ $Y = y + 1 \Rightarrow y + 1 = 0 \Rightarrow y = -1$

So foci $\left(-4 \pm 5\sqrt{2}, -1\right)$

Directrices:

$$X = \pm\frac{a}{c}$$

⇨ $x + 4 = \pm\frac{5}{\sqrt{2}} \Rightarrow x = -4 \pm \frac{5}{\sqrt{2}}$

Vertices:

$$(X = \pm a\ \ , Y = 0)$$
$$X = x + 4 \Rightarrow x + 4 = \pm a$$

⇨ $x = -4 \pm 5 \Rightarrow x = -9, 1$

$$Y = y + 1 \Rightarrow y + 1 = 0 \Rightarrow y = -1$$

So vertices $(1, -1)(-9, -1)$

$(x)\ 9x^2 - y^2 - 36x - 6y + 18 = 0$

Solution:

$9x^2 - 36x - y^2 - 6y = -18$

$9(x^2 - 4x + 4 - 4) - (y^2 + 6y + 9 - 9) = -18$

$9(x - 2)^2 - 36 - (y + 3)^2 + 9 = -18$

$9(x - 2)^2 - (y + 3)^2 = -18 - 9 + 36$

⇨ $9(x - 2)^2 - (y + 3)^2 = 9$

⇨ $(x - 2)^2 - \frac{(y+3)^2}{9} = 1$

Compare with $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$

$X = x - 2, Y = y + 3, a^2 = 1, b^2 = 9$

⇨ $a = 1, b = 3,$

$$\because c^2 = a^2 + b^2$$
$$25 + 25 = 50$$
$$c = \sqrt{50} = 5\sqrt{2}$$

Centre:

$$(X = 0, Y = 0)$$
$$X = x - 2 \Rightarrow x - 2 = 0 \Rightarrow x = 2$$
$$Y = y + 3 \Rightarrow y = -3$$

So Centre$(2, -3)$

Foci:

$$(x = \pm c, Y = 0)$$
$$\boldsymbol{X = x - 2 \Rightarrow x - 2 = \pm c \Rightarrow x - 2 = \pm\sqrt{10}}$$
$$\boldsymbol{x = 2 \pm \sqrt{10}}$$
$$Y = y + 3 \Rightarrow y + 3 = 0 \Rightarrow y = -3$$

Hence foci are $\left(2 \pm \sqrt{10}, -3\right)$

Directrices:

$$X = \pm\frac{a}{e}$$

⇨ $x - 2 = \pm\frac{1}{\sqrt{10}} \Rightarrow x = 2 \pm \frac{1}{\sqrt{10}}$

Vertices:

$$\boldsymbol{X = x - 2 \Rightarrow x - 2 = \pm 1}$$

⇨ $X = x - 2 \Rightarrow x - 2 = \pm 1$

⇨ $x = 2 \pm 1 \Rightarrow x = 3, 1$

$Y = 0 \Rightarrow y + 3 = 0 \Rightarrow y = -3$

Question No.3

Let $0 < c < a\ and\ F(c, 0)$

$F'(-c, 0)\ be\ two\ fixed\ points. show\ that$

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

$$(F\ and\ F'\ are\ foci\ of\ hyperbola)$$

By given condition.

$$|PF| - |PF'| = \pm 2a$$

$$\sqrt{(x - (-c)^2 + (y - 0)^2} - \sqrt{(x - c)^2 + (y - 0)^2} = \pm 2a$$

$$\sqrt{(x - c)^2 + (y)^2} - \sqrt{(x - c)^2 + (y)^2} = \pm 2a$$

$$\sqrt{x^2 + c^2 - 2cx} - \sqrt{x^2 + c^2 - 2cx + y^2} = \pm 2a$$

$$\sqrt{x^2 + c^2 - 2cx} = \pm 2a + \sqrt{x^2 + c^2 - 2cx + y^2}$$

Squaring both sides

$$x^2 + c^2 + 2cx + y^2 = 4a^2 + x^2 + c^2 - 2cx + y^2 \pm 2(2a)\sqrt{x^2 + c^2 - 2cx + x^2 + y^2}$$

$$4cx = 4a^2 \pm 4a\sqrt{x^2 - 2cx + x^2 + y^2}$$

⇨ $4cx = 4a^2 \pm 4a\sqrt{x^2 - 2cx + x^2 + y^2}$

⇨ $cx = a^2 \pm a\sqrt{x^2 - 2cx + x^2 + y^2}$

⇨ $cx - a^2 = \pm\sqrt{x^2 - 2cx + x^2 + y^2}$

$squaring\ both\ sides$

$$c^2x^2 + a^4 - 2cxa^2 = a^2\,(x^2 - 2xc + x^2 + y^2)$$

$$c^2x^2 + a^4 - 2cxa^2 = a^2x^2 - 2exa^2 + a^2x^2 + a^2y^2$$

$c^2x^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$

$\Rightarrow (c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{(c^2 - a^2)} = 1 \div by\ a^2(c^2 - a^2)$$

$$req. eq. of\ hyperbola.$$

Question No. 4

Let $0 < c < a$ and $F(5, 5)F'(-5, -5)$ be foci.

also the set of points $P(x,y)$ such that $|PF|-|PF'|=\pm 2a$ is the hyperbola with
Vertices $A(3\sqrt{2},3\sqrt{2})A'(-3\sqrt{2},-3\sqrt{2})$. *find*
Equation of hyperbola.

Solution:

$$\because 2a=|AA'|$$
$$=\sqrt{(3\sqrt{2}+3\sqrt{2})^2+(3\sqrt{2}+3\sqrt{2})^2}$$
$$\Rightarrow 2a=\sqrt{(6\sqrt{2})^2+(6\sqrt{2})^2}=\sqrt{72+72}$$
$$=\sqrt{144}$$
$$\Rightarrow 2a=12\Rightarrow a=6$$
$$using|PF|-|PF'|=\pm 2a$$
$$|PF|=\pm 2a+|PF^{\wedge\prime}|$$
$$\sqrt{(x+5)^2+(y+5)^2}\pm \mathbf{12}$$
$$+\sqrt{(x-5)^2+(y-5)^2}$$
$$squaring\ both\ sides\ by$$
$$x^2+25x+10x+y^2+25+2$$
$$=144+x^2+25-10x+y^2+25-10y$$
$$\pm 24\sqrt{(x-5)^2+(y-5)^2}$$
$$\Rightarrow 20x+20y=144\pm 24\sqrt{(x-5)^2+(y-5)^2}$$
$$5x+5y-36=\pm 6\sqrt{(x-5)^2+(y-5)^2}$$
$$squaring\ both\ sides\ by$$
$$36(x^2+25-10x+y^2+25-10y)=$$
$$25x^2+25y^2+1296+50xy-360x-360y$$
$$36x^2+36y^2-360x-360y+1800$$
$$=25x^2+25y^2+1296+50xy-360x$$
$$-360y$$
$$11x^2+11y^2-50xy+504=0$$
$$Req.equation\ of\ hyperbola.$$

Question No.5

For any point on the hyperbola the difference of its distance from the points $(2,2)$ *and* $(10,2)$ *is* 6. Find an equation of hyperbola.

Solution:

Let $P(x,y)$ *be any point on hyperbola so*

$$\sqrt{(x-2)^2+(y-2)^2}-\sqrt{(x-10)^2+(y-10)^2}$$
$$=6$$
$$\sqrt{(x-2)^2+(y-2)^2}=\sqrt{(x-10)^2+(y-10)^2}$$
$$+6$$

Squaring both sides

$$x^2+4-4x+y^2+4-4y$$
$$=36+x^2+100-20x$$
$$y^2+4-4y+12\sqrt{(x-10)^2+(y-2)^2}$$

⇨ $-4x+20x+8-36-104=$
$12\sqrt{(x-10)^2+(y-2)^2}$

⇨ $16x-132=$
$12\sqrt{x^2+100-20x+y^2-4y+4}$

⇨ $4x-33=$
$3\sqrt{x^2+y^2-20x+y^2-4y+104}$

Again squaring

$$16x^2+1089-264x$$
$$=9(x^2+y^2-20x-4y+104)$$

⇨ $16x^2+1089-264x=9x^2+9y^2-180x-36y+936$

⇨ $7x^2-9y^2-84x+36y+153=0$

Which is the required equation.

Q6. two listing posts hear the sound of an enemy **Gun. The difference in time is one second. If the listening posts are 1400ft apart, write an equation of the hyperbola passing through the position of the enemy gun. (Sounds) travels at $1080\ ft\backslash sec$.**

Solution:

Let two

listening F_1 *and* F_2 *hear the sound of enemy*
Gun after t and $t-1$ *second respectively here*
listening posts are $1400m$ *apart.i.e* $2c$

$$=1400\Rightarrow c=700$$

If *p is the position of enemy gun.*

So sound travels at 1080 ft/sec so we have

$$|pF_1|-|pF_2|=2a$$

⇨ $1080t-1080(t-1)=2a$

⇨ $1080t-1080t+1080=2a$

$$or\ 2a=1080$$
$$a=540$$

Using $c^2=a^2+b^2$

$$b^2=c^2-a^2$$
$$b^2=(700)^2-(540)^2$$
$$b^2=490000-291600$$
$$b^2=198400$$

The equation of hyperbola is

$$\frac{x^2}{(540)^2}-\frac{y^2}{198400}=1$$
$$or\ \frac{x^2}{291600}-\frac{y^2}{198400}=1$$