Geometry:

The geometry is derived from two Greek words Geo (Earth) and Matron (Measurement). It means Knowledge of measurement of earth.*Geometry is branch of mathematics that deals the shape and size of things.

Analytic geometry:

In analytic geometry or coordinates geometry, points could be represented by numbers, lines and curves represented by equations.

A French philosopher and mathematician Rene Descartes (1596-1650A.D) introduced algebraic methods in geometry named as analytical geometry named (or coordinate geometry.)

Coordinates system:

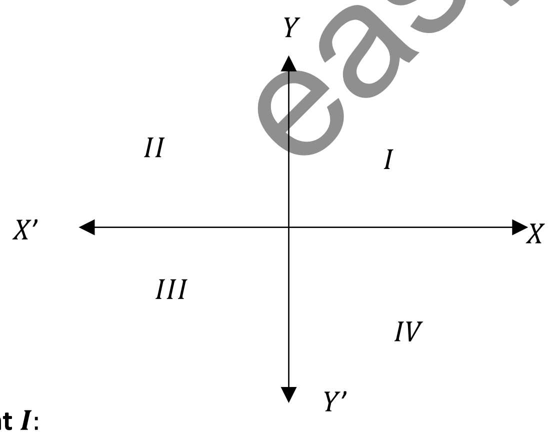
Draw in a plane two mutually number lines XX' and YY'

One horizontal and the other vertical. Let their point of intersection be O called origin and real number O of both lines is represented by O. The two lines are called the coordinate axis. The horizontal line XOX' is called $x-$axis and vertical line YOY' is called $y-$axis.

The plane determined by both $x-$axis and $y-$axis. Is called $xy-$plane or cartesionplane.

*if (x, y) are coordinates of a point p. then the first member of ordered pair$(i.s\ x)$ is called x –coordinate or abscissa of point P. and then second member of ordered pair $(i.s\ y)$ is called y –coordiinate or ordinate of point P.

* The coordinate axis divide the coordinate plane into four equal parts, called quadrants.



Quadrant I:

$$\{(x, y)|x > 0, y > o\}$$

Quadrant II:

$$\{(x, y)|x < 0, y > o\}$$

Quadrant III:

$$\{(x, y)|x < 0, y < o\}$$

Quadrant IV:

$$\{(x, y)|x > 0, y < o\}$$

NOTE: ***on x − axis ordinate is zero i.e y = 0 also on y − axis absissa is zero.***

The distance formula:

The distance between two points $A(x, y)$ and $B(x, y)$ in $xy-plane$ is

$$|AB| = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

NOTE: AB stand for ***$m\overline{AB}$ or $|\overline{AB}|$ and d stands for distance.***

Proof:

let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in $xy = plane$.

Draw $\perp AR$ on BN.

In right $\triangle ABR$ *using pathagoras therrem.*

$|AB|^2 = |AR|^2 + |BR|^2$ $\because$

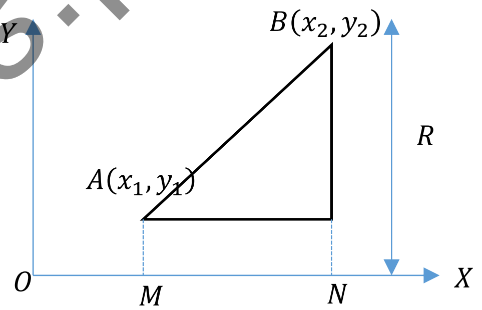
$\begin{pmatrix} |AR| = |MN| \\ = |ON| - |OM| \\ |AR| = x_2 - x_1 \end{pmatrix}$

$\Rightarrow |AB|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$\Rightarrow\ d^2 = |AB|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$|BR| = |BN| - |RN| \Rightarrow y_2 - y_1$

$\Rightarrow d = |AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

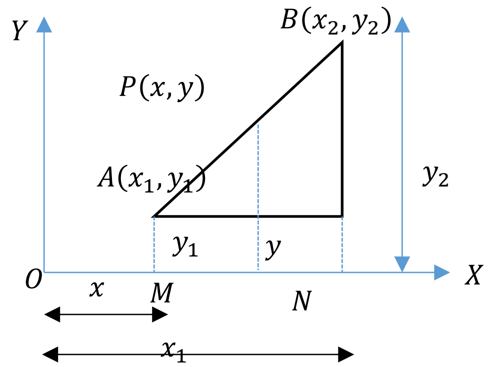


Theorem:

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points in a **Plane. The line segment *AB in the ratio k_1, k_2* are**

$\left(\frac{k_1x_1 + k_2x_2}{k_1 + k_2}, \frac{k_1y_1 + k_2y_2}{k_1 + k_2}\right)$

Proof:



Let $P(x, y)$ be the point. which divides AB in ratio $k_1 : k_2$

Draw $\perp$ ars AM, QM and BN from $A, P,$ and B on $x-axis$ as shown in figure.

$$AP : PB = AS : SR$$

$\Rightarrow\ \frac{AP}{PB} = \frac{AS}{SR} \rightarrow\ i$

$$\because AS = MQ = OQ - OM$$
$$= x - x_1$$
$$SR = QN = ON - OQ$$
$$= x_2 - x$$
$$\because AP : PB = k_1 : k_2$$

So $\frac{k_1}{k_2} = \frac{x - x_1}{x_2 - x}$

⇨ $k_1(x_2 - x) = k_2(x - x_1)$

⇨ $k_1x_2 - k_1x = k_2x - k_2x_1$

⇨ $k_1x_2 + k_2x_1 = k_2x + k_1x$

⇨ $k_1x_2 + k_2x_1 = x(k_1 + k_2$

⇨ $x = \frac{k_1x_2 + k_2x_1}{k_1 + k_2}$

Similarly, by drawing $\perp ars\ from\ A\ P\ and\ B\ on\ y - axis$

$we\ will\ get$

⇨ $y = \frac{k_1y_2 + k_2y_1}{k_1 + k_2}$

Thus $P\left(\frac{k_1x_2 + k_2x_1}{k_1 + k_2}, \frac{k_1y_2 + k_2y_1}{k_1 + k_2}\right)$ is required point.

Note:

i. **Two geometric figures are similar if one is enlargement of other.**

ii. **In two triangles, if two corresponding angles are congruent, then triangles are similar.**

iii. **If the directed distances AP and PB have the same sign, then their ratio is positive and P is said to divide AB internally.**

iv. **If the directed distance AP and PB have opposite signs i.e; p is beyond AB, then their ratio is negative and P is said to divide AB externally.** $\frac{AP}{PB} = \frac{k_1}{k_2}$ **or** $\frac{AP}{PB} = -\frac{K_1}{k_2}$

In this case we can show that

⇨ $x = \frac{k_1x_2 + k_2x_1}{k_1 + k_2}, y = \frac{k_1y_2 + k_2y_1}{k_1 + k_2}$

Thus P is said to divide the line segment AB in ratio $k_1 : k_2$ **internally or externally according as** P **lies b\w AB or beyond AB.**

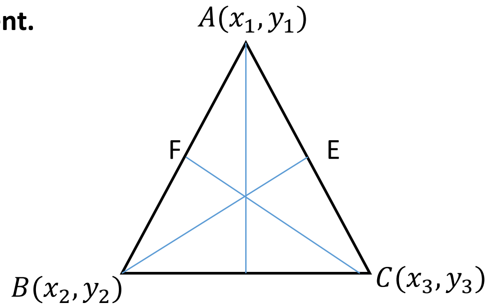
v. **If** $k_1 : k_2 = 1 : 1\ then\ p\ becomes\ mid\ point\ of\ (\overline{AB}$ **and coordinates of p are** $x = \frac{x_1 + x_2}{2}, y = \frac{y_1 + y_2}{2}$

vi. **The above theorem is valid in whichever quadrant A and B lie.**

Remembers:

- Line segment joining one vertex of a triangle to the midpoint of an opposite side of the triangle is called median.
- A point that divides each median in ratio 2: 1 is called centroid.
- The point of concurrency of medians is called centroid.
- When two or more than two lines meet at a point. Then they are said to be concurrent.

Theorem: show that medians of a triangle are concurrent.



$$D\ \frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}$$

Proof:

Let $C(x_3, y_3) be\ vertices\ of\ a\ \triangle$
$ABC\ let\ D, E\ anf\ F\ be$
$mid\ points\ of\ sides\ BC, AC\ and\ AB\ resp.$

So $AD, BE\ and\ CF\ are\ medians\ of\ \triangle ABC.$

$$\because midpoint\ of\ BC\ is\ D\ \frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}$$

Let p be the point dividing BC in ratio 2:1 so using formula

$\frac{k_1x_2 + k_2x_1}{k_1 + k_2}, \frac{k_1y_2 + k_2y_1}{k_1 + k_2}$ $so\ coordinates\ of\ p\ in\ ratio\ 2:1\ are$

$$\left(\frac{1\left(\frac{x_2 + x_3}{2} + 91\right)(x_3}{2 + 1}, \frac{2\left(\frac{y_2 + y_3}{2}\right) + (1)(y_3)}{2 + 1}\right)$$

⇨ $p\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

Similarly it can be proved that coordinates of point that divides medians BE and CF each in 2: 1 are $p\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

Remembers:

- **A line that divides an angle into equal parts is called angle bisector.**
- **An angle bisector divides line opposite to into a ratio, equal to ratio of remaining two sides.**
- **In figure AD is an angle bisector of** $\measuredangle A$ **the sides opposite to** $\measuredangle A\ is\ BC$**.so**$BD : DC = BA : AC$
- $BD : DC = c : b\ (BA = C\quad AC = b$

Theorem:

Bisector of angles of a triangle are concurrent.

Proof:

$let\ A(x_1, y_1), B(x_2, y_2) and\ C(x_3, y_3) be\ vertices\ of\ \triangle ABC\ then\ |AB| = c\ , |BC| = a, \quad |AC| = b$

Let bisector $\angle A\ meet\ BC\ at\ point\ D$

Now

$$\frac{BD}{DC} = \frac{BC}{AC}$$

⇨ $\frac{BD}{DC} = \frac{c}{b} \rightarrow\ i \qquad (\because |BA| = c, |DC| = b$

⇨ $BD:DC = c:b$ it means D divides BC in $c:b$

Using ratio formula coordinates of D are

$\left(\frac{bx_2+cx_3}{b+c}, \frac{by_2+cy_3}{b+c}\right)$

Let angle bisector of

$\angle B$ intersects AD at point I then $\frac{AI}{ID} = \frac{AD}{BD}$

⇨ $\frac{AI}{ID} = \frac{c}{BD} \rightarrow\ ii)\ \because |AB| = c$

Now take reciprocal of eq.(i)

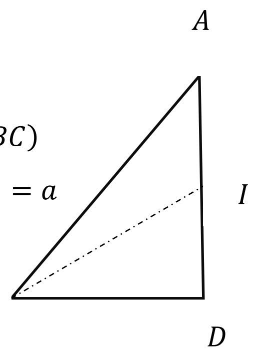
$\frac{DC}{DB} = \frac{b}{c} \Rightarrow 1 + \frac{DC}{DB} = 1 + \frac{b}{c}$

⇨ $\frac{BD+DC}{BD} = \frac{b+c}{c}$ $\quad (\because BD + DC = BC)$

⇨ $\frac{BC}{BD} = \frac{b+c}{a} \Rightarrow \frac{a}{BD} = \frac{b+c}{c}$ $\quad \because |BC| = a$

⇨ $\frac{BD}{a} = \frac{c}{b+c} \Rightarrow BD = \frac{ac}{b+c}$

⇨ So $(ii) \Rightarrow \frac{AI}{ID} = \frac{c}{\frac{ac}{b+c}} = c\left(\frac{b+c}{ac}\right)$



⇨ $\frac{AI}{ID} = \frac{b+c}{a} \Rightarrow AI:AD = (b+c):a$

By ratio formula

$$\Rightarrow I\left(\frac{(b+c)\left(\frac{bx_2+cx_3}{b+c}\right)+ax_1}{a+b+c}, \frac{(b+c)\left(\frac{by_2+cy_3}{b+c}\right)+ay_1}{b+c+a}\right)$$

⇨ $I\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

Similarly, it can be prove that bisector of $\angle c$ will also pass through point I.

⇨ Hence bisector of angles of triangle are concurrent.

Exercise 4.1

Q1. Describe the location in the plane $p(x,y)$ for which

$(i)\ x > 0$ $\ (ii)\ x > 0$ and $y > 0$ $\ (iii)\ x = 0$ $\ (iv)\ y = 0$ $\ (v)\ x < 0$ and $y \geq o$ $\ (vi)\ x = y$ $\ (vii)\ |x| = |y|$ $\ (viii)\ |x| \geq 3$ $\ (ix)\ x > 2$ and $y = 2$

(x) and y have opposite signs.

solution:

$(i)\ x > 0$

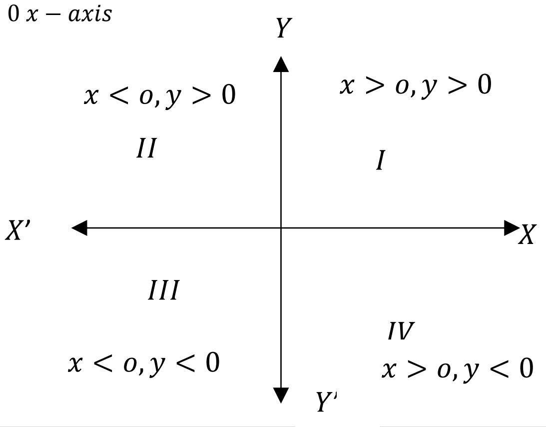
Right half plane.

$(ii)\ x > 0$ and $y > 0$

1st quadrant

$(iii)\ x = 0$ $\quad y-axis.$

$(iv)\ y = 0\ x-axis$



$(v)\ x < 0$ and $y \geq o$

2nd quadrant and $-ve\ x-axis$

$(vi)\ x = y$ it is line bisecting 1st and 3rd quadrant.

$(vii)\ |x| = |y|$

1st and 3rd quadrant.

$(viii)\ \ |x| \geq 3$

on $x-axis$ less than equal to -3 and greater Than equal to -3

$(ix)\ x > 2$ and $y = 2$

In 1st quad x greater than 2 and $y = 2$

(x)

x and y have possible sign (in II $(-2,2)$ and $(IV)(2,-2)$)

Q2. Find in each of the following

(i) the distance between two given points

$(a)\ A(3,1); B(-2,-4)$ $\qquad (b)\ A(-8,3); B(2,-1$

$(c)\ A\left(-\sqrt{5}, \frac{1}{3}\right); B(-3\sqrt{5}, 5)$

Solution:

$(a)\ A(3,1); B(-2,-4)$

$$|AB| = \sqrt{(-2-3)^2 + (-4-1)^2}$$
$$= \sqrt{(-5)^2 + (-5)^2}$$
$$= \sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$$

Midpoint of AB== $\left(\frac{3-2}{2}, \frac{1-4}{2}\right) = \left(\frac{1}{2}, -\frac{3}{2}\right)$

$(b)\ A(-8,3); B(2,-1)$

$$|AB| = \sqrt{(2+8)^2 + (-1-3)^2} = \sqrt{100+16}$$
$$= \sqrt{116} = \sqrt{4 \times 29} = 2\sqrt{29}$$

Midpoint of AB=$\left(\frac{-8+2}{2}, \frac{3-1}{2}\right) = \left(\frac{-6}{2}, \frac{2}{2}\right) = (-4,1)$

$c)\ A\left(-\sqrt{5}, \frac{1}{3}\right); B(-3\sqrt{5}, 5)$

$$|AB| = \sqrt{\left((-3\sqrt{5}) - (-\sqrt{5})\right)^2 + \left(5 + \frac{1}{3}\right)^2}$$
$$= \sqrt{\left(-3\sqrt{5} + \sqrt{5}\right)^2 + \left(\frac{15+1}{3}\right)^2}$$
$$= \sqrt{\left(-2\sqrt{5}\right)^2 + \left(\frac{16}{3}\right)^2}$$
$$= \sqrt{4(5) + \frac{256}{9}} = \sqrt{20 + \frac{256}{9}} = \sqrt{\frac{436}{9}}$$
$$= \sqrt{\frac{4 \times 109}{3}} = \frac{2}{3}\sqrt{109}$$

Midpoint of AB=$\left(\frac{-\sqrt{5}-3\sqrt{5}}{2}, \frac{-\frac{1}{3}+5}{2}\right)$

$$= \left(\frac{-4\sqrt{5}}{2}, \frac{-1+15}{3\times 2}\right) = \left(-2\sqrt{5}, \frac{14}{6}\right.$$

$$= \left(-2\sqrt{5}, \frac{7}{3}\right.$$

Q3. Which of the following points are at a distance of 15 units from the origin?

a) $(\sqrt{176}, 7)$

b) $10, -10$

c) $1, 15$

d) $\left(\frac{15}{2}, \frac{15}{2}\right)$

Solution:

$a)$ $(\sqrt{176}, 7)$ and $O(0,0)$

$$|OA| = \sqrt{(\sqrt{176}-0)^2 + (7-0)^2}$$

$$= \sqrt{176-47} = \sqrt{215} = 15$$

⇨ $|OA| = 15$

so A is at a distance of 15 *units from* Origin.

$(\boldsymbol{b}).(\mathbf{10}, -\mathbf{10})$

Distance of $(10, -10)$ from origin

$= \sqrt{(10-0)^2 + (-10-0)^2} = \sqrt{100+100}$

$= \sqrt{200} = 10\sqrt{2}$

Hence the point $(10, -10)$is not at 15 units away from the origin.

$(\boldsymbol{c}).$ $(\mathbf{1}, \mathbf{15})$

let $C(1,15)$ *and* $O(0,0)$ *so,*

$|OC| = \sqrt{(1-0)^2 + (15-0)^2} = \sqrt{1+225} = \sqrt{226}$

So $|OC| \neq$

15 *Thus C is not at a distance of* 15 *units from orgin.*

$d).\left(\frac{15}{2}, \frac{15}{2}\right)$

Distance of $\left(\frac{15}{2}, \frac{15}{2}\right)$ from origin

$$= \sqrt{\left(\frac{15}{2}-0\right)^2 + \left(\frac{15}{2}-0\right)^2} = \sqrt{\frac{256}{4} + \frac{256}{4}}$$

$$= \sqrt{\frac{2(256)}{4}} = 15$$

Hence the point $\left(\frac{15}{2}, \frac{15}{2}\right)$ is at 15 units away from the origin.

Question.4 Show that

i. The points A (0, 2), $B(\sqrt{3}, -1)$ and C (0, -2) are vertices of a right triangle.

ii. The points A (3, 1), B (-2, -3) and C (2, 2) are vertices of an isosceles triangle.

iii. The points A (5, 2), B (-2, 3), C (-3, -4) and D (4, -5) are vertices of a parallelogram. Is the parallelogram a square?

Solution.

i. **Given that**

$$A(3,1), B(-2,-3) \text{ and } C(2,2)$$

$$|AB| = \sqrt{(\sqrt{3}-0)^2 + (-1-2)^2}$$

$$|AB| = \sqrt{(\sqrt{3})^2 + (-3)^2}$$

$$|AB| = \sqrt{3+9} = \sqrt{12} ==> |AB|^2 = 12$$

$$|AC| = \sqrt{(0-0)^2 + (2+2)^2}$$

$$|AC| = \sqrt{(0)^2 + (4)^2}$$

$$|AC| = \sqrt{0+16} = \sqrt{16} = 4 ==> |AC|^2 = 16$$

$$|BC| = \sqrt{(0-\sqrt{3})^2 + \ \ -2+1)^2}$$

$$|BC| = \sqrt{(-\sqrt{3})^2 + (-1)^2}$$

$$|BC| = \sqrt{3+1} = \sqrt{4} = 2 ==> |BC|^2 = 4$$

Since

$$|AB|^2 + |BC|^2 = 12 + 4 = 16 = |CA|^2$$

Hence by Pythagoras theorem A, B, C are the vertices of the triangle.

Remember

(i)A triangle having two sides equal in length (but not to third side) is called an isoaceles triangles.

(ii) in an isisceles triangle, angles opposite to the equal sides are also equal.

ii. **Given that**

$$A(3,1), B(-2,-3) \text{ and } C(2,2)$$

$$|AB| = \sqrt{(-2-3)^2 + (-3-1)^2}$$

$$|AB| = \sqrt{(-5)^2 + (-4)^2}$$

$$|AB| = \sqrt{25+16} = \sqrt{41} \Rightarrow |AB|^2 = 41$$

$$|AC| = \sqrt{(3-2)^2 + (1-2)^2}$$

$$|AC| = \sqrt{(1)^2 + (-1)^2}$$

$$|AC| = \sqrt{1+1} = \sqrt{2} ==> |AC|^2 = 2$$

$$|BC| = \sqrt{(2+2)^2 + (2+3)^2}$$

$$|BC| = \sqrt{(4)^2 + (5)^2}$$

$$|BC| = \sqrt{16+25} = \sqrt{41} ==> |BC|^2 = 41$$

Since

$$|AB| = |BC| \ \ and \ |BC| + |AC|$$

Hence A, B, C are vertices of an isosceles triangle.

iii. **Given that**

$$A(5,2), B(-2,3), C(-3,-4) \text{ and } D(4,-5$$

$$|AB| = \sqrt{(-2-5)^2 + (3-2)^2}$$

$$|AB| = \sqrt{(-7)^2 + (1)^2}$$

$$|AB| = \sqrt{49+1} = \sqrt{50} = 5\sqrt{2}$$

$$|BC| = \sqrt{(-3+2)^2 + (-4-3)^2}$$

$$|BC| = \sqrt{(-1)^2 + (-7)^2}$$

$$|BC| = \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$$

$$|CD| = \sqrt{(4+3)^2 + (-5+4)^2}$$

$$|CD| = \sqrt{(7)^2 + (-1)^2}$$

$$|CD| = \sqrt{49+1} = \sqrt{50} = 5\sqrt{2}$$

$$|DA| = \sqrt{(5-4)^2 + (2+5)^2}$$

$$|DA| = \sqrt{(1)^2 + (7)^2}$$
$$|DA| = \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$$

Since

$$|AB| = |CD| \ and \ |BC| = |DA|$$

Hence A , B, C are vertices of Parallelogram.

Now

$$|AC| = \sqrt{(-3-5)^2 + (-4-2)^2}$$
$$|AC| = \sqrt{(-8)^2 + (-6)^2}$$
$$|AC| = \sqrt{64+36} = \sqrt{100} = 10$$
$$|BD| = \sqrt{(4+2)^2 + (-5-3)^2}$$
$$|BD| = \sqrt{(6)^2 + (-8)^2}$$
$$|BD| = \sqrt{36+64} = \sqrt{100} = 100$$

Since all sides are equals and also both diagonals are equal therefore A,B, C, D are vertices of a square.

Question.5. the midpoint of the sides of a triangle are $(1,-1), (-4,-3)$ and $(-1,1)$. Find the coordinates of the vertices o a triangle.

Solution.

Let $A(x_1,y_1), B(x_2, y_2) and\ C(x_3,y_3)$ are vertices of triangle ABC and let

$D(1,-1), E(-4,-3)\ and\ F(-1,1)$ are midpoints of sides AB, BC and CA respectively.

Then

$$\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2} = (1,-1)$$

$\Rightarrow x_1 + x_2 = 2 \rightarrow (i \quad and \quad y_1 + y_2 = -2 \rightarrow (ii)$

$$\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2} = (-4,-3)$$

$\Rightarrow x_2 + x_3 = -8 \rightarrow (iii) \quad and \quad y_2 + y_3 = -6 \rightarrow iv$

$$\frac{x_3+x_1}{2}, \frac{y_3+y_1}{2} = (-1,1)$$

$\Rightarrow x_3 + x_1 = 2 \rightarrow (v) \quad and \quad y_3 + y_1 = 2 \rightarrow (vi$

Subtracting (i) and (iii)

$$(x_1+x_2)-(x_2+x_3) = 2+8$$

$x_1 - x_3 = 10 \rightarrow (vii)$

Adding (v) and (vii)

$(x_1+x_3) + (x_1-x_3) = -2+10$

$2x_1 = 8$

$x_1 = 4$

Putting value of $x_1\ in\ (i)$

$4 + x_2 = 2$

$x_2 = 2 - 4$

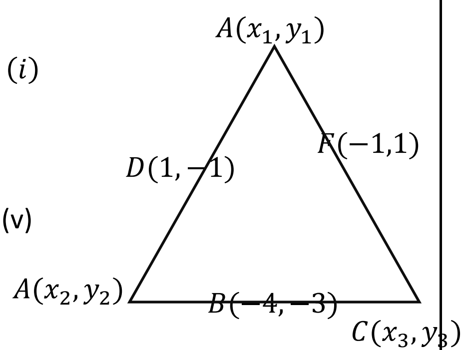
$x_2 = -2$

Putting value of x_1 in (v)

$4 + x_3 = -2$

$x_3 = -2 - 4$

$x_3 = -6$



Subtracting (ii) and (iv)

$(y_1+y_2)-(y_2+y_3) = -2+6$

$y_1 - y_3 = 4 \rightarrow (vii)$

Adding (vi) and (viii)

$(y_1+y_3) + (y_1-y_3) = 2+4$

$2y_1 = 6$

$y_1 = 3$

Putting value of $y_1\ in\ (ii)$

$3 + y_2 = -2$

$y_2 = -2 - 3$

$y_2 = -5$

Putting value of y_1 in (vi)

$3 + y_3 = 2$

$y_3 = 2 - 3$

$y_3 = -1$

Hence vertices of triangle are

$(4,3), (-2,-5)\ and\ (-6,-1)$.

Question.6. Find h such that the point $A(\sqrt{3},-1), B(0,2)$ and $C(h,-2)$ are the vertices of a right angle with right angle at the vertex A.

Solution.

Since ABC is a right angle triangle therefore by Pythagoras theorem

$$|AB|^2 + |CA|^2 = |BC|^2$$

$\left[(0-\sqrt{3})^2 + (2+1)^2\right] + \left[(\sqrt{3}-h)^2 + (-1+2)^2\right]$

$= (h-0)^2 + (-2-2)^2$

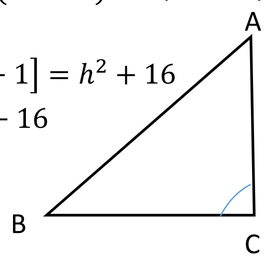
$[3+9] + [3+h^2-2\sqrt{3}h+1] = h^2+16$

$12 + h^2 - 2\sqrt{3}h + 4 = h^2 + 16$

$-2\sqrt{3}h = 0$

$h = 0 \quad \because 2\sqrt{3}$

Which is required.



Remember: (i) points lying on the same line are called collinear points.

(ii) The points

$\boldsymbol{A(x,y)\ and\ B(x,y)\ and\ C(x,y) collinear}$

if shape $\boldsymbol{of\ AB}$

$\boldsymbol{= slop\ of\ AC\ and\ slope\ of\ AB = slope\ of\ AC}$

$$\boldsymbol{\left(and\ slope\ of\ AB = \frac{y_2-y_1}{x_2-x_1}\right.}$$

The points

$\boldsymbol{A(x,y), B(x_1,y_1) and\ C(x_2,y_2) are\ collinear\ if}$

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Question.7. find h such that $A(-1,h), B(3,2)$ and $C(7,3)$ are collinear.

Solution.

Three point

$A(x_1,y_1), B(x_2,y_2)\ and\ C(x_3,y_3)\ are\ said\ to\ be\ collinear\ if$

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Since given points are collinear therefore

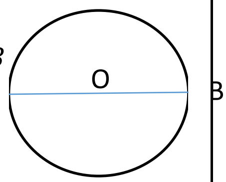
$$\begin{vmatrix} -1 & h & 1 \\ 3 & 2 & 1 \\ 7 & 3 & 1 \end{vmatrix} = 0$$

$$-1(2-3) - h(3-7) + 1(9-14) = 0$$
$$1 + 4h - 5 = 0$$
$$4h - 4 = 0$$
$$4h = 4$$
$$h = 1.$$

Question.8. the points $A(-5,-2)$ *and* $B(5,-4)$ are end of a diameter of a circle. Find the center and radius of the circle.

Solution.

The center of the circle is midpoint of AB



$i.e.\ \ center\ C = \left(\frac{-5+5}{2}, \frac{-2-4}{2}\right)$

$= \left(\frac{0}{2}, -\frac{6}{2}\right) = (0,-3)$

Now radius $= |AC| = \sqrt{(0+5)^2 + (-3+2)^2}$

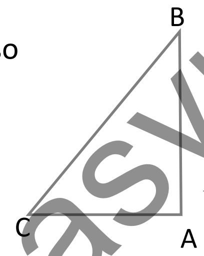
$= \sqrt{25+1} = \sqrt{26}$

Question.9. Find *h* such that the points *A(h,1)*, *B*(2,7) and *C*(– – 6, 7) are vertices of a right triangle with right angle at the vertex *A*

Solution.

$\because A(h,1), B(2,7), C(-6,-7)$

$\because right\ angle\ is\ at\ vertex\ A\ so\ by\ patagoras$ Theorem,



$|BC|^2 = |AC|^2 + |AB|^2 \rightarrow (i)$so

$|AB| = \sqrt{(2-h)^2 + (7-1)^2}$

$= \sqrt{4 - 4h + h^2 + 36}$

$|AB| = \sqrt{40 - 4h + h^2}$

⇨ $|AB|^2 = 40 - 4h + h^2$

$|BC|^2 = \sqrt{(-6-2)^2 + (-7-7)^2} = \sqrt{(-8)^2 + (-14)^2}$

$= \sqrt{64+196} = \sqrt{260} \Rightarrow |BC|^2 = 260$

$$|AC| = \sqrt{(-6-h)^2 + (-7-1)^2}$$
$$= \sqrt{36 + 12h + h^2 + 64}$$
$$|AC| = \sqrt{h^2 + 12h + 100}$$
$$= |AC|^2 = h^2 + 12h + 100$$

So eq (i) become

$$260 = h^2 + 12 + 100 + 40 - 4h + h^2$$

⇨ $2h^2 + 8h + 140 = 260$

⇨ $2h^2 + 8h + 140 - 260 = 0$

⇨ $2h^2 + 8h - 120 = 0$

⇨ $h^2 + 4h - 600 = 0 \quad \div by\ 2$

⇨ $h^2 + 10h - 6h - 60 = 0$

⇨ $h(h+10) - 6(h+10$

⇨ $h+10)(h-6) = 0$

⇨ $h + 10 = 0\ \ or\ h - 6 = 0$

⇨ $h = -10\ \ 0r\ h = 6$

Question.10.

A quadrilateral has the points *A (9,3) B(-7,-7)* , *C (-3 , -7)* and *D* (–5,5) as its vertices. Find the midpoints of its sides. Show that the figure formed by joining the midpoints consecutively is a parallelogram.

Solution.

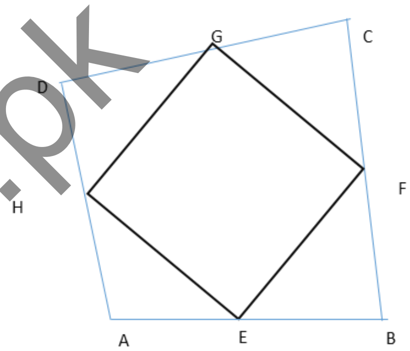
$\because A(9,3), B(-7,7), C(-3,-7), D(5,-5)$

Midpoint of $ABis\ E\left(\frac{-7+9}{2}, \frac{7+3}{2}\right)$

$= E\left(\frac{2}{2}, \frac{10}{2}\right)$

Midpoint $of\ AB\ is\ E\ \left(\frac{-7+9}{2}, \frac{7+3}{2}\right)$

$$= E\ \frac{2}{2}, \frac{10}{2}$$
$$= E(1,5)$$



Midpoint of BC is F$\frac{(-7+(-3))}{3}, \frac{7+(-7}{2} = F\left(\frac{-10}{2}, \frac{0}{2}\right)$

$$F = (-5,0)$$

Midpoint$of CD\ is\ G\ \left(\frac{-3+5}{2}, \frac{-7+(-5}{2}\right) =$

$G\left(\frac{2}{2}, \frac{-7-5}{2}\right)$

$$= G\ \frac{2}{2}, \frac{-12}{2}\ \ = G(1,6)$$

Mid-point $AD\ is\ H\ \left(\frac{9+5}{2}, \frac{3-5}{2}\right) = H\left(\frac{14}{2}, \frac{-2}{2}\right)$

$$= H\left(\frac{14}{2}, -\frac{2}{2}\right.$$

Now point of AD is H $\left(\frac{9+h}{2}, \frac{3-5}{2}\right) = H\left(\frac{14}{2}, -\frac{2}{2}\right)$

$$= (7,-1)$$

Now

$figure\ formed\ by\ midpoint\ E, F, G\ and\ H\ will\ be\ ||gram\ if\ |EF| = |HG|\ and\ |HE| = |GF|\ so$

$$|EF| = \sqrt{(-5-1)^2 + (0-5)^2}$$
$$= \sqrt{(-6)^2 + (5)^2}$$
$$= \sqrt{61}$$
$$|GF| = \sqrt{(1-7)^2 + (-6+1)62}$$
$$= \sqrt{(-6)^2 + (6)^2} = \sqrt{36+25}$$
$$= \sqrt{61}$$

$$|HE| = \sqrt{(1-7)^2 + (5-(-1)^2}$$
$$= \sqrt{(-6)^2 + (6)^2}$$
$$= \sqrt{36+36} = \sqrt{72}$$

Thus $|EF| = |HG|$ *and* $|HE| + |GF|$ *so EFGH is a* $||gram$.

Question.11.

Find h such that the quadrilateral with vertices A (-3,0) B(1,-2) C(5,0) and D(1,h) is a parallelogram. Is it a square?

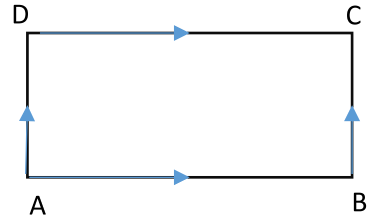
Solution.

Given $A(-3,0), B(1,-2), C(5,0), D(1,h)$

Quadrilateral ABCD is a parallelogram if

$$|AB| = |CD| and\ |BC| = |AD|$$

When $|AB| = |CD|$



$$|AB|^2 = |CD|^2$$
$$(1+3)^2 + (-2-0)^2 = (1-5)^2 + (h-0)^2$$
$$16 + 4 = 16 + h^2$$
$$h^2 = 4$$
$$h = \pm 2$$

When $h = 2$ *then* $D(1,h) = D(1,2)$ then

$$|AB| = \sqrt{(1+3)^2 + (-2-0)^2} = \sqrt{16+4}$$
$$= \sqrt{20}$$
$$|BC| = \sqrt{(5-1)^2 + (0+2)^2} = \sqrt{16+4} = \sqrt{20}$$
$$|CA| = \sqrt{(1-5)^2 + (2-0)^2} = \sqrt{16+4} = \sqrt{20}$$
$$|DA| = \sqrt{(-1-3)^2 + (0-2)^2} = \sqrt{16+4}$$
$$= \sqrt{20}$$

Now for diagonals

$$|AC| = \sqrt{(5+3)^2 + (0-0)^2} = \sqrt{64+0} = \sqrt{64}$$
$$= 8$$
$$|BD| = \sqrt{(1-1)^2 + (2+2)^2} = \sqrt{0+16} = \sqrt{16}$$
$$= 4$$

Hence all sides all equal but diagonals $|AC| \neq |BD|$

Therefore ABCD is a parallelogram but not a square.

Now when $h = -2$ *then* $D(1,h) = D(1,-2)$*but we also have* $B(1,-2)$.B and D represent the same point which cannot happened in quadrilateral. So we cannot take $h = -2$.

Question.12. If two vertices of an equilateral triangle are *A*(–3, 0) and *B*(3, 0) find the third vertex. How many of these triangles are possible?

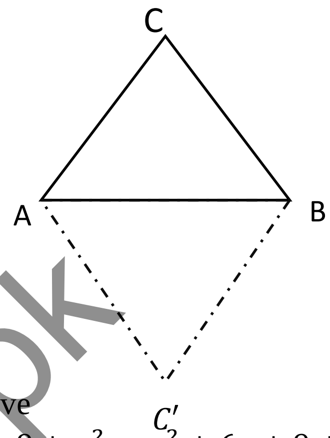
Solution.

Given that$A(-3,0). B(3,0)$.

Let $C(x,y)$ be the third vertex of an equilateral triangle ABC.

Then $|AB| = |BC| = |CA| \rightarrow |AB|^2 = |BC|^2 = |CA|^2$

$$(3+3)^2 + (0-0)^2 = ((x-3)^2 + (y-0)^2$$
$$= (x+3)^2 + (y-0)^2$$
$$36 + 0 = x^2 - 6x + 9 + y^2 = x^2 + 6x + 9 + y^2$$
$$36 = x^2 - 6x + 9 + y^2 = x^2 + 6x + 9 + y^2$$
$$\rightarrow\ i$$



From (i), we have

$$x^2 - 6x + 9 + y^2 = x^2 + 6x + 9 + y^2$$
$$-6x = 6x$$
$$12x = 0$$
$$x = 0$$

Again from the equation (i), we have

$$36 = x^2 + y^2 - 6x + 9$$

Using $x = 0$ *, we have*

$$36 = y^2 + 9$$
$$y^2 = 36 - 9 = 27$$
$$y = \pm 3\sqrt{3}$$

Hence the required third vertex is $C(x,y) = C(0, \pm 3\sqrt{3})$.

Hence two triangles formed.

Question.13.Find the points trisecting the join of $A(-1,4)$ *and* $B(6,2)$.

Solution.

Given that

$$A(-1,4) and\ B(6,2)$$

Let C *and D be the points bisecting A and B.*

Then $AC : CB = 1:2$

So Coordinates of C= $\left(\frac{1(6)+2(-1}{1+2}, \frac{1(2)+2(4)}{1+2}\right)$

$$C = \frac{6-2}{3}, \frac{3+8}{3} = \frac{4}{3}, \frac{10}{3}$$

Also $AD : DB = 2:1$

So Coordinates of D= $\left(\frac{2(6)+1(-1}{2+1}, \frac{2(2)+1(4)}{2+1}\right)$ C $= \left(\frac{12-1}{3}, \frac{4+4}{3}\right) = \left(\frac{11}{3}, \frac{8}{3}\right)$

Hence

$\left(\frac{4}{3}, \frac{10}{3}\right)$ *and* $\left(\frac{11}{3}, \frac{8}{3}\right)$ *are points of trisecting A and B.*

Question.14.
Find the point three-fifth of the way along the line segment from $A(-5,8)$ to $B(5,3)$.
Solution.
Given that

$$A(-5,8) and\ B(5,3)$$

Let $C(x,y) be\ a\ required\ point\ then$

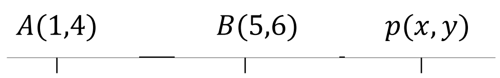
$$AC:CB = 3:2$$

So Coordinates of $C = \left(\frac{3(5)+2(-5}{3+2}, \frac{3(3)+2(8)}{3+2}\right)$

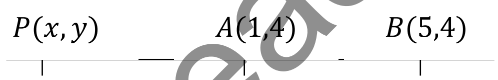
$$C = \frac{15-10}{5}, \frac{9+16}{5} = \frac{5}{5}, \frac{25}{5}$$

$$C = (1,5)$$

Question.15. Find the point P on the joining of A (1, 4) and B (5, 6) that is twice as far from A as B is from A and lies
(i) Lies on the same side of the A and B
(ii) On the opposite side of A as B does.
Solution.
$(i) A(1,-4), B(5,6)$
$\because B\ becomes\ midpoint\ of\ AP\ so$
$5 = \frac{1+x}{2}, 6 = \frac{4+y}{2}$

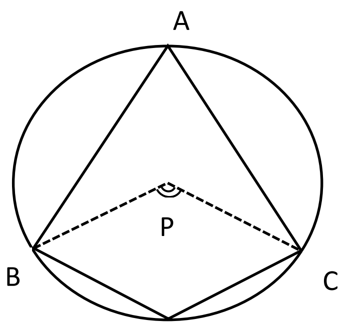


⇨ $10 = 1 + x \quad , \quad 12 = 4 + y$
⇨ $x = 10 - 1 \quad , \quad y = 12 - 4$
⇨ $x = 9 \quad y = 8 \quad so\ P(9,8) is\ the\ required\ point.$
$(ii) \quad A(1,4), B(5,6)$



$\because A\ divides\ PB\ in\ ratio\ 2:1$

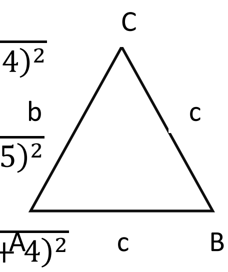
Question.16. Find the point which is equidistant from the points $A(5,3), B(2,-2) and\ C(4,2)$. What is the radius of the circumcircle of the ΔABC.
Solution.
Given that $A(5,3), B(2,-2) and\ C(4,2)$
Let D(x,y) be a point which is equidistant from A , B and C then



$|DA| = |DB| = |DC|$
$|DA|^2 = |DB|^2 = |DC|^2$
$(x-5)^2 + (y-3)^2$
$= (x+2)^2 + (y-2)^2$
$= (x-4)^2 + (y-2)^2 \rightarrow (i)$

From (i), we have
$(x-5)^2 + (y-3)^2 = (x+2)^2 + (y-2)^2$
$x^2 - 10x + 25 + y^2 - 6y + 9$
$= x^2 + 4x + 4 + y^2 - 4y + 4$
$-10x - 6y + 34 = 4x - 4y + 8$
$-10x - 4x - 6y + 4y + 34 - 8 = 0$
$-14x - 2y + 26 = 0$
$7x + y - 13 = 0 \rightarrow (ii)$
Again from (i) , we have
$(x+2)^2 + (y-2)^2 = (x-4)^2 + (y-2)^2$
$x^2 + 4x + 4 + y^2 - 4y + 4$
$= x^2 - 8x + 16 + y^2 - 4y + 4$
$4x - 4y + 8 = -8x - 4y + 20$
$4x + 8x + 8 - 20 = 0$
$12x - 12 = 0$
$12x = 12$
$x = 1$
Using this value in (ii), we have
$7 + y - 13 = 0$
$y - 6 = 0$
$y = 6$
Hence the required point is $D(x,y) = D(1,6)$.
Now Radius of circumcircle $= |DA| =$
$\sqrt{5-1)^2 + (3-6)^2} = \sqrt{16+9} = \sqrt{25} = 5$ units.

Question.17.
The point
$\boldsymbol{(4,-2), (-2,4)\ and\ C(5,5) are\ the\ vertices\ of\ a\ triangle. find\ the\ in-center\ of\ the\ triangle.}$
Solution.
Let $A(4,-2), B(-2,4)$, $C(5,5)$ are the vertices of triangle then



$a = |BC| = \sqrt{(5+2)^2 + (5-4)^2}$
$= \sqrt{49+1} = \sqrt{50} = 5\sqrt{2}$
$b = |CA| = \sqrt{(4-5)^2 + (2-5)^2}$
$= \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$
$c = |AB| = \sqrt{(-2-4)^2 + (2+4)^2}$
$= \sqrt{36+36} = \sqrt{36 \times 2} = 6\sqrt{2}$
Now

$$In - center = \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

$In - center$

$$= \left(\frac{5\sqrt{2}(4)+5\sqrt{2}(-2)+6\sqrt{2}(5)}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}, \frac{5\sqrt{2}(-2)+5\sqrt{2}(4)+6\sqrt{2}(5)}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}\right)$$

$In - center$

$$= \left(\frac{20\sqrt{2}-10\sqrt{2}+30\sqrt{2}}{16\sqrt{2}}, \frac{-10\sqrt{2}+20\sqrt{2}+30\sqrt{2}}{16\sqrt{2}}\right)$$

$$In - center = \left(\frac{40\sqrt{2}}{16\sqrt{2}}, \frac{40\sqrt{2}}{16\sqrt{2}}\right)$$

$$In - center = \frac{5}{2}, \frac{5}{2}$$

Question.18.

Find the points that divide the line segment joining $A(x_1,y_1)$ and $B(x_2,y_2)$ into four equal parts.

Solution.

Given

$A(x_1,y_1)$ and $B(x_2,y_2)$

Let

C, D and E are the points dividing AB into four equal parts.

Since $AC:CB = 1:3$

$Co-ordinate\ of\ C$

$$= \left(\frac{1(x_2)+3(x_1)}{1+3}, \frac{1(y_2)+3(y_1)}{1+3}\right)$$

$$Co-ordinate\ of\ C = \frac{2x_1+x_2}{4}, \frac{2y_1+y_2}{4}$$

Now $AD:DB = 2:2 = 1:1$

$Co-ordinate\ of\ D$

$$= \left(\frac{1(x_2)+1(x_1)}{1+1}, \frac{1(y_2\ +1(y_1}{1+1}\right)$$

$$Co-ordinate\ of D = \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}$$

Now $AE:EB = 3:1$

$Co-ordinate\ of\ E$

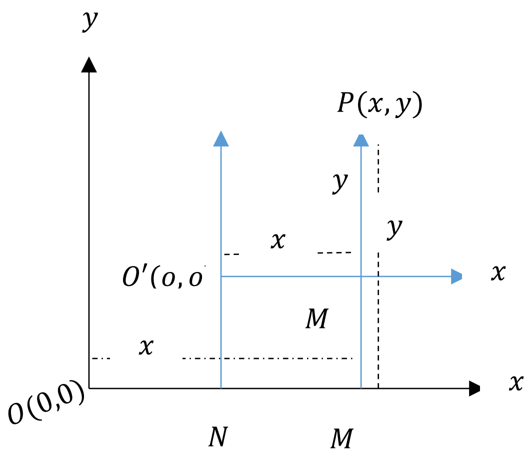
$$= \left(\frac{3(x_2)+1(x_1)}{3+1}, \frac{3(y_2)+1(y_1)}{3+1}\right)$$

$Co-ordinate\ of E$

$$= \frac{x_1+3x_2}{4}, \frac{y_1+3y_2}{4}$$

Hence

$\left(\frac{2x_1+x_2}{4}, \frac{2y_1+y_2}{4}\right)$, $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ and $\left(\frac{x_1+3x_2}{4}, \frac{y_1+3y_2}{4}\right)$ are the points divining AB into four equal parts?



Translation and relation of axis.

Let $P(x,y)$ be any point in $xy-Plane$. Let we Draw two mutually perpendicular lines $O'X$ and $O'Y$

Such that they meet at a point $O'(h,k)$ in $xy-plane$

Here $O'(x)$ and $O'(y)$ are parallel to OX and OY Respectively. The new axis $O'X$ and $O'Y$ are called translation of OX and OY axis through point O'. Let $P(x,y)$ be point in $XY-Plane$. Draw $\perp$ ars PM and $O'N$ from P and O' on $x-axis$. in figure

$OM = x, PM = y\ \ O'N = M'M = k\ X = O'M'$

$= NM$

$= OM - ON = x - h$ and $Y = PM'$

$= PM - M'M$

$PM - O'N = y - k$

thus cordinates of P in $XY-plane$ are (X,Y)

$= (x-h, y-k)$

Important note:

as $X = x - h \Rightarrow x = X + h$ and

$Y = y - k \Rightarrow y = Y + k$

i. If $P(x,y)$ and $O'(h,k)$ are given in $xy-plane$ and we are to find $xy-coordinates$ of p
Then we put $X = x - h$ and $Y = y - k$

ii. if $P(X,Y)$ and $O'(h,k)$ are given in $XY-plane$ and we are to find xy coordinates of P then we put $x = X + h$ and $y = Y + k$

Exercise 4.2

Question.1.

The two points P and O' are given in $xy-coordinates\ system$.Find the $XY-coordinates$ of P referred to the translated axes $O'X$ and $O'Y$.

(i). $P(3,2); O'(1,3)$

Solution.

Since $P(x,y) = P(3,2)$

$$x = 3 \text{ and } y = 2.$$
$$O'(h,k) = O'(1,3)$$
$$h = 1 \text{ and } k = 3.$$

Since

$$X = x - h \quad \text{and} \quad Y = y - k$$
$$X = 3 - 1 \quad \text{and} \quad Y = 2 - 3$$
$$X = 2 \quad \text{and} \quad Y = -1$$

Hence $(2,-1)$ is point P in $XY-coordinates$.

(ii) $P(-2,6)$, $O'(-3,2$

Solution.

Here $x = -2$, $y = 6$ and $h = -3$, $k = 2$

$$P(X,Y) = ? \quad \because X = x - h = -2 - (-3)$$
$$= -2 + 3 = 1$$

And $Y = y - k = 6 - 2 = 4$ so $P(X,Y) = P(1,4)$

(iii). $P(-6,-8)$, $O'(-4,-6)$

Solution.

Here $x = -6, y = -8, h = -4, k = -6$ $\quad P(X,Y) = ?$

$$\because X = x - h = -6 \pm 4 = -6 + 4 = -2$$
$$Y = y - k = -8 - (-6) = -8 + 6 = -2$$

So $P(X,Y) = P(-2,-2)$

(iv). $P\left(\frac{3}{2},\frac{5}{2}\right)$, $O'\left(-\frac{1}{2},\frac{7}{2}\right)$

Solution.

Since $P(x,y) = P\left(\frac{3}{2},\frac{5}{2}\right)$

$$x = \frac{3}{2} \text{ and } y = \frac{5}{2}$$
$$O'(h,k) = O'\left(-\frac{1}{2},\frac{7}{2}\right)$$
$$h = -\frac{1}{2} \text{ and } k = \frac{7}{2}$$

Since

$$X = x - h \quad \text{and} \quad Y = y - k$$
$$X = \frac{3}{2} + \frac{1}{2} \quad \text{and} \quad Y = \frac{5}{2} - \frac{7}{2}$$
$$X = \frac{4}{2} \quad \text{and} \quad Y = -\frac{2}{2}$$
$$X = 2 \quad \text{and} \quad Y = -1$$

Hence $(2,-1)$ is point P in $XY-coordinates$.

Question.2.

The $xy-coordinate$ axes are translated through the point O' whose coordinates are given in $xy-coordinates$. the coordinates of P are given in the $XY-coordinate\ system$.Find the coordinates of P in $xy-coordinates$ system.

(i). $P(8,10)$, $O'(3,4)$

Solution.

Since $P(X,Y) = P(8,10)$

$$X = 8 \text{ and } Y = 10$$
$$O'(h,k) = O'(3,4)$$
$$h = 3 \text{ and } k = 4$$

Since

$$X = x - h \quad \text{and} \quad Y = y - k$$
$$8 = x - 3 \quad \text{and} \quad 10 = y - 4$$
$$x = 8 + 3 \quad \text{and} \quad y = 10 + 4$$
$$x = 11 \quad \text{and} \quad y = 14$$

Hence $(11,14)$ is point P in $xy-coordinates$.

(ii). $P(-5,-3)$, $O'(-2,-6)$

Solution.

Since $P(X,Y) = P(-5,-3)$

$$X = -5, \quad Y = -3$$
$$O'(h,k) = O'(-2,-6)$$
$$h = -2 \text{ and } k = -6$$

Since

$$X = x - h \quad \text{and } Y = y - k$$
$$8 = x - 3 \quad \text{and } 10 = y - 4$$
$$x = 8 + 3 \quad \text{and } y = 10 + 4$$
$$x = 11 \quad \text{and } y = 14$$

(iii). $P\left(-\frac{3}{4},-\frac{7}{6}\right)$, $O'\left(\frac{1}{4},-\frac{1}{6}\right)$

Solution.

Since $P(X,Y) = P\left(-\frac{3}{4},-\frac{7}{6}\right)$

$$X = -\frac{3}{4} \text{ and } Y = -\frac{7}{6}$$
$$O'(h,k) = O' \frac{1}{4}, -\frac{1}{6}$$
$$h = \frac{1}{4} \text{ and } k = -\frac{1}{6}$$

Since

$$X = x - h \quad \text{and} \quad Y = y - k$$
$$-\frac{3}{4} = x - \frac{1}{4} \quad \text{and} \quad -\frac{7}{6} = y + \frac{1}{6}$$
$$x = -\frac{3}{4} + \frac{1}{4} \quad \text{and} \quad y = -\frac{7}{6} - \frac{1}{6}$$
$$x = -\frac{2}{4} \quad \text{and} \quad y = -\frac{8}{6}$$
$$x = -\frac{1}{2} \quad \text{and} \quad y = -\frac{4}{3}$$

Hence $(-\frac{1}{2}, -\frac{4}{3}$ is point P in $xy - coordinates$.

(iv). $\boldsymbol{P(4,-3), O'(-2,3)}$

Solution.

Since $P(X,Y) = P(4,-3)$

$$X = 4 \text{ and } Y = -3$$
$$O'(h,k) = O'(-2,3)$$
$$h = -2 \text{ and } k = 3$$

Since

$$X = x - h \quad and \quad Y = y - k$$
$$4 = x + 2 \quad and \quad -3 = y - 3$$
$$x = 4 - 2 \quad and \quad y = -3 + 3$$
$$x = 2 \quad and \quad y = 0$$
$$so\ P(x,y) = P(2,0)$$

Question.3.

The xy − coordinates -axes are rotated about the origin through the indicated angle. The new axes are *OX* and *OY*. Find the *XY-coordinates* of the point *P* with the *given* xy − coordinates.

(i), $\mathbf{P(5,3)\ ;\theta = 45^0}$

Solution.

Since

$$P(x,y) = P(5,3)$$
$$x = 5 \quad and \quad y = 3 \quad ,\theta = 45^0$$

Since

$$X = xCos\theta + ySin\theta$$
$$X = 5Cos45^0 + 3Sin45^0$$
$$X = 5\left(\frac{1}{\sqrt{2}} + 3\ \frac{1}{\sqrt{2}}\right.$$
$$X = \frac{5+3}{\sqrt{2}}$$
$$X = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

Also

$$Y = yCos\theta - xSin\theta$$
$$Y = 3Cos45^0 - 5Sin45^0$$
$$Y = 3\left(\frac{1}{\sqrt{2}} - 5\ \frac{1}{\sqrt{2}}\right.$$
$$Y = \frac{3-5}{\sqrt{2}}$$
$$Y = \frac{-2}{\sqrt{2}} = -\sqrt{2}$$

Hence the required point is $\left(4\sqrt{2}, -\sqrt{2}\right)$.

(ii). $P(3,-7); \theta = 30^0$

Solution.

$$Here\ x = 3, y = -7, \theta = 30^o, P(X,Y) = ?$$
$$\because X = xcos\theta + ysin\theta = 3cos30^0 - 7sin30^0$$
$$= 3\left(\frac{\sqrt{3}}{2}\right) - 7\left(\frac{1}{2}\right. = \frac{3\sqrt{3}-7}{2}$$

$$Y = ycos\theta - xsin\theta = -7cos30^0 - 7sin30^0$$
$$= \frac{-7\sqrt{3}}{2} - \frac{3}{2} = \frac{-7\sqrt{3}-2}{2}$$

$So\ P(X,Y) = P(\frac{3\sqrt{3}-7}{2}, \frac{-7\sqrt{3}-2}{2}$

(iii). $\boldsymbol{P(11,-15); \theta = 60^0}$

Solution.

$$here\ x = 11, y = -15, \theta = 60^0, P(X,Y) = ?$$
$$\because xcos\theta + ysin\theta = 11cos60^0 - 15sin60^0$$
$$= 11\left(\frac{1}{2}\right) - 15\left(\frac{\sqrt{3}}{2}\right) = \frac{11-15\sqrt{3}}{2}$$
$$Y = ycos\theta - xsin\theta = -15cos60^0 - 11sin60^0$$
$$-\frac{15}{2} - \frac{11\sqrt{3}}{2} = \frac{-15-11\sqrt{3}}{2}$$

$So\ P(X,Y) = P\left(\frac{11-15\sqrt{3}}{2}, \frac{-15-11\sqrt{3}}{2}\right)$

(iv). $\boldsymbol{P(15,10); \theta = \text{arctan}\frac{1}{3}}$

Solution.

Since

$$P(x,y) = \text{P}(15,10)$$
$$x = 15 \quad and \quad y = 10$$
$$\theta = \tan^{-1}\frac{1}{3}$$
$$tan\theta = \frac{1}{3} = \frac{p}{b}$$
$$p = 1 \quad , \quad b = 3$$
$$h^2 = p^2 + b^2 = 1 + 3^2 = 1 + 9 = 10$$
$$h = \sqrt{10}$$
$$Sin\theta = \frac{p}{h} = \frac{1}{\sqrt{10}} \quad and\ Cos\theta = \frac{b}{h} = \frac{3}{\sqrt{10}}$$

Now

$$X = 15\ \frac{3}{\sqrt{10}} + 10\ \frac{1}{\sqrt{10}}$$
$$X = \frac{45}{\sqrt{10}} + \frac{10}{\sqrt{10}}$$
$$X = \frac{55}{\sqrt{10}}$$

Also

$$Y = yCos\theta - xSin\theta$$
$$Y = 10\frac{3}{\sqrt{10}} - 15\frac{1}{\sqrt{10}}$$
$$Y = \frac{30}{\sqrt{10}} - \frac{15}{\sqrt{10}}$$
$$Y = \frac{30-15}{\sqrt{10}}$$
$$Y = \frac{15}{\sqrt{10}}$$

Hence the required point is $\left(\frac{55}{\sqrt{10}}, \frac{15}{\sqrt{10}}\right)$.

Question.4.

The *xy-coordinates* axes are rotated about the origin through the indicated angle and the new axes are *OX* and *OY*, Find the *xy-coordinates* of *P* with the given *XY-coordinates*.

(i). $P(-5,3)\ ;\theta = 30^0$

Solution.

Since

$$P(X,Y) = P(-5,3)$$

$$X = -5 \quad and \quad Y = 3$$

Also

$$\theta = 30^0$$

Therefore $Sin\theta = Sin30^0 = \frac{1}{2}$ and $Cos\theta = Cos30^0 = \frac{\sqrt{3}}{2}$

Now

$$X = xCos\theta + ySin\theta$$

$$-5 = x\frac{\sqrt{3}}{2} + y\,\frac{1}{2}$$

$$-5 = \frac{\sqrt{3}x + y}{2}$$

$$\sqrt{3}x + y = 1 = -10 \rightarrow (i$$

Also

$$Y = yCos\theta - xSin\theta$$

$$3 = y\frac{\sqrt{3}}{2} - x\frac{1}{2}$$

$$3 = \frac{\sqrt{3}y - x}{2}$$

$$\sqrt{3}y - x = 6$$

$$\sqrt{3}y - 6 = x$$

$$x = \sqrt{3}y - 6 \rightarrow (ii)$$

Using (ii) in (i) , we have

$$\sqrt{3}(\sqrt{3}y - 6) + y = -10$$

$$3y - 6\sqrt{3} + y = -10$$

$$4y = -10 + 6\sqrt{3}$$

$$y = \frac{6\sqrt{3} - 10}{4}$$

$$y = \frac{3\sqrt{3} - 5}{2}$$

Using in (ii), we have

$$x = \sqrt{3}\left(\frac{3\sqrt{3} - 5}{2}\right) - 6$$

$$x = \frac{3(3) - 5\sqrt{3} - 12}{2}$$

$$x = \frac{9 - 5\sqrt{3} - 12}{2}$$

$$x = -\frac{5\sqrt{3} + 3}{2}$$

Hence the required point is $\left(-\frac{5\sqrt{3}+3}{2}, \frac{3\sqrt{3}-5}{2}\right)$.

(ii)$P(-7\sqrt{2},\ 5\sqrt{2})\ ;\theta = 45^0$.

Solution.

$$here\ X = -7\sqrt{2}, Y = 5\sqrt{2}, \theta = 45^0$$

$$\because X = xcos\theta + ysin\theta$$

$$-7\sqrt{2} = xcos45^0 + ysiin45^0$$

$$-7\sqrt{2} = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$$

$$\Rightarrow x + y = -7(2) \quad \text{x by } \sqrt{2}$$

$$\Rightarrow x + y = -14 \rightarrow (i$$

$And\ Y = ycos\theta - xsin\theta$

$$\Rightarrow 5\sqrt{5} = ycos45^0 - xsin45^0$$

$$5\sqrt{2} = \frac{y}{\sqrt{2}} - \frac{x}{\sqrt{2}}$$

$\Rightarrow -x + y = 5(2)$ " $\times$ "$by\ \sqrt{2}$

$$\Rightarrow -x + y = 10 \rightarrow (ii$$

$by\ (i) + (ii) \quad \Rightarrow$

$$(x + y = -14)$$

$$-x + y = 10$$

$$2y = -4$$

$\Rightarrow y = -2\ put\ in(I)$

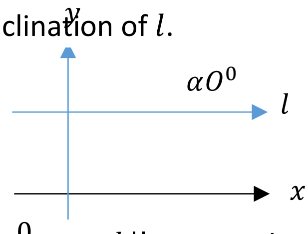
$$x - 2 = -14 \Rightarrow x = -14 + 2$$

$\Rightarrow x = -12\ so\ P(x,y) = (-12,-2)$

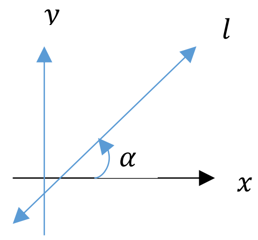
Equations of straight lines:

Inclination of lines: The angle $(O^0 < \alpha < 180^0)$ Measured anti-clockwise from positive $x -$ *axis to*

A non-horizontal straight line l *is called* inclination of l.

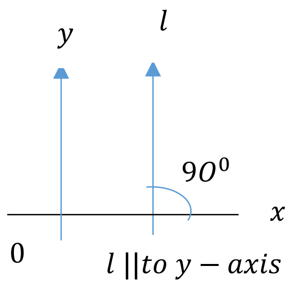


$l\ ||to\ x - axis$



Note:

i. If l *is* $||to\ x - axis, then\ \alpha = o^0$
ii. $if\ l\ is\ parallel\ to\ y - axis.then\ \alpha = 90^0$



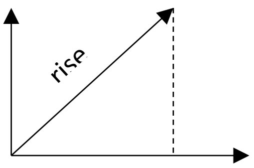
$l\ ||to\ y - axis$

Slope or gradient of a line:

Let α *be inclination of a line, then slope of a line is denoted by m and denoted by m defined as*

$m = tan\alpha$

- The measure of steepness (ratio) of rise to the run is named as slope or gradient.



Note:

- Slope of $x-axis$ or any line parallel to $x-axis$ is zero $(\alpha = O^0 \Rightarrow tano^0 = 0$
- $slope\ of\ y-axis\ or\ any\ line\ parallel\ to\ y-axis\ is\ undefined. (\because \alpha = 90^0 \Rightarrow tan90^0 = \infty$
- If $O^0 < \alpha < 90^0$ then m is positive and if $9o^0 < \alpha < 180^0\ then\ m\ is -ve.$

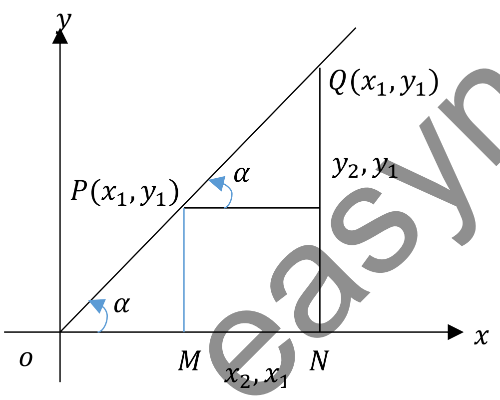
Slope of a straight line joining two points.

Theorem:

$if\ a\ non-verticle\ line\ l\ with\ inclined\ \alpha\ passes\ through\ two\ points\ P(x,y) and\ Q(x,y), then\ the\ slope\ or\ gradient\ m\ of\ l\ is\ given\ by$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = tan\alpha$$

Proof:



Let us draw $\perp ars. PM\ and\ QN\ from\ points\ P\ and\ Q\ on\ x-axis. ALso\ draw\ a \perp ar\ PR\ on\ QN. We\ get\ right\ angled\ \triangle QPR.$

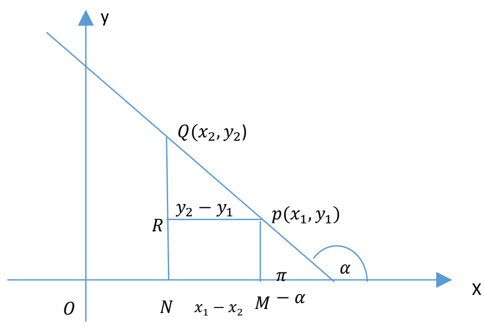
In figure $|PR| = |MN| = |ON| - |OM| = x_2 - x_1$

$$|QR| = |QN| - |RN| = y_2 - y_1$$

$$In\ \triangle QPR\quad m = tan\alpha = \frac{|QR|}{|PR|} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$thus\ m = \frac{y_2 - y_1}{x_2 - x_1} = Tan\alpha$$

Case(ii) $when \frac{\pi}{2} < \alpha < \pi$



$Let\ us\ draw\ \perp ars.\ PM\ and\ QN\ from\ points\ P\ and\ Q\ on\ x-axis\ Also\ draw\ a\perp ar\ PR\ on\ QN. we\ get\ right-angled\ \triangle QPR.$

In figure $|PR| = |MN| = |OM| - |ON| = x_1 - x_2$

$also |PR| = |QN| - |RN| = y_2 - y_1$

$$\triangle QPR, m = Tan(\pi - \alpha) = \frac{|QR|}{|PR|} = \frac{y_2 - y_1}{x_1 - x_2}$$

$$\Rightarrow m = Tan\alpha = \frac{y_2 - y_1}{x_1 - x_2}$$

$$m = Tan\alpha = \frac{y_2 - y_1}{-x_1 - x_2}$$

$m = Tan\alpha = \frac{y_2 - y_1}{x_2 - x_1}$ Hence proved.

Note:

i. $\boldsymbol{m \neq \frac{y_2 - y_1}{x_1 - x_2}\quad and\ m = \frac{y_1 - y_2}{x_2 - x_1}}$

ii. **L is horizontal,** $\boldsymbol{iff\ m = 0 (\because \alpha = o^2}$

iii. $\boldsymbol{l\ is\ vertical, iff\ m\ is\ not\ defined\ \because \alpha = 90^0}$

iv. $\boldsymbol{If\ slope\ of\ AB = slope\ of\ BC, then\ points\ A, B\ and\ C\ are\ collinear}$

Theorem:

The two lines $l_1 and\ l_2 with\ respectively\ slopes\ m_1$ And $m_2\ are\ (i) parallel\ iff\ m_1 = m_2$

$$(ii) perpendicular\ iff\ m_1 = -\frac{1}{m_2} or\ m_1 m_2 = -1$$

Equation of straight lines

✓ Line parallel to $x-axis (or\ perpendicular\ to\ y-axis)\ An\ equation\ of\ the\ form\ y = a$ is called equation of line parallel to $x-axis.$

i. If $a > 0$ then the line l is below the $x-axis.$

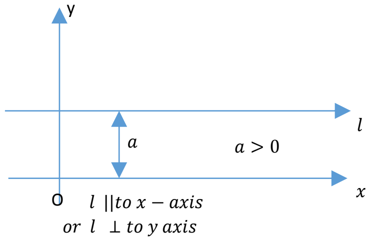
ii. $if\ a = 0\ then\ the\ line\ l\ becomes\ the\ x-axis.$ Thus equations$of\ x-axis\ is\ y = o$

iii. Line parallel to $y-axis (or\ perp. to\ x-axis$

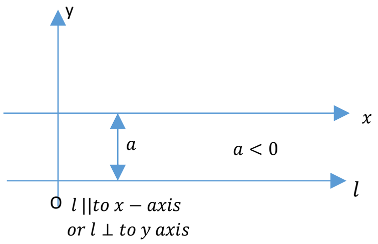
An equation of the form $x = b\ is\ called\ eq.$

of line parallel to $y - axis$.

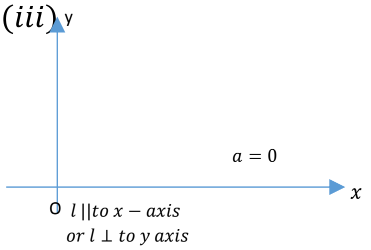
(i)



(ii)



(iii)



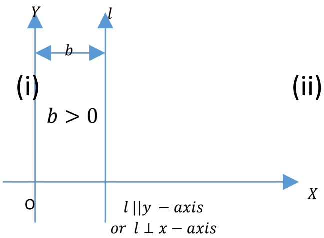
Note:

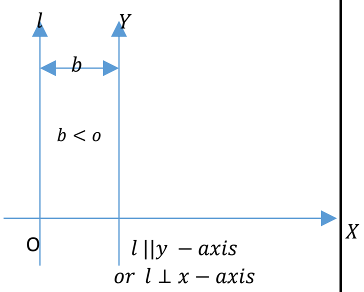
(i) $if\ b > 0\ then\ the\ line\ is\ on\ the\ right\ of\ the\ y - axis$

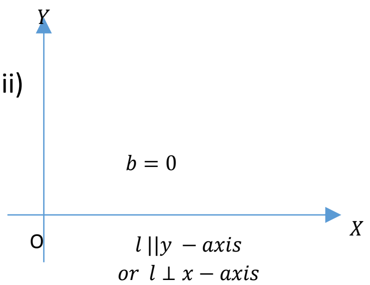
(ii) $if\ b < 0, then\ the\ line\ is\ on\ the\ left\ of\ y - axis.$

(iii) If b=0 then the line becomes the $y - axis. thus$ the equation of $y - axis\ is\ x = 0$

(i)

(i) 

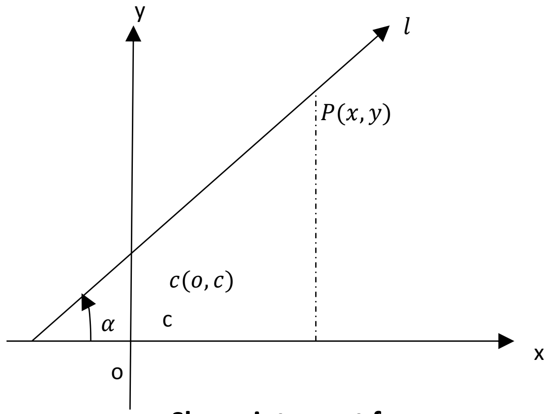
(ii) 

(iii) 

Intercept:

$*\ if\ a\ line\ intersects\ x - axis\ at\ pt. (a, o) then\ a$ Is called $x - intercept\ of\ the\ line.$

$*\ if\ a\ line\ intercept\ y - axis\ at\ pt. (o, b) then\ b$ $is\ called\ y - intercept\ of\ the\ line.$



Slope- intercept form

Theorem: equation of non-vertical straight line with slop m and $y - intercept\ c\ is\ given\ by\ y = mx + c$

Proof:

Since $m\ is\ the$ $slop\ of\ line\ l$

And $y\ intercept\ is\ c$

So point on $y - axis$

Will be $(0, c)$ $let\ p(x, y) be\ any\ point\ on\ the\ line\ l.$

$\because the\ line\ l\ passes\ through\ points\ C(0, c) and\ P(x, y)$

$$so\ using\ m = \frac{y_2 - y_1}{x_2 - x_1}$$

⇨ $m = \frac{y-c}{x-o} = \frac{y-c}{x} \Rightarrow mx = y - c$

⇨ $y = mx + c$ $hence\ proved.$

Note:

The equation of the line for which $c = 0\ is\ y = mx + c$

$in\ this\ case\ the\ line\ passes\ through\ the\ orgin.$

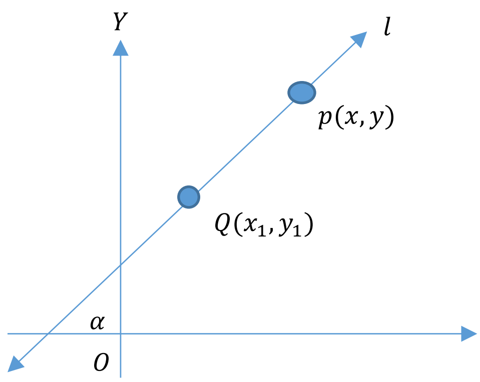
Point –slope form

Theorem:

Equation of a non-vertical straight line l with slope m and passing through a point $Q(x, y) is$

$$y - y_1 = m(x - x_1)$$

Proof:



since m is the slope of line passes throgh point $Q(x_1, y_1)$
Let $P(x, y)$ *be any point on the line* l. *Since the line* l
Passes through the points
$Q(x_1, y_1)$ *and* $P(x, y)$ *so using*

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow m = \frac{y - y_1}{x - x_1} \Rightarrow m(x - x_1) = y - y_1$$

Or $y - y_1 = m(x - x_1)$ *hence proved.*

Symmetric form

we know that

$$m(x - x_1) = y - y_1 \qquad \because m = Tan\alpha = \frac{sin\alpha}{cos\alpha}$$

$$\Rightarrow y - y_1 = \frac{sin\alpha}{cos\alpha}(x - x_1)$$

$$\Rightarrow \frac{y - y_1}{sin\alpha} = \frac{x - x_1}{cos\alpha} \quad \text{thus is called symmetric.}$$

Form of equation of a straight line.

Two point form

Theorem:

Equation of a non- vertical straight line passing through two points $Q(x_1, y_1)$ *and* $R(x_2, y_2)$ *is*

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \quad 0r \quad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Proof:

Let $p(x, y)$ *be any oint on the line* l

$\because$ *line passes through* $R(x_2, y_2)$ *points* $Q(x_1, y_1)$

As P, Q, R *are collinear points. so*

$$slop\ of\ QR = Slop\ of\ QP$$

$$⇨ \frac{y_2 - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\Rightarrow y_2 - y_1)(x - x_1) = (y - y_1)(x_2 - x_1$$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$⇨ \ y(x_2 - x_1) - y(x_2 - x_1) = x(y_2 - y_1) - x_1(y_2 - y_1$

$⇨ \ -x(y_2 - y_1) + y(x_2 - x_1) + x_1(y - y_1) - y_1(x_2 - x_1) = 0$

$⇨ \ x(y_1 - y_2) - y(x_1 - x_2) + x_1y_2 - x_2y_1 = 0$

$$⇨ \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \ \text{hence proved.}$$

if $x_1 = x_2$

then slope becomes undefined so, the line is vertical.

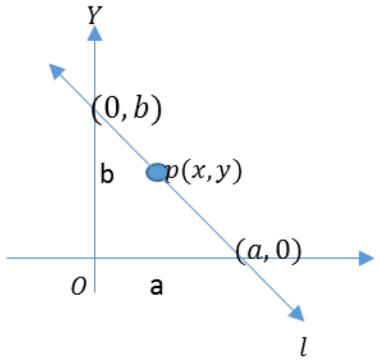
Intercept form

Theorem: Equation of a line whose non-zero *x and y*

Intercept are *a and b resp. is*

$$\frac{x}{a} + \frac{y}{b} = 1$$

Proof:



$\because$ *x intercept is a so point on x − axis is* $(a, 0)$ *and y − axis is* (o, b).

Hence equation of line passing through the *points* $(a, 0)$ *and* $(0, b)$ *is* — *two points form*

$$\frac{y-0}{b-0} = \frac{x-a}{0-a} \qquad \because \frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$⇨ \frac{y}{b} = \frac{x-a}{-a} = \frac{x}{-a} + 1 \Rightarrow \frac{y}{b} + \frac{x}{a} = 1$$

$$⇨ \frac{x}{a} + \frac{y}{b} = 1 \quad \text{hence proved.}$$

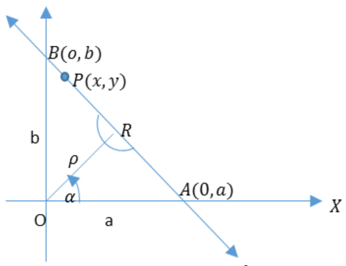
Normal form:

Theorem:

An equation of a non-vertical straight line l, such that length of the perpendicular from the origin to l is P and α is the inclination of this perpendicular is

$$xcos\alpha + ysin\alpha = P$$

Proof:



Let $P(x, y)$ *be any point of AB and let OR be ⊥ ar to line l. then* $|OR| = p$

Let *x − intercept be a and y − intercept be b.*

so equation of line is

$$\frac{x}{a} + \frac{y}{b} = 1 \rightarrow (1)$$

In $\triangle AOR, cos\alpha = \frac{OR}{OB} \Rightarrow \frac{p}{a}$

$\Rightarrow a = \frac{\rho}{cos\alpha}$

In $\Delta BOR, sin\alpha = \frac{OR}{OB} = \frac{\rho}{b}$

$$b = \frac{\rho}{sin\alpha} \quad \text{so (1) becomes as}$$

$$⇨ \frac{x}{\frac{\rho}{cos\alpha}} = \frac{y}{\frac{\rho}{sin\alpha}} = 1 \Rightarrow \frac{xcos\alpha}{\rho} + \frac{ysin\alpha}{\rho} = 1$$

$$⇨ xcos\alpha + ysin\alpha = \rho \ \text{hence proved.}$$

Linear Equation in two variables

General equation of straight line

Theorem: the linear equation $ax + by + c = 0$ In two variables *x and y represents a straight line.*

Proof:

Consider general linear equation in *x and y*

$$ax + by + c = 0 \rightarrow (i)$$

Where a, b, c *are constants and* $a \neq 0, b \neq 0$ *simultaneously.*

So following cases arises.

$case1.\ let\ a \neq 0\ but\ b = 0\ so\ (1) \Rightarrow ax + 0y + c = 0$

$\Rightarrow ax + c = 0 \Rightarrow x = -\frac{c}{a}$ *which is equation of line* $||to\ y - axis.$

Case II.

Let $a = 0$ *but* $b \neq o$ *so*

$$(1) \Rightarrow a(0) + by + c = 0 \Rightarrow by + c = 0 \Rightarrow y = -\frac{c}{b}$$

which is eq. of line $||to\ x - axis.$

Case III.

let $a \neq 0, b \neq o$ *so*

$$(1) \Rightarrow ax + by + c = 0 \Rightarrow by = -ax - c$$

⇨ $y = -\frac{a}{b}x - \frac{c}{a}$ *which is of the form* $y = mx + c$

(a line in slope intercept form) hence in all cases $(ax + by + c = 0$ *represents a line.*

Transform the general linear equation to standard form

Theorem: to transform the equation $ax + by + c = 0$ *in the standard form .*

1. Slope intercept form $y = mx + c$

$$\because ax + by + c = 0$$

$$\Rightarrow by = -ax - c \Rightarrow y = \left(-\frac{a}{b}\right)x + \left(-\frac{c}{b}\right)$$

2. Point slope form :$y - y_1 = m(x - x_1$

A point on the line is $\left(-\frac{c}{b}, o\right)$ *and slope is* $-\frac{a}{b}$ *so*

$$y - 0 - -\frac{a}{b}\left(x + \frac{c}{b}\right) \text{ this is point slope form.}$$

3. Symmetric form: $\left(\frac{x-x_1}{cos\alpha}, \frac{y-y_1}{sin\alpha}\right)$

⇨ $sin\alpha = \frac{a}{\sqrt{a^2+b^2}}, cos\alpha = \frac{b}{\sqrt{a^2+b^2}} \because tan\alpha = -\frac{a}{b}$ *and point an* $ax + by + c = 0$ *is* $\left(-\frac{c}{a}, 0\right)$ *so*

$$\frac{x - \left(-\frac{c}{a}\right)}{\frac{b}{\sqrt{a^2 + b^2}}} = \frac{y - o}{b/\sqrt{a^2 + b^2}}$$

is required symmetric form and sign of radical to be property chosen.

4. **Two point form** $\left(\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}\right)$

We take two points on $ax + by + c = 0$ *are* $\left(-\frac{c}{a}, 0\right)$ *and* $\left(0, -\frac{c}{b}\right)$*. so required transformed equation is*

$$\frac{y - o}{0 + \frac{c}{b}} = \frac{x + \frac{c}{a}}{-\frac{c}{a} - 0} \qquad \left(\because \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}\right.$$

5. Intercept form $\left(\frac{x}{a} + \frac{y}{b} = 1\right)$

$$\because ax + by + c = 0 \Rightarrow ax + by = -c$$

$$\Rightarrow \frac{a}{-c}x + \frac{b}{-c}y = 1$$

which is equation of required two intersects form.

6. Normal Form

$$\because ax + by + c = 0 \rightarrow (1)$$

$$and\ xcos\alpha + ysin\alpha = \rho \rightarrow (2)\ Normal\ form$$

As $(1) and\ (2) are\ identicle\ so$

$$\frac{a}{cos\alpha} = \frac{b}{sin\beta} = -\frac{c}{\rho} \rightarrow 3$$

$$\because m = tan\alpha = -\frac{a}{b} \quad so \quad sin\alpha = \frac{a}{\sqrt{a^2 + b^2}}$$

$$cos\alpha = \frac{b}{\sqrt{a^2 + b^2}} \quad so$$

$$\because \frac{\rho}{-c} = \frac{cos\alpha}{a} = \frac{sin\alpha}{b}$$

$$\frac{\sqrt{\cos^2\alpha + \sin^2\beta}}{\pm\sqrt{a^2 + b^2}} = \frac{1}{\pm\sqrt{a^2 + b^2}}$$

So (3) $\Rightarrow \frac{ax+by}{\pm\sqrt{a^2+b^2}} = -\frac{c}{\sqrt{a^2+b^2}}$

(sign of radicle to be property chosen)

Position of a point respect to a line

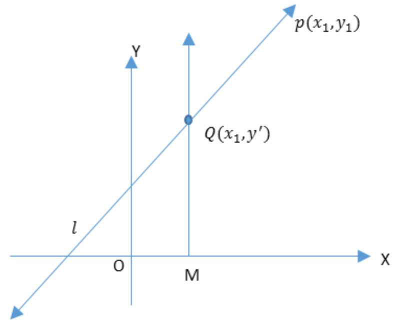
Theorem: let

$P(x, y)$ *be a point in the plane ont lying*

On $\boldsymbol{l}$ $\boldsymbol{ax + by + c = 0}$ ***then p lies***

a) **Above the line** $\boldsymbol{l\ if\ ax_1 + by_1 + c > o}$
b) $\boldsymbol{below\ the\ line\ l\ if\ ax_1 + by_1 + c < 0}$

Proof:



a) Let we draw $\perp PM$ *from point* P *on* $x - axis.$ S. that it meets the line l *at point* $Q(x_1, y_1)$ *the point P will lie*

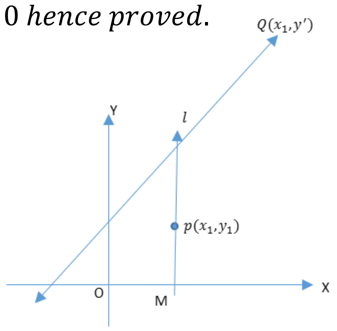
Above line l *if* $y_1 > y_1'$ *or* $y_1 - y' > 0 \rightarrow (x$

As the point $Q(x_1, y_1')$*lies on the line* $l;$

$$ax + by + c = 0 \Rightarrow ax_1 + by' + c = 0$$

$\Rightarrow by' = -ax_1 - c \Rightarrow y' = -\frac{a}{b}x_1 - \frac{c}{b}$ *put in*(1)

Or $y_1 + by_1 + c > 0$ *hence proved.*



b) Let us draw $\perp QM$ $from\ point\ Q\ on\ x-axis.$

The point P will be lie below the line $l\ \ if\ y' > y_1$

$$or\ y_1 - y' < 0 \rightarrow (i$$

$as\ the\ point\ Q(x_1, y')lie\ on\ the\ line\ l$

$$l; ax + by + c = 0 \Rightarrow ax_1 + by_1 + c = 0$$

$$\Rightarrow by' = -ax_1 - c$$

$$=> y' = -\frac{a}{b}x_1 - \frac{c}{a}\ \ put\ (i)$$

$$y_1 - \left(-\frac{a}{b}x_1 + \frac{c}{a}\right) < 0$$

$$\Rightarrow y_1 - \frac{a}{b}x_1 + \frac{c}{a} + \frac{c}{a} < 0 \Rightarrow by_1 + ax_1 + c < 0$$

$$\Rightarrow ax_1 + by_1 + c < 0\ hence\ proved$$

Corollary 1. The point p above or below l respectively if $ax_1 + by_1 + c$ and b have the same signs or have opposite signs.

Proof:

If $P(x_1, y_1)above\ l\ then\ y_1 - y' > 0\ \Rightarrow then$

$y_1 - y' > 0\ \ y_1 - \frac{a}{b}x_1 - \frac{c}{a} > 0$

- ⇨ $y_1 + \frac{a}{b}x_1 + \frac{c}{b} > 0 \Rightarrow \frac{ax_1 + by_1 + c}{b} > o$
- ⇨ It is only possible if$ax_1 + by_1 + c\ and\ b\ have\ some\ signs.$

$Similarly, P(x, y)below\ l\ then$

$y_1 - y' < 0 \Rightarrow y_1 - \left(-\frac{a}{b}x_1 - \frac{c}{b}\right) < 0$

$\Rightarrow y_1 + \frac{a}{b}x_1 + \frac{c}{a} < 0 \Rightarrow \frac{(ax_1 + by_1 + c)}{b} < o$

It is possible if $if\ ax_1 + by_1 + c\ and\ b\ have\ possible$ Sign.

Corollary 2. The point $P(x, y)$ and orgin are (i)on The same side of l according as $ax_1 + by_1 + c$ and c have the same sign.

(ii)*on the opposite side of l according as $ax_1 + by_1 + c$ and have opposite sign*.

Proof:

The point $P(x_1, y_1)and\ O(0,0)are\ same\ side\ of\ l\ if\ ax_1 + by_1 + c\ have\ same\ sign.$

(ii)the point $P(x_1, y_1)and\ O(0,0)are\ opposite\ side\ of\ l$

$ax_1 + by_1 + c\ and\ a(0) + b(0) + c\ have\ opposite\ sign$

Two and three straight lines

For any two distance lines l_1, l_2

$l_1; a_1x + b_1y + c_1 = 0\ \ and\ l_2; a_2x + b_2y + c_2 = 0$

One and only one of following holds.

$(i)l_1 || l_2\quad (ii)\ \ l_1 \perp l_2\quad (iii)l_1 and\ l_2\ are\ not\ related$

As $(i)and\ (ii)$

$$slope\ of\ line\ l_1 = m_1 = -\frac{a_1}{b_1}$$

$$xlpoe\ of\ l_2 = m_2 = -\frac{a_2}{b_2}$$

(i)$l_1 || l_2$

$$\because for\ parallel\ lines\ slopes\ are\ equal\ so$$

$$\Rightarrow slope\ of\ l = slope\ of\ l_2$$

$$\Rightarrow m_1 = m_2 \Rightarrow -\frac{a_1}{b_1} = -\frac{a_1}{b_2}$$

$$\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} \Rightarrow a_1b_2 = a_2b_1$$

$$\Rightarrow a_2b_2 - a_2b_1 = 0$$

(ii)$l_1 \perp l_2$

$\because\perp lines\ , product\ of\ their\ slopes\ equal\ to\ -1\ so$

$$(slopes\ of\ l_1)(slope\ of\ l_2) = -1$$

- ⇨ $m_1m_2 = -1\ \Rightarrow \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right) = -1$
- ⇨ $\frac{a_1a_2}{b_1b_2} = -1 \Rightarrow a_1a_2 = -b_1b_2$
- ⇨ $a_1a_2 + b_1b_2 = 0$

$iii)\ if\ l_1\ and\ l_2\ are\ not\ realtd\ as\ in\ (i)or\ (ii$
$then\ is\ no\ simple\ relation\ of\ the\ above\ forms.$

The point of intersection of two straight lines:

Let $l_1\ ; a_1x + b_1y + c_1 = 0 \rightarrow (i)$

$l_2; a_2x + b_2y + c_2 = 0 \rightarrow (ii)be\ two\ non-parallel\ lines$

Remember;

Two non para;;e; lines intersect each other at one and only one points.

Let

$P(x_1, y_1)be\ the\ points\ of\ intersction\ of\ lines\ l_1, l_2$

Solving

$(i)and\ (ii)by\ cross\ multiplication\ method\ we\ have$

$$\frac{x_1}{b_1c_2 - b_2c_1} = \frac{y_2}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

- ⇨ $\frac{x_1}{b_1c_2 - b_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$ and $\frac{y_2}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$
- ⇨ $x_1 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$ and $y_1 = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$
- ⇨ $thus\ p(x, y) = \left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$

Note $a_1b_2 - a_2b_2 \neq 0\ otherwise\ l_1 || l_2$

Condition of concurrency of three straight lines:

Three non –parallel lines

$l_1\ ; a_1x + b_1y + c_1 = 0 \rightarrow (i)$

$l_2; a_2x + b_2y + c_2 = 0 \rightarrow (ii)$

$l_3; a_3x + b_3y + c_3 = 0\ are\ concurrent\ iff$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Proof:

We know that point of intersection of lines $l_1\ and\ l_2$

Is $P\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$ $\quad\because\ the\ lines\ are$

Concurrent so

l_3 *will also pass through this point.*

then l_3 becomes

⇨ $a_3\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)+c_3=0$

⇨ $a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)$

it can be written in determinent form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$$

This is a necessary and sufficient condition of concurrency of the given three lines.

Equation of lines through the point of intersection of two lines.

Consider

$l_1\ ; a_1x+b_1y+c_1=0\rightarrow(i)$

$l_2; a_2x+b_2y+c_2=0\rightarrow(ii)$

Let

$P(x,y)$ *be the point of intersection of lines* l_1

And l_2 so (i) and (ii) becomes

$as,\quad a_1x_1+b_1y_1+c_1=0\rightarrow(iii)$

$a_2x_1+b_2y_1+c_2=0\rightarrow(iv)$

Consider $l_1+kl_2=0$

⇨ $a_1x+b_1y+c_1+k(a_2x+b_2y+c_2=0\rightarrow(v)$

⇨ $a_1x+b_1y+c_1+ka_2+kb_2y+kc_2=0$

⇨ $a_1x+ka_2x+b_1y+kb_2y+c_1kc_2=0$

⇨ $a_1+ka_2)x+(b_1+kb_2)y+(c_1+kc_2)=0$

Which is of the form $ax+by+c=0$

Hence

(v) *represents a straight line. for different values of* k, (v) *represents different lines. so it is also* Called family of lines.

Note:

Now lines (v) will pass through the point $P(x,y)$ *if it satisfied* the eq. of line (v) i.e

$$a_1x_1+b_1y_1+c_1+k(a_2x_2+b_2y_2+c_2)=0$$

$$\because a_1x_1+b_1y_1+c_1=0 \ \ and\ a_2x_2+b_2y_2+c_2=0$$

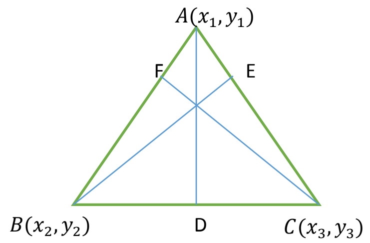
So

$$L.H.S=a_1x_1+b_1y_1+c_1+k(a_2x_2+b_2y_2+c_2$$
$$=0+k(0)=0\ L.H.S$$

Theorem:

Altitudes of a triangle are concurrent

Proof:



Let

$A(x,y), B(x_2,y_2)$ *and* $C(x_3,b_3)$ *be vertics of* ΔABC

Draw $\perp$ *ars* AD, BE *and* AB *resp.* AD, BE *and* CF

Are altitudes of ΔABC.

$$\because slope\ of\ side\ BC=\frac{y_3-y_2}{x_3-x_2}$$

⇨ *slope of* altitude AD= $-\left(\frac{x_3-x_2}{y_3-y_2}\right)$ $(\because AD'\perp BC)$

so eq. of altidue AD is

$y-y_1=-\left(\frac{x_3-x_2}{y_3-y_2}\right)(x-x_1)$ (point –slope form)

⇨ $y-y_1=-\left(\frac{x_3-x_2}{y_3-y_2}\right)(x-x_1)=0$

⇨ $y-y_1)(y_3-y_2)+(x_3-x_2)(x-x_1)=0$

⇨ $x(x_3-x_2)+y(y_3-y_2)-x_1(x_3-x_2)-y_1(y_3-y_2$

⇨ *so eq.*s of altitude BE and CF respectively

By symmmetery

$$x(x_3-x_2)+y(y_3-y_2)-x_2(x_3-x_1-y_2(y_3-y_1)=0$$

$$x(x_2-x_1)+y(y_2-y_1)-x_3(x_2-x_1)-y_3(y_2-y_1)=0$$

How altitude will be concurrent if

$$\begin{vmatrix} x_3-x_2 & y_3-y_2 & -x_1(x_3-x_2)-y_1(y_3-y_2) \\ x_3-x_1 & y_3-y_1 & -x_2(x_3-x_1)-y_2(y_3-y_1) \\ x_2-x_1 & y_2-y_1 & -x_3(x_2-x_1)-y_3(y_2-y_1) \end{vmatrix}=0$$

Now taking (-1) as common from R_2

$$=(-1)\begin{vmatrix} x_3-x_2 & y_3-y_2 & -x_1(x_3-x_2)-y_1(y_3-y_2) \\ x_1-x_3 & y_1-y_3 & x_2(x_3-x_1)+y_2(y_3-y_1) \\ x_2-x_1 & y_2-y_1 & -x_3(x_2-x_1)-y_3(y_2-y_1) \end{vmatrix}=0$$

$$by\ R_1+(R_1+R_3)$$

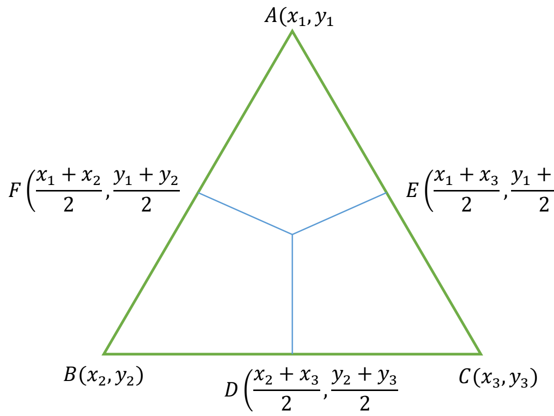
$$=(-1)\begin{vmatrix} 0 & 0 & 0 \\ x_1-x_3 & y_1-y_3 & x_2(x_3-x_1)+y_2(y_3-y_1) \\ x_2-x_1 & y_2-y_1 & -x_3(x_2-x_1)-y_3(y_2-y_1) \end{vmatrix}=0$$

$$=0\ (\because R_1=0)$$

Thus altitude of a triangle are concurrent.

Theorem: Right bisectors of a triangle are concurrent.

Proof:



let $A(x_1, y_1), B(x_2, y_2)$ *and* $C(x_3, y_3)$ *be vertices of* ΔABC. *let* D, E *and* F *be mid points of the sides* BC, AC *and* AB *respectively. so* OD, DE *and* DF *are right bisectors.*

$$\text{coordinates of } D \text{ are } \frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}$$

$$\text{coordinates of } E \text{ are } \frac{x_1+x_3}{2}. \frac{y_1+y_3}{2}$$

$$\text{coordinates of } F \text{ are } \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}$$

Slope of side $BCC = \frac{y_3-y_2}{x_3-x_2}$ $\quad \because OD \perp BC$

⇨ Slope of right bisector OD$-\left(\frac{x_3-x_2}{y_3-y_2}\right)$

So eq. of right bisector OD is

$y-\left(\frac{y_2+y_3}{2}\right) = -\left(\frac{x_3-x_2}{y_3-y_2}\right)\left(x-\left(\frac{x_2+x_3}{2}\right)\right)$ point slope form

$$\Rightarrow \; y-\left(\frac{y_2+y_3}{2}\right)+\left(\frac{x_3-x_2}{y_3-y_2}\right)\left(x-\left(\frac{x_2+x_3}{2}\right)\right)=0$$

$$y(y_3-y_2)-(y_3-y_2)\left(\frac{y_2+y_3}{2}\right)+(x_3-x_2)\left(x-\left(\frac{x_2+x_3}{2}\right)\right)=0$$

$$\Rightarrow \; x(x_3-x_2)+y(y_3-y_2)-\frac{1}{2}(x_3-x_2)(x_3+x_2)-\frac{1}{2}(y_3+y_2)(y_3-y_2)=0$$

Equations of the other two rights bisectors *OE and OF are* *(by smmetry)*

$$\Rightarrow x(x_3-x_1)+y(y_3-y_1)-\frac{1}{2}\ x_3^2-x_1^2)-\frac{1}{2}\ y_3^2-y_1^2 = 0$$

And

$$x(x_2-x_1)+y(x_3-x_1)-\frac{1}{2}(x_2^2-x_1^2\ -\frac{1}{2}\ y_2^2-y_1^2)=0$$

Right bisectors will be concurrent if

$$\begin{vmatrix} x_3-x_2 & y_3-y_2 & -\frac{1}{2}\ x_3^2-x_2^2)-\frac{1}{2}\ y_3^2-y_2^2 \\ x_3-x_1 & y_3-y_1 & -\frac{1}{2}\ x_3^2-x_1^2)-\frac{1}{2}\ y_3^2-y_1^2 \\ x_2-x_1 & y_2-y_1 & -\frac{1}{2}\ x_2^2-x_1^2)-\frac{1}{2}\ x_2^2-x_1^2 \end{vmatrix} = 0$$

Now taking (-1) as common from R_2

$$-1\begin{vmatrix} x_3-x_2 & y_3-y_2 & -\frac{1}{2}\ x_3^2-x_2^2)-\frac{1}{2}\ y_3^2-y_2^2 \\ x_1-x_3 & y_1-y_3 & \frac{1}{2}\ x_3^2-x_1^2)+\frac{1}{2}\ y_3^2-y_1^2 \\ x_2-x_1 & y_2-y_1 & -\frac{1}{2}\ x_2^2-x_1^2)-\frac{1}{2}\ x_2^2-x_1^2 \end{vmatrix} = 0$$

$$By\ R_1+(R_2+R_3)$$

$$-1)\begin{vmatrix} o & o & o \\ x_1-x_3 & y_1-y_3 & \frac{1}{2}\ x_3^2-x_1^2)+\frac{1}{2}\ y_3^2-y_1^2 \\ x_2-x_1 & y_2-y_1 & -\frac{1}{2}\ x_2^2-x_1^2)-\frac{1}{2}\ x_2^2-x_1^2 \end{vmatrix} = 0$$

$$=0 \quad (\because R_1 = 0)$$

Thus right bisectors of triangle are concurrent.

Note:

If equations of sides of the triangle are given, then intersection of any two lines gives a vertex of the triangle.

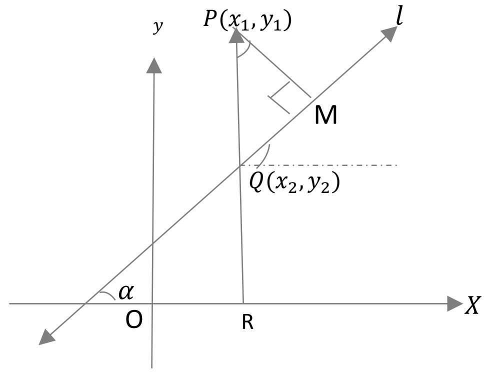
Distance of a point from a line:

Theorem: the distance d from the point $P(x, y)$ to

The line l; $ax+by+c=0$ is given by

$$d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

Proof:



Let α *be the inclination of the line* $l; ax+by+c=0$

Draw $\perp$ *ars PR from point P on* $x-$*axis such that it meets line* l *at point Q. ALso draw a* $\perp$ *rPM on line* l.

In ΔPQM $\quad$, $m\angle QPM = \alpha$

$$|PM| = d, \quad |PQ| = |y_1-y_2|$$

Cos$\alpha = \frac{|PM|}{|PQ|} \Rightarrow |PM| = |PQ|cos\alpha$

$$\Rightarrow d = |y_1-y_2|cos\alpha \rightarrow (i$$

$\because Q(x_2-y_2)$*lies on line* $l: ax+by+c=0$

So $ax_2+by_2+c=0 \Rightarrow by_2 = -ax_2-c$

$$\Rightarrow y_2 = -\frac{a}{b}x_2-\frac{c}{b}$$

Given eq. of line is $ax+by+c=0$

$$\Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$$

- ⇨ Slope of given line$= m = -\frac{a}{b}$
- ⇨ $\because m = \tan\alpha \Rightarrow \tan\alpha = -\frac{a}{b}$
- ⇨ $1 + \tan^2\alpha = \sec^2\alpha \Rightarrow 1 + \left(-\frac{a}{b}\right)^2 = \sec^2\alpha$
- ⇨ $1 + \frac{a^2}{b^2} = \sec^2\alpha \Rightarrow \sec\alpha = \frac{\sqrt{a^2+b^2}}{b}$
- ⇨ $\cos\alpha = \frac{b}{\sqrt{a^2+^2}}$ *put all vales in* (2)
- ⇨ $d = \left|y_1 + \frac{ax_1+c}{b}\right|\left(\frac{b}{\sqrt{a^2+b^2}}\right)$
- ⇨ $\left|\frac{ax_1+by_1+c}{b}\right|\left(\frac{b}{\sqrt{a^2+b^2}}\right)$
- ⇨ $d = \left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ hence proved

Distance b/w two parallel lines

The distance between two parallel lines is the distance from any point on one of the lines to the other.

Corollary: if the points P, Q, R are collinear then$\Delta = 0$

Trapezium:

A quadrilateral having two sides parallel and two non -parallel is called trapezium. Its area is

$\frac{1}{2}(sum\ of\ length\ of\ ||sides\ )(distance\ b/w||sides)$

Exercise 4.3

Exercise 4.3

Question no.1: Find the slope and inclination of the line joining the plane.

i: (-2, 4) ; (5,11)

Solution: slope of $\overline{AB}$=m= $\frac{11-4}{5-(-2)}$

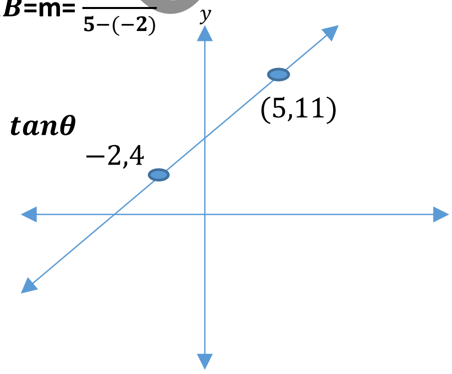
$= \frac{7}{7}$

$m = 1 \qquad \therefore m = \tan\theta$

So $\tan\theta = 1$

$\theta = \tan^{-1} 1$

$\theta = 45° = \frac{\pi}{4}$



ii: (3, -2) ; (2,7)

sol: **slope of $\overline{AB}$** $= m = \frac{7- -2)}{2-3}$

$= \frac{9}{-1}$

$m = -9$

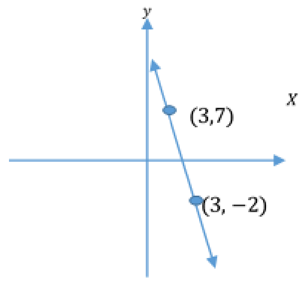
$\tan\theta = -9$

$\theta = \tan^{-1} -9$

$\theta = 180° - \tan^{-1} -9$

$\therefore \theta\ lies\ in\ II - quadrant$

$= 180 - 83.6 = 96.34°$



iii) (4,6) ; (4,8)

Sol: slope of $\overline{AB}$ $= m = \frac{8-6}{4-4}$

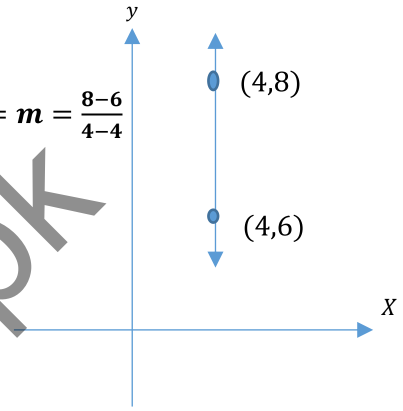
$= \frac{2}{0}$

$= \infty$

$\tan\theta = \infty$

$\theta = \tan^{-1}\infty$

$\theta = 90°$



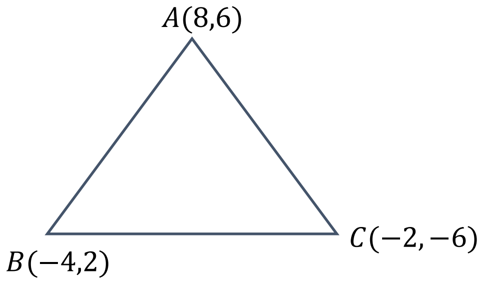
Question no.2: In a triangle A(8,6) B(-4,2), C(-2,-6), find the slopes of (i) each side of the triangle

(ii) Each median of the triangle.

(iii) Each altitude of the triangle.

Solution:

i. A(8,6) B(-4,2), C(-2,-6),



Slope of $\overline{AB} = \frac{2-6}{-4-8} = -\frac{4}{-12} = \frac{1}{3}$

Slope of $\overline{BC} = \frac{-6-2}{-2+4} = -\frac{8}{2} = -4$

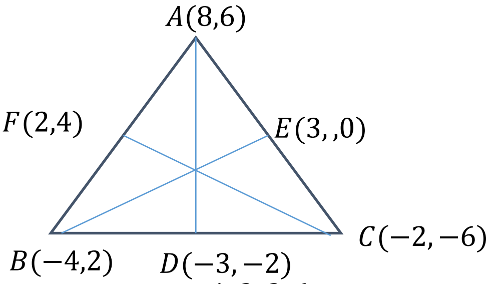
Slope of $\overline{AC} = \frac{-6-6}{-2-8} = -\frac{12}{-10} = \frac{6}{5}$

SOLUTION:

ii.

L, M, N be the mid points of $\overline{AB}, \overline{BC}\ \overline{AC}$ respectively.

Midpoint of side $\overline{AB} = L(\frac{8-4}{2}, \frac{6+2}{2} = L(2,4)$



Midpoint of side $\overline{BC}$, $M(\frac{-4-2}{2},\frac{2-6}{2} = M(-3,-2)$

Midpoint of side $\overline{AC}$ $= N(\frac{8-2}{2},\frac{6-6}{2} = N(3,0)$

Slope of the median $\overline{AM}$ $=\frac{-2-6}{-3-8} = -\frac{8}{-11} = \frac{8}{11}$

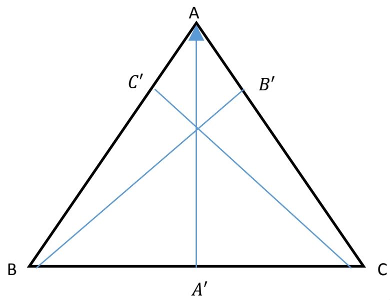
Slope of the median $\overline{BN}$ $=\frac{-0-2}{3+4} = \frac{-2}{7}$

Slope of the median $\overline{CL}$ $=\frac{4+6}{2+2} = \frac{10}{4} = \frac{5}{2}$

Solution:

iii.

Slope of side $\overline{AB}$=m_1 $= \frac{2-6}{-4-8} = -\frac{4}{-12} = \frac{1}{3}$



Slope of side $\overline{BC}$=$m_2 = \frac{-6-2}{-2+4} = \frac{-8}{2} = \frac{-4}{1}$

Slope of side $\overline{AC}$=$m_3 = \frac{-6-6}{-2-8} = \frac{-12}{-10} = \frac{6}{5}$

Let $\overline{AP}$ $\overline{BQ}$ $\overline{CR}$ be the altitude of ΔABC

Slope of altitude $\overline{AP}$= $\frac{-1}{m_2} = -\frac{1}{-4} = \frac{1}{4}$

Slope of altitude $\overline{BQ}$= $\frac{-1}{m_3} = -\frac{1}{\frac{6}{5}} = -\frac{5}{6}$

Slope of altitude $\overline{CR}$= $\frac{-1}{m_1} = -\frac{1}{\frac{1}{3}} = -3$ $(\because CC' \perp AB))$

Question no.3: By means of slopes, show that the following points lie on the same line.

(a) $\mathbf{-1,-3)\,;\,(1,5)\,;\,(2,9}$

(b) $\mathbf{4,-5)\,;\,(7,5)\,;\,(10,15}$

(c) $\mathbf{-4,6)\,;\,3,8)\,;\,(10,10}$

(d) $\mathbf{a,2b)\,;(c,a+b)\,;\,(2c-a,2a)}$

Solution: $(a)A\ (\mathbf{-1,-3})\,;\ \mathbf{B(1,5)}\,;\ \mathbf{C\ (2,9)}$

$$Slope\ of\overline{AB} = \frac{5+3}{1+1} = \frac{8}{2} = 4$$

$$Slope\ of\overline{BC} = \frac{9-5}{2-1} = \frac{4}{1} = 4$$

$$Slope\ of\overline{AB} = \ Slope\ of\overline{BC}$$

So the $points\ A, B, C\ lie\ on$ the same line.

Solution: (b) $\mathbf{A(4,-5)\,;\ B(7,5)\,;\ C(10,15)}$

$$Slope\ of\overline{AB} = \frac{5+5}{7-4} = \frac{10}{3}$$

$$Slope\ of\overline{BC} = \frac{15-5}{10-7} = \frac{10}{3}$$

$$Slope\ of\overline{AB} = \ Slope\ of\overline{BC}$$

So the points A, B, C lie on the same line

.

Solution: (c)

$\mathbf{A(-4,6)\,;\ B(3,8)\,;\ C(10,10)}$

$$Slope\ of\overline{AB} = \frac{8-6}{3+4} = \frac{2}{7}$$

$$Slope\ of\overline{BC} = \frac{10-8}{10-3} = \frac{2}{7}$$

$$Slope\ of\overline{AB} = \ Slope\ of\overline{BC}$$

So the points A, B, C lie on the same line.

Solution: (d)

$(a,2b)\,;(c,a+b)\,;\ (2c-a,2a)$

$$Slope\ of\overline{AB} = \frac{a+b-2b}{c-a} = \frac{a-b}{c-a}$$

$$Slope\ of\overline{BC} = \frac{2a-a-b}{2c-a-c} = \frac{a-b}{c-a}$$

$$Slope\ of\overline{AB} = \ Slope\ of\overline{BC}$$

So the points A, B, C lie on the same line.

Question no.4: Find k so that the line joining A(7,3), B(K,6) and the line joining C(-4,5) D(-6,4) are

i) Parallel

ii) Perpendicular

Solution: $A(7,3), B(K,6)\ \ C(-4,5)\ D(-6,4)$

$$m_1 = slope\ of\overline{AB} = \frac{-6-3}{K-7} = -\frac{9}{K-7}$$

$$m_2 = slope\ of\overline{CD} = \frac{4-5}{-6+4} = -\frac{1}{-2} = \frac{1}{2}$$

(i) *As $\overline{AB}$ and $\overline{CD}$ are parallel therefore*

$$m_1 = m_2$$

$$\frac{-9}{k-7} = \frac{1}{2}$$

−18=k-7

$k = -11$

(ii) As $\overline{AB}$ and $\overline{CD}$ are Perpendicular, therefore

$$m_1.m_2 = -1$$

$$\left(\frac{-9}{k-7}\right).\left(\frac{1}{2}\right) = -1$$

$$-9 = -2(k-7$$

$$-9 = -2k+14$$

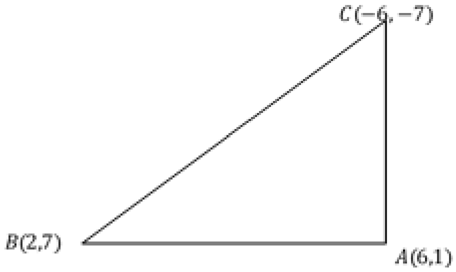
$$2k = 23$$

$$\mathbf{K = 23/2}$$

Question no.5: Using slopes, show that the triangle with *its vertices* $\mathbf{A(6,1), B(2,7)}$ and $\mathbf{C(-6,-7)}$ is a right triangle.

Soltion: $\mathbf{A(6,1), B(2,7)\ , C(-6,-7)}$

Slope of $\overline{AB}$ = $m_1 = \frac{7-1}{2-6} = \frac{6}{-4} = -\frac{3}{2}$



Slope of $\overline{BC}$ = $m_2 = \frac{-7-7}{-6-2} = \frac{-14}{-8} = \frac{7}{4}$

Slope of $\overline{AC}$ = $m_3 = \frac{-7-1}{-6-6} = \frac{-8}{-12} = \frac{2}{3}$

Since $m_1.m_2 = \left(-\frac{3}{2}\right)\left(\frac{2}{3}\right)$

$m_1.m_2$ = -1, therefore

$\overline{AB} \perp \overline{AC}$ so ΔABC *is a right triangle*

Question No.6: The three points A(7,-1) B(-2,2) and C(1,4) are consecutive vertices of a parallelogram. Find the fourth vertex.

Solution: $\boldsymbol{A(7,-1)\ B(-2,2)\ and\ C(1,4)}$

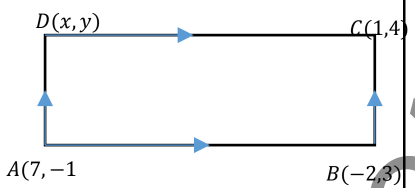
Let fourth vertex $= D(x,y)$

Since ABCD is a parallelogram, therefore

Slope of $\overline{AB}$ = *Slope of* $\overline{CD}$

$\frac{2+1}{=2-7} = \frac{Y-4}{X-1}$

$\frac{3}{-9} = \frac{Y-5}{x-1}$



$3(x-1) = -9(y-4)$

$3x - 3 = -9y + 36$

$3x + 9y - 3 - 36 = 0$

$3x + 9y - 39 = 0$

Dividing by 3 on both sides

$x + 3y - 13 = 0 ------(i)$

Now,

Since ABCD is a parallelogram, therefore

Slope of $\overline{AD}$ = *Slope of* $\overline{BC}$

$$\frac{y+1}{x-7} = \frac{4-2}{1+2}$$

$$\frac{y+1}{x-7} = \frac{2}{3}$$

$$3(y+1) = 2(x-7)$$

$$3y + 3 = 2x - 14$$

$$0 = 2x - 14 - 3y - 3$$

$$2x - 3y - 17 = 0 ------ii$$

By Adding i and ii we get

$$x + 3y - 13 = 0$$

$$\underline{2x - 3y - 17 = 0}$$

$$3x - 30 = 0$$

$$x = 10$$

Put value of x in eq. i

$$10 + 3y - 13 = 0$$

$$3y - 3 = 0$$

$$y = 1$$

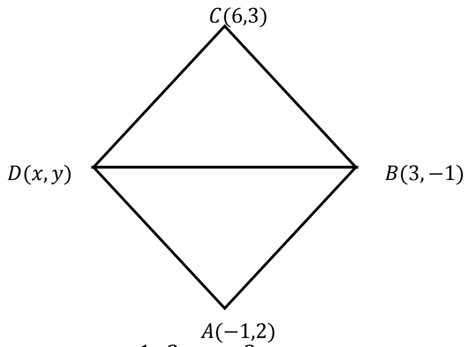
Hence fourth vertex $= D(x,y) = D(10,1)$

Question no.7:

The point A(-1,2) B(3,-1) and C(6,3) are consecutive vertices of rhombus. Find the fourth vertex and show that the diagonal of the rhombus are perpendicular to each other.

Solution:

Let $D(a,b)$ be the fourth vertex of rhombus



Slope of$\overline{AB}$ = $\frac{-1-2}{3+1} = -\frac{3}{4}$

Slope of$\overline{BC}$ = $\frac{3+1}{6-3} = \frac{4}{3}$

Slope of$\overline{CD}$ = $\frac{b-3}{a+6}$

Slope of$\overline{DA}$ = $\frac{2-b}{-1-b}$

Since ABCD is a rhombus therefore

Slope of $\overline{AB}$ = *Slope of* $\overline{CD}$

$$\frac{-3}{4} = \frac{b-3}{a-6}$$

$$-3(a-6) = 4(b-3$$

$$-3a + 18 = 4b - 12$$

$$-3a - 4b = -12 - 18$$

$$-3a - 4b + 30 = 0 ----i$$

Slope of $\overline{BC}$ = *Slope of* $\overline{DA}$

$$\frac{4}{3} = \frac{2-b}{-1-a}$$

$$4(-1-a) = 3(2-b)$$

$$-4 - 4a = 6 - 3b$$

$$-4a + 3b - 10 = 0 ----ii$$

multiply i by 3 and ii by 4 and then adding both

$$-9a - 12b + 90 = 0$$

$$\underline{-16a + 12b - 40 = 0}$$

$$-25a + 50 = 0$$

$$25a = 50$$

$$a = \frac{50}{25}$$

$$a = 2$$

Putting value of a *in ii*

$$-4(2) + 3b - 10 = 0$$

$$-8 + 3b - 10 = 0$$

$$3b - 18 = 0$$

$$b = 6$$

Hence $D(2,6)$ is the fourth vertex of rhombus.

QUESTION NO.8:

Two pairs of points are given. Find whether the two lines determined by these points are

i) Parallel

ii) Perpendicular

iii) None

a) (1,-2) (2,4) and (4,1) (-8,2)

b) (-3,4) (6,2) and (4,5) (-2,-7)

Solution: (a) slope of joining (1,-2) and (2,4

$$= m_1 = \frac{4+2}{2-1} = 6$$

Slope of joining (4,1)and(-8,2)= $m_2 = \frac{2-1}{-8-4} = \frac{1}{-12}$

Sine $m_1 \neq m_2$ aand also

$$m_1.m_2 = 6.\frac{1}{-12} \neq -1$$

So the lines are neither parallel nor perpendicular.

Solution: (b) : (a) slope of joining(-3,4)and (6,2)

$$= m_1 = \frac{2-4}{6+3} = -\frac{2}{9}$$

slope of joining(4,5) and (-2,-7)= $m_2 = \frac{-7-4}{-2-4} = \frac{-11}{-6}$

Sine $m_1 \neq m_2$ aand also

$$m_1.m_2 = -\frac{2}{9}.\frac{-11}{-6} \neq -1$$

So the lines are neither parallel not perpendicular.

Question no.9: find an equation of

a) The horizontal line through (7,-9)

b) The vertical line through (-5,3)

c) The line bisecting the first and third quadrant.

d) The line bisecting the second and fourth quadrants.

Solution: (a) slope of horizontal line m=0

And $(x_1, y_1) = (7, -9$

Therefore equation of line

$$y - (-9) = 0(x - 7)$$
$$y + 9 = 0$$

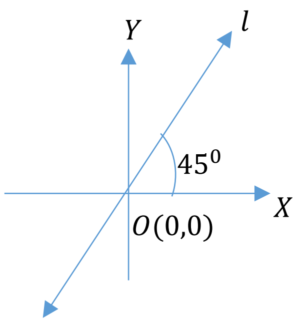
(b) Since the slope of vertical line $m = \infty = \frac{1}{0}$

And $(x_1, y_1) = (-5, 3)$

Therefore required equation of line

$$y - 3 = \infty(x - (-5))$$
$$y - 3 = \frac{1}{0}(x + 5)$$
$$x + 5 = 0$$

(c) The line bisecting the first and third quadrant makes an angle of 45°with the x-axis therefore

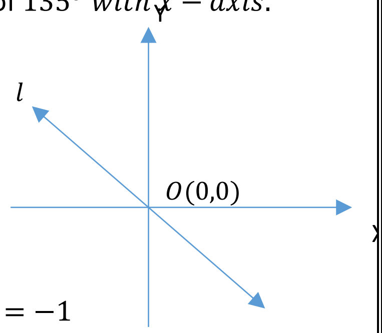


Slope of line= $m = tan45° = 1$

Also it passes through origin (0,0), so its equation

$$y - 0 = 1(x - 0)$$
$$y = x$$
$$y - x = 0$$

(d) The line bisecting the 2nd and 4th quadrant makes an angle of 135^o $with\ x - axis.$



So slope $= m = tan135^0 = -1$

$$\because it\ passes\ through\ orgin\ so\ eq\ is$$
$$y - o = (-1)(x - 0)$$
$$\Rightarrow \quad y = -x \Rightarrow y + x = 0$$

Question no 10: find an equation of line

a) Through A(-6,5) and slope 7

b) Through (8,-3) and slope 0

c) Through (-8,5) having slope undefined

d) Through (-5,-3) and (9,-1)

e) Y-intercept -7 and slope -5

f) X-intercept -3 and y-intercept -4

g) X-intercept -9 and slope -4

Solution: (a) $(x_1, y_1) = (-6, 5)$

And slope of line $= m = 7$

So required equation

$$y - 5 = 7(x - (-6))$$
$$y - 5 = 7x + 42$$
$$7x - y + 42 + 5 = 0$$
$$7x - y + 47 = 0$$

Solution: (b) $(x_1, y_1) = (8, -3)$

And slope of line $= m = 0$

So required equation

$$y - (-3) = 0(x - (8))$$
$$y + 3 = 0$$

Solution: (c) (x_1, y_1) **=(-8,5)**

And slope of line $= m = \infty$

So required equation

$$y - 5 = \infty(x - (-8))$$
$$y - 5 = \frac{1}{0}(x + 8)$$
$$0 = x + 8$$
$$x + 8 = 0$$

Solution: (d) Through (-5,-3) and (9,-1)

$$y - (-3) = \frac{-1 - (-3}{9 - \ -5)}(x - (-5))$$
$$y + 3 = \frac{2}{14}(x + 5)$$
$$y + 3 = \frac{1}{7}(x + 5)$$
$$7(y + 3) = (x + 5)$$
$$7y + 21 - x - 5 = 0$$
$$-x + 7y + 16 = 0$$
$$x - 7y - 16 = 0$$

Solution: (e) Y-intercept -7 and slope -5

$(0, -7)$ Lies on a required line

And slope of line $= m = -5$

So required equation

$$y-(-7)=-5\big(x-(0)\big)$$
$$y+7=-5x$$
$$5x+y+7=0$$

Solution: (F) X-intercept -3 and y-intercept 4

$\boldsymbol{-3,0)}$ ***and*** $\boldsymbol{(0,-4)}$ ***lies on required line***

here $a=-3$ and b=4

we use here two intercept form

$$\frac{x}{a}+\frac{y}{b}=1$$
$$\frac{x}{-3}+\frac{y}{4}=1$$

Multiplying by -12

$$4x-3y=-12$$
$$4x-3y+12=0$$

g) $x-intercept\colon -9\ and\ slope -4$

$x-intercept=-9\ so\ point\ on\ x-axis\ is\ (-9,0)$

$and\ let\ A(x_1,y_1)=A(-9,0)$ and slope $m=-4$

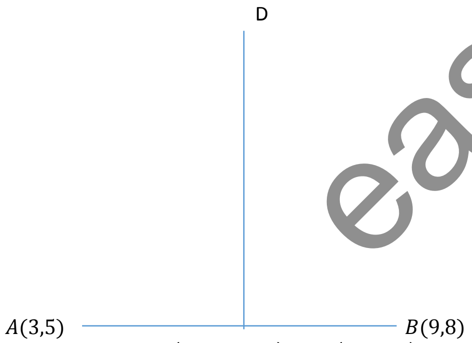
$eq\ of\ line\ is\ y-y_1$

$$=m(x-x_1)(point\ slope\ form)$$
$$\Rightarrow y-0=-4\big(x-(-9)\big)\Rightarrow y=-4(x+9$$
$$y=-4x-36\Rightarrow 4x+y+36=0$$

Rhombus: A ||gram having equal sides is called rhombus.

Question no 11: find an equation of perpendicular bisector of the segment joining the points A(3,5) and B(9,8).

Solution: Given point A(3,5) and B(9,8).



Midpoint of $\overline{AB}$=$\left(\frac{3+9}{2},\frac{5+8}{2}\right)=\left(\frac{12}{2},\frac{13}{2}\right)=(6,\frac{13}{2})$

Slope of $\overline{AB}=m=\frac{8-5}{9-3}=\frac{3}{6}=\frac{1}{2}$

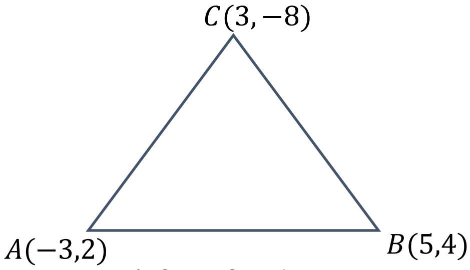
Slope of line is $\perp$ to $\overline{AB}$=$-\frac{1}{m}=\frac{-1}{\frac{1}{2}}=-2$

Noe equation of $\perp$ bisector having slope -2 through$(6,\frac{13}{2})$

$$y-\frac{13}{2}=-2(x-6)$$
$$y-\frac{13}{2}=-2x+12$$
$$2y-13=-4x+24$$
$$4x+2y-13-24=0$$
$$4x+2y-37=0$$

Question no 12: find equation of the side's altitudes and medians of the triangle whose vertices are A(-3,2) B(5,4) and C(3,-8).

Solution: given vertices of triangle are A(-3,2) B(5,4) and C(3,-8).



Slope of $\overline{AB}$=$m_1=\frac{4-2}{5-(-3)}=\frac{2}{8}=\frac{1}{4}$

Slope of $\overline{BC}$=$m_2=\frac{-8-4}{-3-5}=\frac{-12}{-8}=\frac{3}{2}$

Slope of $\overline{CA}$=$m_3=\frac{2-(-8}{-3-3}=\frac{10}{-6}=-\frac{5}{3}$

Now equation of side $\overline{AB}$ having slope ¼ passing through A(-3,2) . (you may take B(5,4) instead of A(-3,2))

$$y-2=\frac{1}{4}\big(x-(-3)\big)$$
$$4(y-2)=x+3$$
$$4y-8-x-3=0$$
$$-x+4y-11=0$$
$$x-4y+11=0$$

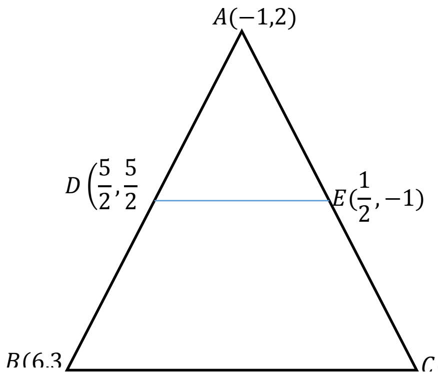
Now equation of side $\overline{BC}$ having slope 6 passing through B(5,4)

$$y-4=6(x-5)$$
$$y-4=6x-30$$
$$-6x+y-4+30=0$$
$$6x-y-26=0$$

Now equation of side $\overline{CA}$ having slope $-\frac{5}{3}$ passing through B(3,-8)

$$y-(-8)=-\frac{5}{3}(x-3)$$
$$-3(y+8)=5(x-3$$
$$-3y-24=5x-15$$
$$-5x-3y-24+15=0$$
$$-5x-3y-9=0$$
$$5x+3y+9=0$$

Equation of Altitudes:



Since altitudes are perpendicular to the sides of the triangle therefore

Slope of altitude on $\overline{AB}=-\frac{1}{m_1}=\frac{-1}{\frac{1}{4}}=-4$

Equation of altitude from C(3,-8) having slope -4

$$y+8=-4(x-3)$$
$$y+8=-4x+12$$
$$4x+y+8-12=0$$
$$4x+y-4=0$$

Slope of altitude on $\overline{AB}=-\frac{1}{m_2}=\frac{-1}{6}$

Equation of altitude from C(-3,2) having slope -1/6

$$y-2=-\frac{1}{6}(x+3)$$
$$6(y-2)=-1(x+3)$$
$$6y-12=-x-3$$
$$x+6y-12+3=0$$
$$x+6y-9=0$$

Slope of altitude on $\overline{CA}=-\frac{1}{m_3}=\frac{-1}{-\frac{5}{3}}=\frac{3}{5}$

Equation of altitude from B(5,4) having slope 3/5

$$y-4=\frac{3}{5}(x-5)$$
$$5(y-4)=3(x-5)$$
$$5y-20=3x-15$$
$$-3x+5y-20+15=0$$
$$-3x+5y-5=0$$
$$3x-5y+5=0$$

Equation of medians:

Suppose D, E and F are the medians of $\overline{AB}$,$\overline{BC}$ *and* $\overline{CA}$ respectively

The coordinate D=$\left(\frac{-3+5}{2},\frac{2+4}{2}\right)=\left(\frac{2}{2},\frac{6}{2}\right)=(1,3)$

The coordinate E=$\left(\frac{5+3}{2},\frac{4-8}{2}\right)=\left(\frac{8}{2},\frac{-4}{2}\right)=(4,-2)$

The coordinate F=$\left(\frac{3-3}{2},\frac{-8+2}{2}\right)=\left(\frac{0}{2},\frac{-6}{2}\right)=$ $(0,-3)$

Equation of $\overline{AE}$ BY two point form

$$y-2=\frac{-2-2}{4-(-3)}(x-(-3))$$
$$y-2=-\frac{4}{7}(x+3)$$
$$7(y-2)=-4(x+3)$$
$$7y-14=-4x-12$$
$$4x+7y-14+12=0$$
$$4x+7y-2=0$$

Equation of $\overline{BF}$ BY two point form

$$y-4=\frac{-3-4}{0-5}(x-5)$$
$$y-4=\frac{-7}{-5}(x-5)$$
$$5(y-4)=7(x-5)$$
$$5Y-20=7x-35$$
$$-7X+5Y-20+35=0$$
$$7x-5y-15=0$$

Equation of $\overline{CD}$ BY two point form

$$y-(-8)=\frac{3-(-8)}{1-3}(x-3)$$
$$y+8=\frac{11}{-2}(x-3)$$
$$-2(y+8)=11(x-3$$
$$-2y-16=11X-33$$
$$-11X-2Y-16+33=0$$
$$11x+2y-17=0$$

Question no 13: find an equation of the line through (-4,-6) and perpendicular to the line having slope$-\frac{3}{2}$.

Solution: slope of line = $-\frac{3}{2}$

Slope of required line= $m=\frac{-1}{-\frac{3}{2}}=\frac{2}{3}\because$

$\textit{line is perpendicular}$
$\textit{point on required line}=A(-4,-6)$
$\textit{equation of required line is}$

$$y+6=\frac{2}{3}(x+4)$$
$$3(y+6)=2(x+4)$$
$$3y+18=2x+8$$
$$-2x+3y+18-8=0$$
$$2x-3y-10=0$$

Question no. 14: find an equation of the line through (11,-5) and parallel to a line with slope -24.

Solution: Slope of required line= $m=-24$

$\therefore$ *line is parallel*
$\textit{point on required line}=A(11,-5)$
$\textit{equation of required line is}$

$$y+5=-24(x-11)$$
$$(y+5)=-24(x-11)$$
$$y+5=-24x+264$$
$$24x+y+5-264=0$$
$$24x+y-259=0$$

Question no.15: the points A(-1,2) B(6,3) and C(2,-4) are vertices of triangle. Show that the line joining the midpoint D of AB and the midpoint E of AC is parallel to BC and DE=$\frac{1}{2}BC$

Solution: A(-1,2) B(6,3) and C(2,-4)

Midpoint of $\overline{AB}=D\left(\frac{-1+6}{2},\frac{2+3}{2}\right)=D\left(\frac{5}{2},\frac{5}{2}\right)$

Midpoint of $\overline{AC}=E\left(\frac{-1+2}{2},\frac{2-4}{2}\right)=E\left(\frac{1}{2},-1\right)$

Slope of $\overline{DE}=m_1=\frac{-1-\frac{5}{2}}{\frac{1}{2}-\frac{5}{2}}=\frac{\frac{-2-5}{2}}{\frac{1-5}{2}}=-\frac{7}{-4}=\frac{7}{4}$

Slope of $\overline{BC}=m_2=\frac{-4-3}{2-6}=-\frac{7}{-4}=\frac{7}{4}$

As $m_1=m_2$, so $\overline{DE}$ is parallel to $\overline{BC}$

Now

$$\overline{DE}=\sqrt{\left(\frac{5}{2}-\frac{1}{2}\right)^2+\left(\frac{5}{2}+1\right)^2}$$
$$\overline{DE}=\sqrt{\left(\frac{5-1}{2}\right)^2+\left(\frac{5+2}{2}\right)^2}$$
$$=\sqrt{\frac{16}{4}+\frac{49}{4}}$$
$$=\sqrt{\frac{65}{4}}$$

$$\overline{DE}=\frac{1}{2}\sqrt{65}$$
$$\overline{BC}=\sqrt{(6-2)^2+(3+4)^2}$$
$$=\sqrt{16+49}$$
$$=\sqrt{65}$$

$clearly\ ,\overline{DE}=\frac{1}{2}\overline{BC}$ As required.

Question No.16,17,18,19,20 (Not solved)

Question no.21: convert each of the following into

I) Slope intercept form

II) Two intercept form

III) Normal form

(a) $2x-4y+11=0$ (b) $4x+7y-2=0$ (c) $15y-8x+3=0$

Solution: (a) $2x-4y+11=0$

Slope intercept form: $(y=mx+c)$
$$-4y=-2x-11$$
$$y=\frac{1}{2}x+\frac{11}{4}$$
$$m=\frac{1}{2},\qquad c=\frac{11}{4}$$

Two intercept form: $(\frac{x}{a}+\frac{y}{b}=1)$
$$2x-4y+11=0$$
$$2c-4y=-11$$
$$dividing\ both\ sides\ by-11$$
$$\frac{2}{-11}x+\frac{4}{11}y=1$$
$$\frac{x}{-\frac{11}{2}}+\frac{y}{\frac{11}{4}}=1$$
$$a=-\frac{11}{2}\quad,\quad b=\frac{11}{4}$$

Normal form: $(xcos\alpha+ysin\alpha=p)$
$$2x-4y+11=0$$
$$2x-4y=-11$$
$$\sqrt{2^2+(-4)^2}$$
$$\sqrt{4+16}$$
$$\sqrt{20}=2\sqrt{5}$$
$$\frac{2}{2\sqrt{5}}x-\frac{4}{2\sqrt{5}}y=\frac{-11}{2\sqrt{5}}$$
$$\frac{1}{\sqrt{5}}x-\frac{2}{\sqrt{5}}y=\frac{-11}{2\sqrt{5}}$$

Multiplying both sides by -1
$$-\frac{1}{\sqrt{5}}x+\frac{2}{\sqrt{5}}y=\frac{11}{2\sqrt{5}}$$

Where $cos\alpha=-\frac{1}{\sqrt{5}},sin\alpha=\frac{2}{5},p=\frac{11}{2\sqrt{5}}$
$$\alpha\ lies\ in\ 2nd\ quadrant\ ,so$$
$$\alpha=\cos^{-1}-\frac{1}{\sqrt{5}}=116.57°$$

Length of perpendicular form (0,0) to line $2x-4y+12=0\ is\ p=\frac{11}{2\sqrt{5}}$

$(b)\ 4x+7y-2=0$

Slope intercept form: $(y=mx+c)$
$$-7y=-4x+2$$
$$y=\frac{-4}{7}x+\frac{2}{7}$$
$$m=\frac{-4}{7},\qquad c=\frac{2}{7}$$

Two intercept form: $(\frac{x}{a}+\frac{y}{b}=1)$
$$4x-7y-2=0$$
$$4x-7y=2$$
$$dividing\ both\ sides\ by\ 2$$
$$\frac{4}{2}x-\frac{7}{2}y=1$$
$$2x+\frac{7y}{2}=1$$
$$\frac{x}{\frac{1}{2}}+\frac{y}{\frac{2}{7}}=1$$
$$a=\frac{1}{2}\quad,\quad b=\frac{2}{7}$$

Normal form: $(xcos\alpha+ysin\alpha=p)$
$$4x-7y-2=0$$
$$4x-7y=2$$

Dividing both sides by
$$\sqrt{4^2+(7)^2}$$
$$\sqrt{16+49}$$
$$\sqrt{65}=$$

$\frac{4}{\sqrt{65}}x-\frac{7}{\sqrt{65}}y=\frac{2}{\sqrt{65}}$ Normal form

Where $cos\alpha=\frac{4}{\sqrt{65}},sin\alpha=\frac{2}{\sqrt{65}},p=\frac{2}{\sqrt{65}}$
$$\alpha\ lies\ in\ 1st\ quadrant\ ,so$$
$$\alpha=\cos^{-1}\frac{4}{\sqrt{65}}=60.26°$$

Length of perpendicular form (0,0) to line $4x-7y-2=0\ is\ p=\frac{2}{\sqrt{65}}$

(C)

(i) Slope-intercept form: $y=mx+c$
$$\because 15y-8x+3=0\Rightarrow 15y=8x-3$$
$$\Rightarrow y=\frac{8}{15}x-\frac{3}{15}\rightarrow y=mx+c$$

Where $m=\frac{8}{15},\ c=-\frac{3}{15}=-\frac{1}{5}$

$$ii)\ intercept\ form:\ \left(\frac{x}{a}+\frac{y}{b}=1\right)$$
$$\because 15y-8x+3=0\Rightarrow 15y-8x=-3$$
$$\Rightarrow \frac{15y}{-3}-\frac{8x}{-3}=1\Rightarrow -5y+\frac{8}{3}x=1$$
$$\Rightarrow \frac{x}{\frac{3}{8}}+\frac{y}{-\frac{1}{5}}=1\rightarrow\frac{x}{a}+\frac{y}{b}=1$$
$$\Rightarrow where\ a=\frac{3}{18},b=-\frac{1}{5}$$

(iii)Normal line $(xcos\alpha+ysin\alpha)=p$
$$\because 15y-8x+3=0\Rightarrow 15y-8x=-3$$
$$\Rightarrow -8x+15y=-3$$
$$\Rightarrow -8x+15y=-3$$
$$\div by\ \sqrt{(-8)^2+(15)^2}=\sqrt{64+225}=\sqrt{89}$$
$$=17$$

$$\Rightarrow -\frac{8}{17}x+\frac{15}{17}y=-\frac{3}{17}\Rightarrow\frac{8}{17}x-\frac{15}{17}=\frac{3}{17}$$

$$\Rightarrow x\ \frac{8}{17}\ +y\left(-\frac{15}{17}\right)=\frac{3}{17}\rightarrow xcos\alpha+ysin\alpha=\rho$$

$$where\ cos\alpha=\frac{8}{17},sin\alpha=-\frac{15}{17}$$

$$tan\alpha=\frac{sin\alpha}{cos\alpha}=-\frac{\frac{15}{17}}{\frac{8}{17}}\because cos\alpha>o\ \ sin\alpha<o$$

$$\Rightarrow \alpha lies\ in\ (iv)quadrant$$

$$\Rightarrow tan\alpha=-\frac{15}{8}\Rightarrow\alpha=tan6-1\left(-\frac{15}{8}\right)=-61.93^0$$

$$\alpha=360^0-61.93^0=298.07^0$$

$$thus\ xcos298.07^0+ysin289.07^0$$

Thus length of $\perp$ $from\ (0,0)\ is\ \rho=\frac{3}{17}$

QUESTION NO.22: IN each of the following check whether the two lines are

I: parallel

Ii: perpendicular

Iii: neither parallel nor perpendicular

a) $\boldsymbol{2x+y-3=0\ ;4x+2y+5=0}$

b) $\boldsymbol{3y=2x+5\ ;\ \ 3x+2y-8=0}$

c) $\boldsymbol{4y+2x-1=0\ ;x-2y-7=0}$

d) $\boldsymbol{4x-y+2=0\ ;\ \ 12x-3y+1=0}$

e) $\boldsymbol{12x+35y-7=0\ ;\ 105x-36y+11=0}$

Solution: (a) $2x+y-3=0;4x+2y+5=0$

$$slope\ of\ line\ 1=m_1=-\frac{2}{1}=-2$$

$$slopeof\ line\ 2=m_2=-\frac{4}{2}=-2$$

Since $m_1=m_2$ therefore given lines are parallel

(b)

$$3y=2x+5\ ;\ \ 3x+2y-8=0$$

$$slope\ of\ line\ 1=m_1=-\frac{2}{-3}=\frac{2}{3}$$

$$slopeof\ line\ 2=m_2=\frac{3}{2}$$

Since $m_1.m_2=\left(\frac{2}{3}\right)\left(-\frac{3}{2}\right)$ therefore given lines are perpendicular.

(c) $4y+2x-1=0;x-2y-7=0$

Solution:

$$2x+4y-1=0,x-2y-7=0$$

$$\because m_1=-\frac{a}{b}=-\frac{2}{4}=-\frac{1}{2},m_2=-\frac{a}{b}=-\frac{1}{-2}=\frac{1}{2}$$

$$\because m_1\neq m_2 so\ given\ lines\ are\ neither\ ||nor\ \perp ar$$

$$d)\ 4x-y+2=0;12x-3y+1=0$$

$$\because m_1=-\frac{a}{b}=-\frac{4}{-1}=4,m_2=-\frac{a}{b}=-\frac{2}{-3}=4$$

$$\because m_1=m_2\ \ so\ given\ lines\ are\ parallel.$$

$(e)12x+35y-7=0;\ \ 105x-36y+11=0$

Solution:

$$\because m_1=-\frac{a}{b}=-\frac{12}{35},m_2=-\frac{a}{b}=-\frac{105}{-36}=\frac{35}{12}$$

$$\because m_1m_2=\left(-\frac{12}{35}\ \ \frac{35}{12}\right)=-1$$

so gives lines are perpendiuclar.

question No.23

find the distance between the given parallel **Lines sketch the lines. Also find an equation of the parallel line lying midway between them.**

a) $\boldsymbol{3x-4y+3-0;3x-4y+7=0}$

Solution:

$$\boldsymbol{l_1;3x-4y+3=0;l_2=3x-4y+7=0}$$

For $l_1;put\ x=0,3(0)-4y+3=0\Rightarrow -4y=-3$

$$y=\frac{3}{4}$$

Put$y=0,3x-4(0)+3=0\Rightarrow 3x=-3$

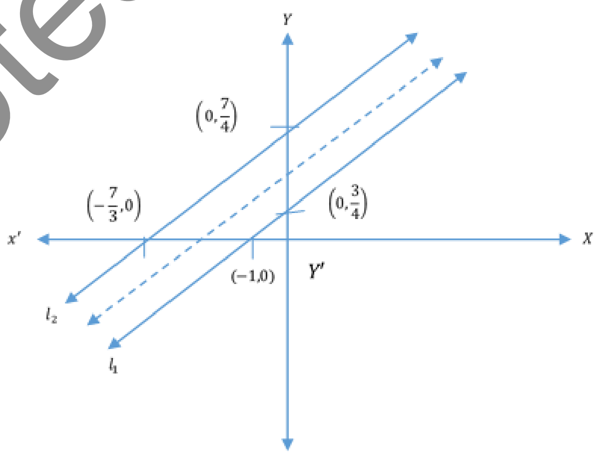
$$\Rightarrow x=-1\ so\ \left(o,\frac{3}{4}\right)and\ (-1,0)\ on\ l_1$$

$$for\ l_2,putx=0,3(0)-4y+7=0\Rightarrow -4y=-7$$

$$y=\frac{7}{4}$$

$$puty=0,3x-4(0)+7=0\Rightarrow 3x=-7$$

$$\Rightarrow x=-\frac{7}{3}so\ \left(o,\frac{7}{4}\ \ and\ \left(-\frac{7}{3},0\right)on\ l_2\right.$$



Now distance d from $(-1,0)to\ l_2\ is$

$$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

$$=\frac{|3(-1)-4(0)+7|}{\sqrt{\ 3)^2+)-(4)^2}}$$

$$d=\frac{|-3+7|}{\sqrt{9+16}}=\frac{4}{\sqrt{25}}=\frac{4}{5}$$

$$\Rightarrow d=\frac{4}{5}\ thus\ distancs\ between\ the\ parallel\ lines\frac{4}{5}$$

Now midpoint of $(-1,0)and\ \left(-\frac{7}{3},0\right)is$

$$=\left(\frac{-1-\frac{7}{3}}{2},\frac{0+0}{2}\right)=\ \frac{-3-7}{6},0\Big)=\ \frac{-10}{6},0\Big)$$

$$=\ \frac{-5}{3},0\Big)$$

Slope $= m = -\frac{a}{b} = -\frac{3}{-4} = \frac{3}{4}$

Now required equation of line passing through point $\frac{-5}{3}, 0)$ and slope =3/4 is

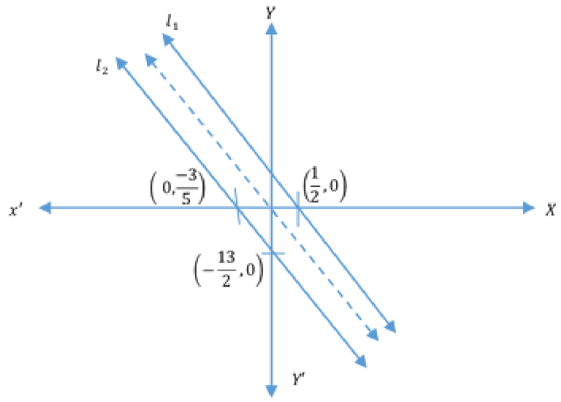
$$y - 0 = \frac{3}{4}\left(x + \frac{5}{3}\right. \quad (\because y - y_1 = m(x - x_1)$$

⇨ $4y = 3x + 5 \Rightarrow 3x - 4y + 5 = 0$

b)

Solution:

$$\boldsymbol{l_1; 12x + 5y - 6 = 0; l_2 = 12x + 5y + 13 = 0}$$



For $l_1; put\ x = 0, 12(0) + 5y - 6 = 0 \Rightarrow 5y = 6$

$$y = \frac{6}{5}$$

Put$y = 0, 12x + 5(0) - 6 = 0 \Rightarrow 12x = 6$

$$\Rightarrow x = \frac{1}{2}\ so\ \left(0, \frac{6}{5}\right) and\ \left.\frac{1}{2}, 0\right)\ on\ l_1$$

$for\ l_2, put x = 0, 12(0) + 5y + 13 = 0 \Rightarrow 5y = -13$

$$y = -\frac{13}{5}$$

put$y = 0, 12x + 5(0) + 13 = 0 \Rightarrow 12x = -13$

$$\Rightarrow x = -\frac{13}{12} so\ \left(o, \frac{-13}{5}\ \ and\ \ \frac{-13}{12}, 0\right)\ on\ l_2$$

Now distance d from $\left(\frac{1}{2}, 0\right)$ $to\ l_2\ is$

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{\left|12\left(\frac{1}{2}\right) - 5(0) + 13\right|}{\sqrt{\ 12)^2 + (5)^2}}$$

$$d = \frac{|6 + 13|}{\sqrt{144 + 25}} = \frac{19}{\sqrt{169}} = \frac{19}{13}$$

$\Rightarrow d$
$= \frac{19}{13}$ $thus\ distancs\ between\ the\ parallel\ lines$ $\frac{19}{13}$

Now midpoint of $\left(\frac{1}{2}, 0\right)$ $and\ \left(-\frac{13}{12}, 0\right)$ is

$$= \left(\frac{\frac{1}{2} - \frac{13}{12}}{2}, \frac{0 + 0}{2}\right) = \left(\frac{\frac{6 - 13}{6}}{2}, 0\right) = \left.\frac{-7}{24}, 0\right)$$

$$= \left.\frac{-7}{24}, 0\right)$$

Slope $= m = -\frac{a}{b} = \frac{-12}{5}$

Now required equation of line passing through point $\frac{-7}{24}, 0)$ and slope $-\frac{12}{5} is$ is

$$y - 0 = -\frac{12}{5}\left(x + \frac{7}{24}\right. \quad (\because y - y_1 = m(x - x_1)$$

⇨ $\boldsymbol{5y = -12x - \frac{7}{2} \Rightarrow 12x + 5y + \frac{7}{2} = 0}$

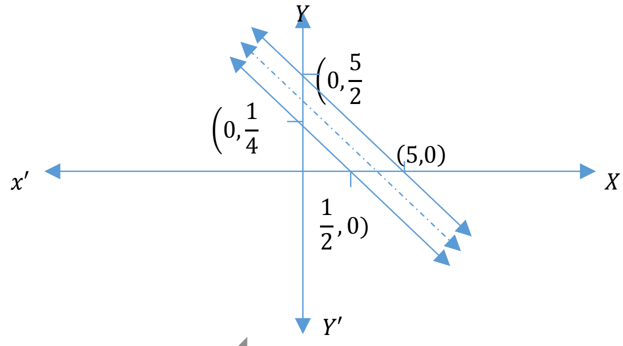
c) $\boldsymbol{x + 2y - 5 = 0; 2x + 4y = 1}$

Solution:

$$l_1; x + 2y - 5 = 0 \quad and\ l_2; 2x + 4y = 1$$

For $l_1; Pitx = 0 \Rightarrow 0 + 2y - 5 = 0$

⇨ $2y = 5 \Rightarrow y = \frac{5}{2}$



Put$y = 0 \Rightarrow x + 2(0) - 5 = 0 \Rightarrow x = 5$

$$so\ \left(0, \frac{5}{2}\ \ and\ (5,0) on\ l_1\right.$$

$$for\ l_2; put\ x = 0, 2(0) + 4y = 1 \Rightarrow y = \frac{1}{4}$$

$$put\ y = 0, 2x + 4(0) = 1 \Rightarrow x = \frac{1}{2}$$

$$so\ \left(0, \frac{1}{4}\ \ and\ \ \frac{1}{2}, 0\right)\ on\ l_2$$

$now\ distance\ d\ from\ (5,0)\ to\ l_2 is$

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|2(5) + 4(0) + 1|}{\sqrt{\ 2)^2 + (4)^2}}$$

$$d = \frac{|10 - 1|}{\sqrt{4 + 16}} = \frac{9}{\sqrt{20}} = \frac{9}{\sqrt{5}}$$

$\Rightarrow d$
$= \frac{9}{2\sqrt{5}}$ $thus\ distancs\ between\ the\ parallel\ lines$ $\frac{9}{2\sqrt{5}}$

Now midpoint of $(5,0) and$ $\left(\frac{1}{2}, 0\right)$ is

$$= \left(\frac{5 + \frac{1}{2}}{2}, \frac{0 + 0}{2}\right) = \left.\frac{10 + 1}{4}, 0\right) = \left.\frac{11}{4}, 0\right)$$

Slope $= m = -\frac{a}{b} = \frac{-1}{2}$

Now required equation of line passing through point$(\frac{11}{4}, 0)$ and slope $-\frac{1}{2} is$ is

$$y - 0 = -\frac{1}{2}\left(x - \frac{11}{4}\right. \quad (\because y - y_1 = m(x - x_1)$$

⇨ $\boldsymbol{2y = -x - \frac{11}{4} \Rightarrow x + 2y - \frac{11}{4} = 0}$

QUESTION NO.24: Find an equation of the line through

(-4,7) and parallel to the line $2x - 7y + 4 = 0$.

Solution: given that $2x - 7y + 4 = 0$

Slope of given line =$-\frac{2}{-7} = \frac{2}{7}$

Slope of required line $= m = \frac{2}{7}$

Point on the required line $= A(-4,7)$

Equation of the line through A(-4,7) is

$$y - y_1 = m(x - x_1)$$
$$y - 7 = \frac{2}{7}(x - 4)$$
$$7y - 49 = 2x + 8$$
$$2x - 7y + 57 = 0$$

Question no.25:

Find an equation of the line through (5,-8) and perpendicular to the join of A(-15,8) B(10,7).

Solution: points on given line= A(-15,8) B(10,7)

Slope of given line $= \frac{7+8}{10+15} = \frac{15}{25} = \frac{3}{5}$

Slope of required line $= m = \frac{-1}{\frac{3}{5}} = -\frac{5}{3}$

Point on required line = p(5,-8)

Equation of required line through p(5,-8) is

$$y + 8 = -\frac{5}{3}(x - 5)$$
$$3y + 24 = -5x + 25$$
$$5x + 3y + 24 - 25 = 0$$
$$5x + 3y - 1 = 0$$

Question no.26:

Find equation of two parallel lines perpendicular to $2x - y + 3 = 0$ such that the product of the x and y-intercept of each is 3.

Solution:

Given line $= 2x - y + 3 = 0$

Any line perpendicular to given line is

$$x + 2y + c = 0(\text{required line})$$

For x-intercept put y=0

$$x + c = 0$$
$$x = -c$$

According to given condition

X-intercept × y-intercept =3

$$-c \times -\frac{c}{2} = 3$$
$$c^2 = 6$$
$$c = \pm\sqrt{6}$$

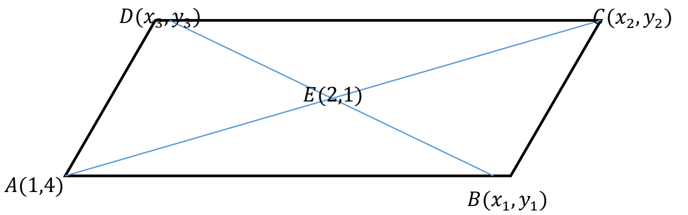
Putting c in required line

$$x + 2y \pm \sqrt{6} = 0$$

Question no 27:

One vertex of a parallelogram is (1,4), *the diagonals intersect at* (2,1)*and the sides have slopes* 1 *and* $\frac{1}{7}$ *find the other* Three vertices.

Solution:



let $(1,4)$*be given vertex and* $B(x_1,y_1)C(x_2,y_2)$*and* $D(x_3,y_3)$ Be required vertices.

$\because$ *E is midpoint of AC so*

$$(2,1) = \frac{1+x_2}{2}, \frac{4+y_2}{2}$$

⇨ $2 = \frac{1+x_2}{2}, 1 = \frac{4+y_2}{2}$

⇨ $1 + x_2 = 4 \ \ , 1 = \frac{4+y_2}{2}$

⇨ $x_2 = 3 \ \ , y_2 - 2$

So $C(c_2, y_2) = (3,-2)$

Now

Slope of AD=$\frac{y_3-4}{x_3-1} \Rightarrow 1 = \frac{y_3-4}{x_3-1}$

⇨ $x_3 - 1 = y_3 - 4 \Rightarrow x_3 - y_3 - 1 + 4 = 0$

⇨ $x_3 - y_3 + 5 = 0 \rightarrow (i$

Slope of BC= $\frac{-2-y_1}{3-x_1} \Rightarrow 1 = \frac{-2-y_1}{3-x_1}$

⇨ $3 - x_1 = -2 - y_1 \Rightarrow x_1 - 3 - 2 - y_1 = 0$

⇨ $x_1 - y_1 - 5 = 0 \rightarrow (ii$

Slope of AB $= \frac{y_1-4}{x_1-1} \Rightarrow -\frac{1}{7} = \frac{y_1-4}{x_1-1}$

⇨ $-x_1 + 1 = 7y_1 - 28 \Rightarrow x_1 + 7y_1 - 1 - 28 = 0$

⇨ $x_1 + 7y_1 - 29 = 0 \rightarrow (iii$

$$slope\ of\ DC = \frac{-2-y_3}{3-x_2} \Rightarrow -\frac{1}{7} = \frac{-2-y_3}{3-x_3}$$
$$\Rightarrow -3 + x_3 = -14 - 7y_3$$
$$\Rightarrow -3 + x_3 + 14 + 7y_3 = 0$$
$$-3 + x_3 + 14 + 7y_3 = 0$$
$$\Rightarrow x_3 + 7y_3 + 11 = 0 \rightarrow (iv$$

$by(iv) - (i) \ \ \Rightarrow x_1 + 7y_3 + 11 = 0$
$\quad -x_3 \mp y_3 \pm 3 = 0$
$\quad 8y_3 + 8 = 0$

$$\Rightarrow 8y_3 = -8 \Rightarrow y_3 = -1\ put\ in\ (1$$
$$x_3 - (-1) + 3 = 0 \Rightarrow x_3 + 1 + 3 = 0$$
$$x_3 + 4 = 0 \Rightarrow x_3 = -4$$

$by\ \ (iii) - (ii) \Rightarrow x_1 + 7y + 29 = 0$
$\quad \pm x_1 \mp y_1 \mp 5 = 0$
$\quad 8y_1 - 24 = 0$

$$8y_1 = 24 \Rightarrow y_1 = 3\ put\ in\ (ii)$$
$$x_1 - 3 - 5 = 0 \Rightarrow x_1 - 8 = 0$$
$$x_1 = 8$$

Hence required vertices are $B(x_1,y_1) = B(8,3)$, $C(x_2,y_2) = C(3,-2), D(x_3,y_3) = D(-4,-1)$

Remember ***above line****: if sign y in given equation and* Our answer is same.

Below line: if sign y in given equation and our answer is different

Question no 28: find whether the given point lies above or below the given line.

a) $\quad 5, 8); 2x - 3y + 6 = 0$

b) $\quad -7, 6); 4x + 3y - 9 = 0$

Solution: (a) given line $2x - 3y + 6 = 0 \rightarrow (i)$

$-2x + 3y - 6 = 0 \qquad \therefore b > 0$

Given point $\quad p(5,8)$

Put $x = 5\ y = 8$ ***in L.H.S. in (i)***

$$2(5) - 3(8) + 6 = \ -10 + 24 - 6$$
$$-16 + 28 = 8 > 0$$

So the point p(5,8) lies above the line

Solution: (b) given line $4x + 3y - 9 = 0 - - - i \quad \therefore b > 0$

$$Given\ point \quad p(-7,6)$$
$$Put\ x = -7\ and\ y = 6\ in\ i$$

$$4(-7)+3(6)-9=-28+18-9$$
$$\mathbf{-37+18=-19=-ve}$$

So the point (-7, 6) lies below the line

Question no 29: check whether the given points are on the same or opposite sides of the given line.

Solution:

a) **(0,0) and (-4,7) ;$6x-7y+70=0$**

b) **(2,3)and (-2,3) ; $3x-5y+8=0$**

Solution: (a) given line $\mathbf{6x-7y+70=0}$

$-6x+7y-70=0 \qquad \therefore b>0$

$Given\ point\ p(0,0)\ ,Q(-4,7)$

$For\ point\ p(0,0):$

$Put\ x=0\ y=0\ in\ above\ equation$

$-6(0)+7(0)-70=\ 0+0-70=-70<0$

$So\ the\ p(0,0)\ lies\ below\ the\ given\ line$

$For\ point\ Q(-4,7):$

$Put\ x=-4\ and\ y=7\ in\ above\ equation$

$-6(-4)+7(7)-70=24+49-70$

$$3>0$$

So Q lies above the line.

$\boldsymbol{Solution}: (\boldsymbol{b})\ \boldsymbol{given\ line}\quad \mathbf{3x-5y+8=0}$

$$3x-5y+8=0$$

$$\because sign\ of\ coeffeiceient\ of\ y=-5=-ve$$

$$Given\ points\quad p(2,3)\ Q(-2,3)$$

$$For\ point\ p(2,3):$$

$$Put\ x=2\ and\ y=3\ in\ above\ equation$$

$$3(2)-5(3)+8=6-15+8$$
$$=\ -1=-ve$$

$$So\ point\ P\ lies\ above\ the\ line$$

$$For\ point\ Q(-2,3):$$

$$3(-2)-5(3)+8=-6-15+8$$
$$=\ -15+2$$
$$=-13=-ve$$

$So\ point\ Q\ lies\ above\ the\ line.$

$\because$ $both\ points\ are\ above\ points\ are\ on\ the\ same\ sides$

Question No 30: find the distance from the point p (6,-1) in the line$6x-4y+9=0$.

Solution: given point p (6,-1)

Line $\mathbf{6x-4y+9=0}$.

As we know that distance from the points p $(x_1,y_1)$to line $ax+by+c=0$

$$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

Here a=6 b=-4 c=9 and $x_1=6, y_1=-1$

$$d=\frac{|6(6)-4(-1)+9|}{\sqrt{6^2+(-4)^2}}$$

$$d=\frac{|36+4+9|}{\sqrt{36+16}}$$

$$d=\frac{49}{\sqrt{52}}$$

Question no 31: find the area of triangular region whose vertices are $\boldsymbol{A}\ (\mathbf{5},\mathbf{3})\ \boldsymbol{B}(-\mathbf{2},\mathbf{2})\ \boldsymbol{C}(\mathbf{4},\mathbf{2})$.

$\boldsymbol{Solution}$: $A\ (5,3)\ B(-2,2)\ C(4,2)$.

AREA of triangular region= $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$= \frac{1}{2}\begin{vmatrix} 5 & 3 & 1 \\ -2 & 2 & 1 \\ 4 & 2 & 1 \end{vmatrix}$

$=\frac{1}{2}[5(2-2)-3(-2-4)+1(-4-8)]$

$$\frac{1}{2}(0+18-12)$$

$$\frac{1}{2}(6)=3sq.unit$$

Question no32: the coordinates of three points are $\boldsymbol{A}(\mathbf{2},\mathbf{3}), \boldsymbol{B}(-\mathbf{1},\mathbf{1})$ and $\boldsymbol{C}(\mathbf{4},-\mathbf{5})$ by comparing the area bounded by ABC check whether the points are collinear.

Solution:

AREA of triangular region= $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$= \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 4 & -5 & 1 \end{vmatrix}$

$= 1/2\ [2(1+5)-3(-1-4)+1(5-4)]$

$$=\frac{1}{2}(12+15+1$$

$$=\frac{1}{2}(28)=14\neq 0$$

$$so\ the\ points\ A,B,C\ are\ not\ collinear.$$

Angle between two lines

Theorem: $let\ l_1\ and\ l_2 be\ two\ non-verticle\ lines\ such\ that\ they\ are\ not\ \perp ar\ to\ each\ other.$

$if\ m_1 and$
$m_2\ are\ the\ slopes\ of\ l_1\ and\ l_2\ respectively, then\ the\ angle\ \theta\ from\ l_1 to\ l_2 is\ given\ by$

$$tan\theta=\frac{m_2-m_1}{1+m_1m_2}$$

Proof:

$\because sum\ of\ all\ three\ angles\ is\ equal\ to\ 180^0\ so$

$$\alpha_1+\theta+180^0$$

⇨ $\alpha_1-\alpha_2+\theta=180^0-\alpha_2=180^0$

⇨ $\alpha_1-\alpha_2+\theta=0$

⇨ $\theta=\alpha_2-\alpha_1$

$$tan\theta=\tan(\alpha_2-\alpha_1)$$

$$tan\theta=\frac{tan\alpha_2-tan\alpha_1}{1+tan\alpha_1 tan\alpha_2}$$

$\because \tan(\alpha-\beta)=\frac{tan\alpha-tan\beta}{1+tan\alpha tan\beta}$

⇨ $tan\theta=\frac{m_2-m_1}{1+m_1m_2}$

$$\because m_1=tan\alpha_1=slope\ of\ l_1$$
$$m_2=tan\alpha_2=slope\ of\ l_2$$

Corollary 1. If two lines are parallel then their slopes are equal.

$$\boldsymbol{i.e\ l_1||l_2\ if\ and\ only\ if\ m_1=m_2}$$

Proof:

$let\ m_1 and\ m_2\ be\ slopes\ of\ lines\ l_1 and\ l_2\ resp.$

$$let\ \theta\ be\ angle\ from\ l_1 to\ l_2$$
$$\because lines\ are\ ||so\ \theta = 0$$

We know that $tan\theta = \frac{m_2-m_1}{1+m_1m_2}$

$$tan0 = \frac{m_2 - m_1}{1 + m_1m_2} \Rightarrow 0 = \frac{m_2 - m_1}{1 + m_1m_2}$$
$$\Rightarrow m_2 - m_1 = 0 \Rightarrow m_1 = m_2\ \ hence\ proved.$$

Corollary 2.

$if\ two\ lines\ are$ $\perp ar\ then\ product\ of\ their\ slopes$

Is equal to -1

$$i.e\ l_1 \perp l_2\ iff\ 1 + m_1m_2 = 0$$

Proof:

$let\ m_1\ and\ m_2\ be\ slopes\ of\ l_1\ and\ l_2\ respectively.$

$$let\ \theta be\ an\ angle\ from\ l_1 and\ l_2$$
$$\because lines\ are \perp ar\ so\ \theta = 90^0$$

We know that

$$tan\theta = \frac{m_2 - m_1}{1 + m_1m_2}$$
$$tan90^0 = \frac{m_2 - m_1}{1 + m_1m_2}$$
$$\infty = \frac{m_2 - m_1}{1 + m_1m_2} \Rightarrow \frac{1}{0} = \frac{m_2 - m_1}{1 + m_1m_2}$$
$$\Rightarrow 1 + m_1m_2 = 0 \Rightarrow m_1m_2 = -1$$

Hence proved.

Equation of a straight line in matrix form

One linear equation:

A linear equation $l; ax + by + c = 0\ in\ two\ variables$ $x\ and\ y\ has\ its\ matrix\ form\ as$

$$[ax + by] = [-c]$$
$$0r\ [1\ \ b]\begin{bmatrix}x\\y\end{bmatrix} = [-c]$$
$$\Rightarrow AX = B \quad A = [a \quad b]\ , X = \begin{bmatrix}x\\y\end{bmatrix}, B = [-c]$$

A system of linear equation:

A system of two linear equations

$$l_1; a_1x + b_1y + c_1 = 0$$
$$l_2; a_2x + b_2y + c_2 = 0\ in\ two\ variables.$$
$$x\ and\ y\ can\ be\ written\ in\ the\ form\ as$$
$$\begin{bmatrix}a_1x & b_1y\\a_2x & b_2y\end{bmatrix} = \begin{bmatrix}-c_1\\-c_2\end{bmatrix}$$
$$\Rightarrow \begin{bmatrix}a_1 & b_1\\a_2 & b_2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-c_1\\-c_2\end{bmatrix}$$
$$AX = C$$

Where $A = \begin{bmatrix}a_1 & b_1\\a_2 & b_2\end{bmatrix}, C = \begin{bmatrix}-c_1\\-c_2\end{bmatrix}, X = \begin{bmatrix}x\\y\end{bmatrix}$

A system of three linear equations;

A system of three linear equations

$$l_1; a_1x + b_1y + c_1 = 0$$
$$l_2; a_2x + b_2y + c_2 = 0$$
$$l_3; a_3x + b_3y + c_3 = 0$$

In two variables $x\ and\ y\ takes\ the\ form$

As

$$\begin{bmatrix}a_1x + b_1y + c_1\\a_2x + b_2y + c_2\\a_3x + b_3y + c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$\Rightarrow \begin{bmatrix}a_1 & b_1 & c_!\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Exercise 4.4

Q#1) Find the point of intersection of the lines:

i) $x - 2y + 1 = 0$ and $2x - y + 2 = 0$

ii) $3x + y + 12 = 0$ and $x + 2y - 1 = 0$

iii) $x + 4y - 12 = 0$ and $x - 3y + 3 = 0$

(i) $x - 2y + 1 = 0$ and $2x - y + 2 = 0$

Sol: Let $x - 2y + 1 = 0$ →(1), $2x - y + 2 = 0$ →(2)

From Eq.1, we have

$x - 2y + 1 = 0 \ \Rightarrow x = 2y - 1$

Put in Eq. (2)

$2x - y + 2 = 0 \ \Rightarrow 2(2y - 1) - y + 2 = 0$

$\Rightarrow 4y - 2 - y + 2 = 0$

$\Rightarrow\ 3y$=0 $\Rightarrow y = 0$ put in Eq. (1)

$x = 2y - 1 \Rightarrow x = 2(0) - 1 \ \Rightarrow x = -1$

Hence point of intersection of Eq. (1) and (2) is $A(-1,0)$.

(ii) $3x + y + 12 = 0$ and $x + 2y - 1 = 0$

Sol: Let $3x + y + 12 = 0$ →(1), $x + 2y - 1 = 0$ →(2)

From Eq.1, we have

$3x + y + 12 = 0 \ \Rightarrow y = -3x - 12$

put in Eq. (2)

$x + 2y - 1 = 0 \ \Rightarrow\ x + 2(-3x - 12) - 1 = 0$

$\Rightarrow x - 6x - 24 - 1 = 0$

$\Rightarrow\ -5x - 25$=0 $\Rightarrow y = \frac{25}{-5} = -5$ put in Eq. (1)

$y = -3x - 12 \Rightarrow y = -3(-5) - 12$

$\Rightarrow x = 15 - 12 = 3$

Hence point of intersection of Eq. (1) and (2) is $B(-5,3)$.

(iii) $x + 4y - 12 = 0$ and $x - 3y + 3 = 0$

Sol: Let $x + 4y - 12 = 0 = 0$ →(1), $x - 3y + 3 = 0$ →(2)

From Eq.1, we have

$x + 4y - 12 = 0 \ \Rightarrow x = -4y + 12$

Put in Eq. (2)

$x - 3y + 3 = 0 \ \Rightarrow\ (-4y + 12) - 3y + 3 = 0$

$\Rightarrow -4y + 12 - 3y + 3 = 0$

$\Rightarrow\ -7y + 15$=0 $\Rightarrow y = \frac{-15}{-7} = \frac{15}{7}$ put in Eq. (1)

$x = -4y + 12 \Rightarrow x = -4\left(\frac{15}{7}\right) + 12$

$\Rightarrow x = -\frac{60}{7} + 12 = \frac{-60+84}{7} = \frac{24}{7}$

Hence point of intersection of Eq. (1) and (2) is $C(\frac{24}{7}, \frac{15}{7})$.

Q#2) Find an equation of the line through

(i) the point $(2,-9)$ and the intersection of the lines
$2x+5y-8=0$ And $3x-4y-6=0$

Sol: Let$l_1: 2x+5y-8=0,\ l_2: 3x-4y-6=0$
and $(2,-9)$
Equation of line through the intersection of l_1 and $l_2$is given by
$l:\ l_1+kl_2=0$
$(2x+5y-8)+k(3x-4y-6)=0 \rightarrow(1)$
$2x+5y-8+3kx-4ky-6k=0$
Put $x=2$ and $y=-9$ in above
$\Rightarrow\ 2(2)+5(-9)-8+3k(2)-4k(-9)-6k=0$
$\Rightarrow\ 4-45-8+6k+36k-6k=0$
$\Rightarrow\ -49+36k=0 \ \Rightarrow\ k=\frac{49}{36}$
Put in (1)
$(2x+5y-8)+\frac{49}{36}(3x-4y-6)=0$
$\Rightarrow 36(2x+5y-8)+49(3x-4y-6)=0$
$\Rightarrow\ 72x+180y-288+147x-196y-294=0$
$\Rightarrow\ 219x-16y-582=0$

(ii) the intersection f the lines
$x-y-4=0\ and\ 7x+y+20=0\ and$
$(a) parallel\ \ (b) perpendicualr\ to\ be\ line\ 6x+y-14=0$

solution:
$x-y-4=0 \rightarrow (i)$
$7x+y+20=0 \rightarrow (ii)$
By $(i)+(ii) \Rightarrow 8x+16=0 \Rightarrow 8x=-16$
$\Rightarrow x=-2\ put\ in\ (i) \Rightarrow -2-y-4=0$
$\Rightarrow -y-6=0 \Rightarrow -y=6 \Rightarrow y=-6$
So point of intersection is $(-2,-6)$
Given line is
$6x+y-14=0\ slope\ of\ given\ line=-6$
(a) slope of required line is $=-6$
$(\because line\ is\ ||to\ given\ line)$
$thus\ eq. of\ line\ through\ (-2,-6) and\ slope\ is-6$
$y-(-6)=-6(x-(-2)\ \ \because y-y_1=m(x-x_1)$
⇨ $y+6=-6(x-(-2$
⇨ $y+6=-6(x+2$
⇨ $y+6=-6x-12$
⇨ $6x+y+6+12=0$
⇨ $6x+y+18=0 (req. lines$

(b) $\because slope\ of\ given\ line\ is-6$
Slope of required line $=\frac{1}{6}(\because req. line\ is\ \perp ar\ to\ given\ line)$
So eq. of lines through
$-2,-6) line\ and\ slope \frac{1}{6} is$
$y-(-6)=\frac{1}{6}(x+2)\ \ \because y-y_1=m(x-x_1)$
$\Rightarrow y+6=\frac{1}{6}(x+2)$
$6y+36=x+2$
⇨ $x-6y+2-36=0$
⇨ $x-6y-34=0\ req. line$

(iii) Through the intersection of the lines
$x+2y+3=0\ , 3x+4y+7=0\ and\ making$
Equal intercepts on the axes.
Solution:
$any\ line\ through\ intersection\ of$
$x+2y+3=0\ \ and\ \ 3x+4y+7=0\ is$
$x+2y+3+k(3x+4y+7)=0 \rightarrow (i)$
⇨ $x+2y+3kx+7ky+7k=0$
$\left(\because a\ line\ passing\ through\ intersection\ of\ l_1\ and\ l_2\ is\ l_1+kl_2\right)$
⇨ $3k+1)x+(2+4k)y+3+7k=0$
For $x-intercept\ , y=0-$
So $(3k+1)x+3+7k=0$
⇨ $x=\frac{-(3+7k}{3k+1}$
$for\ y-intercept, x=0$
⇨ $2+4k)y+3+7k=0$
⇨ $y=\frac{-(3+7k}{2+4k}$
$\because both\ intercepts\ are\ equal\ so$
$\frac{-\ 3+7k}{3k+1}=\frac{-\ 3+7k}{2+4k}$
$\Rightarrow \frac{1}{3k+1}=\frac{1}{2+4k}$
$\Rightarrow 3k+1=2+4k$
⇨ $4k-3k+2-1=0$
⇨ K+1=0
⇨ $k=-1\ \ \ \ so\ (i) becomes$
As $x+2y+3+(-1)(3x+4y+7)=0$
⇨ $x+2y+3-3x-4y-7=0$
⇨ $-2x-2y-4=0$
⇨ $2x+2y+4=0$
⇨ $x+y+2=0 (\div by\ 2$

Q#3) Find an equation of the line through the intersection of
$16x-10y-33=0; 12x+14y+29=0$
And the intersection of $x-y+4=0; x-7y+2=0$

Solution:
Let$l_1: 16x-10y-33=0,\ l_2: 12x+14y+29=0$
And$l_3: x-y+4=0,\ l_4: x-7y+2=0$
First, we find the intersection of l_1 and l_2
Let $16x-10y-33=0$ →(1), $12x+14y+29=0$ →(2)
From Eq.1, we have
$16x-10y-33=0\ \Rightarrow x=\frac{10y+33}{16}$

put in Eq. (2)

$$12x + 14y + 29 = 0$$

$\Rightarrow 12(\frac{10y+33}{16} + 14y + 29 = 0$

$\Rightarrow 3(\frac{10y+33}{4} + 14y + 29 = 0$

$\Rightarrow 30y + 99 + 56y + 116 = 0$

$\Rightarrow$ $86y+215=0 \Rightarrow y = -\frac{215}{86} = -\frac{5}{2}$ put in Eq. (1)

$x = \frac{10y+33}{16} \Rightarrow x = \frac{10(-\frac{5}{2}+33}{16} \Rightarrow x = \frac{1}{2}$

Hence point of intersection of Eq. (1) and (2) is $A(\frac{1}{2}, -\frac{5}{2})$.

Equation of line through the intersection of l_3 and l_4 is given by

$l:\ l_3 + kl_4 = 0$

$(x - y + 4) + k(x - 7y + 2) = 0 \rightarrow(3)$

Put $x = \frac{1}{2}$ and $y = -\frac{5}{2}$ in above

$\Rightarrow \quad \frac{1}{2} - -\frac{5}{2} + 4) + k \ \frac{1}{2} - 7(-\frac{5}{2} + 2) = 0$

Multiply by 2, we get

$\Rightarrow \quad 1 + 5 + 8 + k(1 + 35 + 4) = 0$

$\Rightarrow \quad 14 + 40k = 0 \ \Rightarrow \ k = \frac{-14}{40} = -\frac{7}{20}$

Put in (3)

$(x - y + 4) - \frac{7}{20}(x - 7y + 2) = 0$

$\Rightarrow 20(x - y + 4) - 7(x - 7y + 2) = 0$

$\Rightarrow 20x - 20y + 80 - 7x + 49y - 14 = 0$

$\Rightarrow 13x - 29y + 66 = 0$ (Required line)

Q#4) Find the condition that the lines $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent.

Sol: Let $l_1: m_1x - y + c_1 = 0$, $l_2: m_2x - y + c_2 = 0$

And $l_3: m_3x - y + c_3 = 0$

As we know that the line are concurrent if

$$\begin{vmatrix} m_1 & -1 & c_1 \\ m_2 & -1 & c_2 \\ m_3 & -1 & c_3 \end{vmatrix} = 0$$

$\begin{vmatrix} m_1 & -1 & c_1 \\ m_2 - m_1 & 0 & c_2 - c_1 \\ m_3 - m_1 & 0 & c_3 - c_1 \end{vmatrix} = 0$ By $R_2 - R_1$ and $R_2 - R_3$

Expanding by R_1, we have

$$\Rightarrow m_1(0-0) + 1\big((m_2 - m_1)(c_3 - c_1) - (m_3 - m_1)(c_2 - c_1)\big) - c_1(0-0) = 0$$

$\Rightarrow \quad m_2 - m_1)(c_3 - c_1) = (m_3 - m_1)(c_2 - c_1$

(Which is the required condition)

Q#5) Determine the value p such that $2x - 3y - 1 = 0$, $3x - y - 5 = 0$ and $3x + py + 8 = 0$

Sol: Let$l_1: 2x - 3y - 1 = 0$, $l_2: 3x - y - 5 = 0$

And $l_3: 3x + py + 8 = 0$

As we know that the line are concurrent if

$$\begin{vmatrix} m_1 & -1 & c_1 \\ m_2 & -1 & c_2 \\ m_3 & -1 & c_3 \end{vmatrix} = 0$$

Put values

As we know that the line are concurrent if

$$\begin{vmatrix} 2 & -3 & -1 \\ 3 & -1 & -5 \\ 3 & p & 8 \end{vmatrix} = 0$$

$\Rightarrow \ 2(-8 + 5p) + 3(24 + 15) - 1(3p + 3) = 0$

$\Rightarrow \ -16 + 10p + 72 + 25 - 3p - 3 = 0$

$\Rightarrow \ 7p + 98 = 0$

$\Rightarrow \ p = -\frac{98}{7} = -14$

Q#6) Show that the lines $4x - 3y - 8 = 0$, $3x - 4y - 6 = 0$ and $x - y - 2 = 0$ are concurrent and the third line bisects the angle formed by first two lines.

Sol: Let$l_1: 4x - 3y - 8 = 0$, $l_2: 3x - 4y - 6 = 0$

And $l_3: x - y - 2 = 0$

To check l_1, l_2 and l_3 are concurrent, we take

$$\begin{vmatrix} 4 & -3 & -8 \\ 3 & -4 & -6 \\ 1 & -1 & -2 \end{vmatrix} = 4(8-6) + 3(-6+6) - 8(-3+4)$$

$$= 8 + 0 - 8 = 0$$

Hence, the given lines are concurrent.

Now, we find the slopes of these line i.e.

$Slope\ of\ l_1 = m_1 = -\frac{4}{-3} = \frac{4}{3}$

$Slope\ of\ l_2 = m_2 = -\frac{3}{-4} = \frac{3}{4}$

$Slope\ of\ l_3 = m_3 = -\frac{1}{-1} = 1$

Let θ_1, be the angle between l_1 and l_3

$tan\theta_1 = \frac{m_3 - m_1}{1 + m_1m_3}$

$= \frac{1 - \frac{4}{3}}{1 + (1)\ \frac{4}{3}} = \frac{\frac{3-4}{3}}{\frac{3+4}{3}}$

$= -\frac{1}{7} \rightarrow(1)$

Let θ_1, be the angle between l_3 and l_2

$tan\theta_2 = \frac{m_2 - m_3}{1 + m_2m_3}$

$= \frac{\frac{3}{4} - 1}{1 + (1)\ \frac{3}{4}} = \frac{\frac{3-4}{4}}{\frac{4+3}{4}}$

$= -\frac{1}{7} \rightarrow(2)$

From eq. (1) and (2)

$tan\theta_1 = tan\theta_2$

$\Rightarrow \ \theta_1 = \theta_2$

$\Rightarrow l_3$ Bisect the angle formed by first two lines.

Q#7) the vertices of a triangle are $A(-2, 3)$, $B(-4, 1$ and $C(3, 5)$. Find the coordinates of (i) centroid (ii) orthocenter (iii) circumcenter. Are these three points collinear?

(I) Sol: <u>Centroid</u>

Centroid of a triangle is the point of concurrency of its three medians.

Let D and E be the mid points of $\overline{BC}$ and $\overline{AC}$ respectively.

Midpoint of $\overline{BC} = D\left(\frac{-4+3}{2}, \frac{1+5}{2}\right) = D(-\frac{1}{2}, 3)$

Mid-point of $\overline{AC} = E\left(\frac{-2+3}{2}, \frac{5+3}{2}\right) = E(\frac{1}{2}, 4)$

Equation of the median $\overline{BE}$

The points on $\overline{BE}$ are $B(-4,1), D(\frac{1}{2}, 4)$

Slope of $\overline{BE} = \frac{4-1}{\frac{1}{2}+4} = \frac{3}{\frac{9}{2}} = \frac{2}{3}$

Now, $y - y_1 = m_1(x - x_1) \Rightarrow y - 1 = \frac{2}{3}(x+4)$

using the point $B(-4,1)$.

$3y - 3 = 2x + 8 \Rightarrow \; 2x - 3y + 11 = 0 \rightarrow (1)$

Equation of the median $\overline{AD}$

The points on $\overline{AD}$ are $A(-2,3), D(-\frac{1}{2}, 3$

Slope of $\overline{AD} = \frac{3-3}{-\frac{1}{2}+2} = 0$

Now, $y - y_1 = m_2(x - x_1) \Rightarrow y - 3 = 0(x+2)$

using the point $A(-2,3)$.

$y - 3 = 0 \Rightarrow y = 3$ put in eq. (1)

$2x - 3y + 11 = 0 \Rightarrow \; 2x - 3(3) + 11 = 0$

$\Rightarrow \; 2x - 9 + 11 = 0 \Rightarrow x = -1$

Hence Centroid is $(-1,3)$.

(II) Sol: Orthocenter

Orthocenter of a triangle is the point of concurrency of its three altitudes.

Let $\overline{AP} \perp \overline{BC}$ and $\overline{BQ} \perp \overline{AC}$ be the altitudes of the triangleABC.

Slope of $\overline{BC} = m_1 = \frac{5-1}{3+4} = \frac{4}{7}$

Slope of $\overline{AC} = m_2 = \frac{5-3}{3+2} = \frac{2}{5}$

Equation of the Altitude $\overline{AP}$

Slope of $\overline{AP} = -\frac{1}{m_1} = -\frac{7}{4}$

Now, $y - y_1 = m_1(x - x_1) \Rightarrow y - 3 = -\frac{7}{4}(x+2)$

using the point $A(-2,3)$ on $\overline{AP}$.

$4y - 12 = -7x - 14 \Rightarrow 7x + 4y + 2 = 0$...... (1)

Equation of the Altitude $\overline{BQ}$

Slope of $\overline{BQ} = -\frac{1}{m_2} = -\frac{5}{2}$

Now, $y - y_1 = m_2(x - x_1) \Rightarrow y - 1 = -\frac{5}{2}(x+4)$

using the point $B(-4,1)$ on $\overline{BQ}$.

$2y - 2 = -5x - 20 \Rightarrow 5x + 2y + 18 = 0$...... (2)

From eq. (2)

$5x + 2y + 18 = 0 \Rightarrow x = \frac{-2y-18}{5}$

put in Eq. (1)

$7x + 4y + 2 = 0 \Rightarrow 7(\frac{-2y-18}{5} + 4y + 2 = 0$

$\Rightarrow -14y - 126 + 20y + 10 = 0$

$\Rightarrow$ 6y-116=0 $\Rightarrow y = \frac{116}{6} = \frac{58}{3}$ put in Eq. (2)

$x = \frac{-2y-18}{5} \Rightarrow x = \frac{-2(\frac{58}{3} - 18}{5} \Rightarrow x = -\frac{34}{3}$

Hence, the orthocenter is $(-\frac{34}{3}, \frac{58}{3})$.

(III) Sol: Circumcenter

Circumcenter of a triangle is the point of concurrency of right bisectors of its sides.

Let $\overline{PQ}$ and $\overline{RS}$ be the right bisectors $\overline{BC}$ and $\overline{AC}$ respectively.

Slope of $\overline{AC} = m_1 = \frac{5-3}{3+2} = \frac{2}{5}$

Slope of $\overline{AC} = m_2 = \frac{5-1}{3+4} = \frac{4}{7}$

Midpoint of $\overline{BC} = D\left(\frac{-4+3}{2}, \frac{1+5}{2}\right) = L(-\frac{1}{2}, 3)$

Midpoint of $\overline{AC} = E\left(\frac{-2+3}{2}, \frac{5+3}{2}\right) = M(\frac{1}{2}, 4)$

Equation of the Altitude $\overline{RS}$

Slope of $\overline{RS} = -\frac{1}{m_1} = -\frac{5}{2}$

Now, $y - y_1 = m(x - x_1) \Rightarrow y - 4 = -\frac{5}{2}\left(x - \frac{1}{2}\right)$

using the point $M\left(\frac{1}{2}, 4\right)$ on $\overline{RS}$.

$2y - 8 = -5x + \frac{5}{2} \Rightarrow 4y - 16 = -10x + 5$

$\Rightarrow 10x + 4y - 21 = 0$...... (1)

Equation of the Bisector $\overline{PQ}$

Slope of $\overline{PQ} = -\frac{1}{m_2} = -\frac{7}{4}$

Now, $y - y_1 = m(x - x_1) \Rightarrow y - 3 = -\frac{7}{4}\left(x + \frac{1}{2}\right)$

using the point $L\left(-\frac{1}{2}, 3\right)$ on $\overline{PQ}$.

$4y - 12 = -7x - \frac{7}{2} \Rightarrow 14x + 8y - 17 = 0$...... (2)

From eq. (2)

$14x + 8y - 17 = 0 \Rightarrow x = \frac{-8y+17}{14}$

put in Eq. (1)

$10x + 4y - 21 = 0 \Rightarrow 10(\frac{-8y+17}{14} + 4y - 21 = 0$

$\Rightarrow -80y + 170 + 56y - 294 = 0$

$\Rightarrow \; -24y - 124 = 0 \Rightarrow y = \frac{124}{-24} = -\frac{31}{6}$ put in Eq. (2)

$x = \frac{-8y+17}{14} \Rightarrow x = \frac{-8(-\frac{31}{6} + 17}{14} \Rightarrow x = \frac{25}{6}$

Hence, the Circumcenter is $\frac{25}{6}, -\frac{31}{6})$.

(IV) Now, we check whether centroid, orthocenter and circumcenter are collinear or not.

Centroid is $(-1,3)$, orthocenter is $(-\frac{34}{3}, \frac{58}{3}$ and Circumcenter is $\frac{25}{6}, -\frac{31}{6})$.

Let

$$\begin{vmatrix} -1 & 3 & 1 \\ -\frac{34}{3} & \frac{58}{3} & 1 \\ \frac{25}{6} & -\frac{31}{6} & 1 \end{vmatrix}$$

$$= -1\left(\frac{58}{3} + \frac{31}{6} \; - 3\left(-\frac{34}{3} - \frac{25}{6}\right.\right.$$

$$+ 1 \; \frac{1054}{18} - \frac{1450}{18}$$

$$= -1\left(\frac{116+31}{6} + 3\ \frac{68+25}{6} + 1\ \frac{1054-1450}{18}\right.$$

$$= -\frac{49}{2} + \frac{93}{2} - 22$$

$$= \frac{-49+93-44}{2} = 0$$

Thus, all the points are collinear (lying on a straight line).

Q#8) Check whether the lines $4x - 3y - 8 = 0$, $3x - 4y - 6 = 0$ and $x - y - 2 = 0$ are concurrent. If so, find the point where they meet.

Sol: Let$l_1: 4x - 3y - 8 = 0$, $l_2: 3x - 4y - 6 = 0$ and $l_3: x - y - 2 = 0$

To check l_1, l_2 and l_3 are concurrent, we take

$$\begin{vmatrix} 4 & -3 & -8 \\ 3 & -4 & -6 \\ 1 & -1 & -2 \end{vmatrix} = 4(8-6) + 3(-6+6) - 8(-3+4)$$

$$= 8 + 0 - 8 = 0$$

Hence, the given lines are concurrent.

For the point of concurrency, we solve , l_2 andl_3.

Let $3x - 4y - 6 = 0 \rightarrow(1)$, $x - y - 2 = 0 \rightarrow(2)$

From Eq.1, we have

$3x - 4y - 6 = 0 \Rightarrow x = \frac{4y+6}{3}$

Put in Eq. (2)

$x - y - 2 = 0 \Rightarrow \frac{4y+6}{3} - y - 2 = 0 \Rightarrow 4y + 6 - 3y - 6 = 0$

$\Rightarrow$ y=0 put in Eq. (1)

$x = \frac{4y+6}{3} \Rightarrow x = \frac{4(0)+6}{3} \Rightarrow x = \frac{6}{3} = 2$

Hence point of intersection of Eq. (1) and (2) is $B(2,0)$.

Q#9.find the coordinates of the vertices of the triangle formed by the lines $x - 2y - 6 = 0$; $3x - y + 3 = 0$; $2x + y - 4 = 0$ *also find* Measures of the angles of the triangle.

Solution:

$$x - 2y - 6 = 0 \rightarrow (i)$$
$$3x - y + 3 = 0 \rightarrow (ii)$$
$$2x + y - 4 = 0 \rightarrow (iii)$$

Solving (i) and (ii)

$$\frac{x}{-6-6} = \frac{y}{-18-3} = \frac{1}{-1+6}$$

⇨ $\frac{x}{-12} = \frac{y}{-21} = \frac{1}{5}$

⇨ $\frac{x}{-12} = \frac{y}{-21} = \frac{1}{5}$

⇨ $x = -\frac{12}{5}$ $and\ y = -\frac{21}{5}$

Solving (ii) and (iii)

$$\frac{x}{4-3} = \frac{y}{6+12} = \frac{1}{3+2}$$

⇨ $\frac{x}{1} = \frac{y}{18} = \frac{1}{5}$

⇨ $\frac{x}{1} = \frac{y}{18} = \frac{1}{5}$

⇨ $\frac{x}{1} = \frac{1}{5}$ $and\ \frac{y}{18} - \frac{1}{5}$

⇨ $x = \frac{1}{5} and\ y = \frac{18}{5}$

Solving (i) and (iii)

$$\frac{x}{8+6} = \frac{y}{-12+4} = \frac{1}{1+4}$$

$$\frac{x}{14} = \frac{1}{5}\ and\ \frac{y}{-8} = \frac{1}{5}$$

$$y = \frac{14}{5}\ and\ y = -\frac{8}{5}$$

So vertices of triangle are

$$A\left(-\frac{14}{8}, -\frac{8}{5}\right), B\ \frac{1}{5}, \frac{18}{5}\Big), C\ \ -\frac{12}{5}, -\frac{21}{5}$$

Now

$$m_1 = Slope\ of\ AB = \frac{\frac{18}{5} - \left(-\frac{18}{5}\right)}{\frac{1}{5} - \left(\frac{14}{5}\right)} = \frac{\frac{18+8}{5}}{\frac{1-14}{5}}$$

$$m_1 = \frac{26}{-13} = -2$$

$$m_2 = Slope\ of\ BC = \frac{-\frac{21}{5} - \frac{18}{5}}{-\frac{12}{5} - -\frac{1}{5}} = \frac{-\frac{39}{5}}{-\frac{13}{5}}$$

$$= -\frac{39}{-13}$$

⇨ $m_2 = 3$

$$m_3 = Slope\ of\ CA = \frac{-\frac{3}{5} + \frac{21}{5}}{\frac{14}{5} + \frac{12}{5}} = \frac{\frac{13}{5}}{\frac{26}{5}} = \frac{13}{26} = \frac{1}{2}$$

$$m_2 = \frac{1}{2}$$

$$Tan\theta_1 = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$(\because \theta\ is\ the\ angle\ from\ l_2 to\ l_1)$

$$= \frac{-2-(-3)}{1+(-2)(3)} = \frac{-5}{1-6} = -\frac{5}{-5} = 1$$

⇨ $Tan\theta_1 = 1$

⇨ $\theta_1 = \tan^{-1}(1) = 45^0$

$$tan\theta_2 = \frac{m_2 - m_1}{1 + m_2 m_3}$$

$(\because \theta_2\ is\ the\ angle\ from\ l_3\ to\ l_2)$

$$Tan\theta_2 = \frac{3 - \frac{1}{2}}{1 + 3\left(\frac{1}{2}\right)} = \frac{\frac{6-1}{2}}{\frac{2+3}{2}} = \frac{5}{5} = 1$$

$$Tan\theta_2 = 1$$

$$\theta_2 = Tan^{-1}(1) = 45^0$$

$$Tan\theta_3 = \frac{m_3 - m_1}{1 + m_3 m_1} = \frac{\frac{1}{2} - (-2}{1 + (\frac{1}{2})\ -2} = \frac{\frac{1}{2} + 2}{1-1}$$

$$= \infty$$

⇨ $Tan\theta_3 = \infty$

⇨ $\theta_3 = Tan^{-1}(\infty) = 90^0$

$(\because \theta_3 \text{ is the angle from } l_1 \text{ to } l_3)$

Q#10) Find the angle measured from the line l_1 to the line l_2 where

a) l_1; joining $(2,7)$ and $(7, 10$
l_2; joining $(1,1)$ and $(-5, 3$

Sol: (a)

Let l_1 : joining $(2,7)$ and $(7, 10)$

Slope of $l_1 = m_1 = \frac{10-7}{7-2} = \frac{3}{5}$

Let l_2: joining $(1,1)$ and $(-5, 3)$

Slope of $l_2 = m_2 = \frac{3-1}{-5-1} = \frac{2}{-6} = -\frac{1}{3}$

Let θ be the angle from $l_1 \rightarrow l_2$, then

$$tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$= \frac{\frac{-1}{3} - \frac{3}{5}}{1 + \frac{-1}{3}\ \frac{3}{5}} = \frac{\frac{-5-9}{15}}{\frac{15-3}{15}}$$

$= -\frac{7}{6}$....

$\theta = \tan^{-1}\left(-\frac{7}{6}\right. = 130.6^o$

Acute angle

$$tan\theta = \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right| = \left|-\frac{7}{6}\right| = \frac{7}{6}$$

$\theta = \tan^{-1}\left(\frac{7}{6}\right. = 49.4^o$

b) l_1; joining $(3,-1)$ and $(5,7$
l_2; joining $(2,4)$ and $(-8, 2$

Sol: (b)

Let l_1 : joining $(3,-1)$ and $(5,7)$

Slope of $l_1 = m_1 = \frac{7+1}{5-3} = \frac{8}{2} = 4$

Let l_2: joining $(2,4)$ and $(-8, 2$

Slope of $l_2 = m_2 = \frac{2-4}{-8-2} = \frac{-2}{-10} = \frac{1}{5}$

Let θ be the angle from $l_1 \rightarrow l_2$, then

$$tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$= \frac{\frac{1}{5} - 4}{1 + \frac{1}{5}\ (4)} = \frac{\frac{1-20}{5}}{\frac{5+4}{5}}$$

$= -\frac{19}{5}$....

$\theta = \tan^{-1}\left(-\frac{19}{5}\right.$

$= 180^o - \tan^{-1}\left(\frac{19}{5}\right.$

$= 115.35^o$

Acute angle

$$tan\theta = \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right| = \left|-\frac{19}{5}\right| = \frac{19}{5}$$

$\theta = \tan^{-1}\left(\frac{19}{5}\right. = 64.65^o$

c) l_1; joining $(1,-7)$ and $(6,-4$
l_2; joining $(-1, 2)$ and $(-6,-1$

Sol: (c)

Let l_1 : joining $(1,-7)$ and $(6,-4)$

Slope of $l_1 = m_1 = \frac{-4+7}{6-1} = \frac{3}{5}$

Let l_2: joining $(-1, 2)$ and $(-6,-1)$

Slope of $l_2 = m_2 = \frac{-1-2}{-6+1} = \frac{-3}{-6} = \frac{3}{5}$

Let θ be the angle from $l_1 \rightarrow l_2$, then

$$tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$= \frac{\frac{3}{5} - \frac{3}{5}}{1 + \frac{3}{5}\ \frac{3}{5}} = 0$$

$= 0$....

$\theta = \tan^{-1}(0) = 0^o$

Acute angle

$$tan\theta = \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right| = |0| = 0$$

$\theta = \tan^{-1}(0) = 0^o$

d) l_1; joining $(-9,-1)$ and $(3,-5$
l_2; joining $(2,7)$ and $(-6,-7$

Sol: (d)

Let l_1 : joining $(-9,-1)$ and $(3,-5)$

Slope of $l_1 = m_1 = \frac{-5+1}{3+9} = \frac{-4}{12} = -\frac{1}{3}$

Let l_2: joining $(2,7)$ and $(-6,-7)$

Slope of $l_2 = m_2 = \frac{-7-7}{-6-2} = \frac{-14}{-8} = \frac{7}{4}$

Let θ be the angle from $l_1 \rightarrow l_2$, then

$$tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$= \frac{\frac{7}{4} + \frac{1}{3}}{1 + \frac{-1}{3}\ (\frac{7}{4})} = \frac{\frac{21+4}{12}}{\frac{12-7}{12}}$$

$= \frac{25}{5} = 5$....

$\theta = \tan^{-1}(5) = 78.69^o$

Acute angle

$\theta = \tan^{-1}(5) = 78.69^o$

Q#11) Find the interior angle of the triangle, whose vertices are

a) $A(-2,11), B(-6,-3)$ and $C(4,-9$

Sol: $A(-2,11), B(-6,-3)$ and $C(4,-9$

$$Slope\ of\ \overline{AB} = m_1 = \frac{-3-11}{-6+2} = \frac{14}{4} = \frac{7}{2}$$

$$Slope\ of\ \overline{BC} = m_2 = \frac{-3+9}{-6-4} = \frac{6}{-10} = \frac{-3}{5}$$

$$Slope\ of\ \overline{AC} = m_3 = \frac{-9-11}{4+2} = \frac{-20}{6}$$

$$= \frac{-10}{3}$$

Let α, β and γ be the angles from $\overline{AB}$ to $\overline{AC}$, $\overline{BC}$ to $\overline{BA}$ and $\overline{CA}$ to $\overline{CB}$ respectively.

$$tan\alpha = \frac{m_3 - m_1}{1 + m_3 m_1}$$

$$= \frac{-\frac{10}{3} - \frac{7}{2}}{1 + \left(-\frac{10}{3}\right)\left(\frac{7}{2}\right)} = \frac{\frac{-20-21}{6}}{\frac{6-70}{6}}$$

$$= \frac{41}{64}$$

$$\alpha = \tan^{-1}\frac{41}{64} = 32.64^o$$

$$tan\,\beta = \frac{m_1 - m_2}{1 + m_2 m_1}$$

$$= \frac{\frac{7}{2} + \frac{3}{5}}{1 + \left(-\frac{3}{5}\right)\left(\frac{7}{2}\right)} = \frac{\frac{35+6}{10}}{\frac{10-21}{10}}$$

$$= \frac{-41}{11}$$

$$\beta = \tan^{-1}\left(\frac{-41}{11}\right) = 180^o - \tan^{-1}\left(\frac{41}{11}\right) = 105.02^o$$

$$tan\gamma = \frac{m_2 - m_3}{1 + m_3 m_2}$$

$$= \frac{-\frac{3}{5} + \frac{10}{3}}{1 + \left(-\frac{10}{3}\right)\left(-\frac{3}{5}\right)} = \frac{\frac{-9+50}{15}}{\frac{15+30}{15}}$$

$$= \frac{41}{45}$$

$$\gamma = \tan^{-1}\left(\frac{41}{45}\right) = 42.34^o$$

Q#12) Find the interior angle of the triangle, whose vertices are

$A(5,2), B(-2,3),\ C(-3,-4)$ and $D(4,-5)$.

Sol:

$$Slope\ of\ \overline{AB} = m_1 = \frac{3-2}{-2-5} = -\frac{1}{7}$$

$$Slope\ of\ \overline{BC} = m_2 = \frac{-4-3}{-3+2} = \frac{-7}{-1} = 7$$

$$Slope\ of\ \overline{CD} = m_3 = \frac{-5+4}{4+3} = \frac{-1}{7}$$

$$Slope\ of\ \overline{AD} = m_4 = \frac{-5-2}{4-5} = \frac{-7}{-1} = 7$$

Let α, β, γ and δ be the angles from $\overline{AB}$ to $\overline{AD}$, $\overline{BC}$ to $\overline{BA}$, $\overline{CD}$ to $\overline{CB}$ and $\overline{AD}$ to $\overline{CD}$ respectively.

$$tan\alpha = \frac{m_4 - m_1}{1 + m_4 m_1}$$

$$= \frac{7 + \frac{1}{7}}{1 + (7)\left(\frac{-1}{7}\right)} = \frac{\frac{49+1}{7}}{\frac{0}{7}}$$

$$= \infty$$

$$\alpha = \tan^{-1}(\infty) = 90^o$$

$$tan\,\beta = \frac{m_1 - m_2}{1 + m_2 m_1}$$

$$= \frac{\frac{-1}{7} - 7}{1 + \left(\frac{-1}{7}\right)(7)} = \frac{\frac{-1-49}{7}}{\frac{0}{7}} = -\infty$$

$$\beta = \tan^{-1}(-\infty) = 180^o - \tan^{-1}(\infty) = 90^o$$

$$tan\gamma = \frac{m_2 - m_3}{1 + m_3 m_2}$$

$$= \frac{7 + \frac{1}{7}}{1 + (7)\left(\frac{-1}{7}\right)} = \frac{\frac{49+1}{7}}{\frac{0}{7}}$$

$$= \infty$$

$$\gamma = \tan^{-1}(\infty) = 90^o$$

$$tan\delta = \frac{m_3 - m_4}{1 + m_3 m_4}$$

$$= \frac{\frac{-1}{7} - 7}{1 + \left(\frac{-1}{7}\right)(7)} = \frac{\frac{-1-49}{7}}{\frac{0}{7}} = -\infty$$

$$\delta = \tan^{-1}(-\infty) = 180^o - \tan^{-1}(\infty) = 90^o$$

$(\because\ \theta_4\ is\ angle\ from\ l_4 to\ l_3)$

Q#13) Show that the points $A(0,0)$, $B(2,1)$, $C(3,3)$ and $D(1,2)$ are vertices of the rhombus. Find the its interior angles.

Sol:

$$Slope\ of\ \overline{AB} = m_1 = \frac{1-0}{2-0} = \frac{1}{2}$$

$$Slope\ of\ \overline{BC} = m_2 = \frac{3-1}{3-2} = \frac{2}{1} = 2$$

$$Slope\ of\ \overline{CD} = m_3 = \frac{2-3}{1-3} = \frac{-1}{-2} = \frac{1}{2}$$

$$Slope\ of\ \overline{AD} = m_4 = \frac{2-0}{1-0} = \frac{2}{1} = 2$$

Let α, β, γ and δ be the angles from $\overline{AB}$ to $\overline{AD}$, $\overline{BC}$ to $\overline{BA}$, $\overline{CD}$ to $\overline{CB}$ and $\overline{AD}$ to $\overline{CD}$ respectively.

$$tan\alpha = \frac{m_4 - m_1}{1 + m_4 m_1}$$

$$= \frac{2 - \frac{1}{2}}{1 + (2)\left(\frac{1}{2}\right)} = \frac{\frac{4-1}{2}}{1+1} = \frac{3}{4}$$

$$\alpha = \tan^{-1}\frac{3}{4} = 36.87^o$$

$$tan\,\beta = \frac{m_1 - m_2}{1 + m_2 m_1}$$

$$= \frac{\frac{1}{2} - 2}{1 + \left(\frac{1}{2}\right)(2)} = \frac{\frac{1-4}{2}}{2} = -\frac{3}{4}$$

$$\beta = \tan^{-1}\left(-\frac{3}{4}\right) = 180^o - \tan^{-1}\left(\frac{3}{4}\right) = 143.13^o$$

$$tan\gamma = \frac{m_2 - m_3}{1 + m_3 m_2}$$

$$= \frac{2 - \frac{1}{2}}{1 + (2)\left(\frac{1}{2}\right)} = \frac{\frac{4-1}{2}}{1+1} = \frac{3}{4}$$

$$\gamma = \tan^{-1}\left(\frac{3}{4}\right) = 36.87^o$$

$$tan\delta = \frac{m_3 - m_4}{1 + m_3 m_4}$$

$$= \frac{\frac{1}{2}-2}{1+\left(\frac{1}{2}\right)(2)} = \frac{\frac{1-4}{2}}{2} = -\frac{3}{4}$$

$$\delta = \tan^{-1}\left(-\frac{3}{4}\right) = 180^o - \tan^{-1}\left(\frac{3}{4}\right) = 143.13^o$$

For rhombus As $m_1 = m_3$ and $m_2 = m_4$

$\Rightarrow \overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$

Thus, $ABCD$ is a parallelogram.

$$Slope\ of\ diagonal\ \overline{AC} = m_5 = \frac{3-0}{3-0} = 1$$

$$Slope\ of\ diagonal\ \overline{BD} = m_6 = \frac{2-1}{1-2} = \frac{-1}{1} = -1$$

$\Rightarrow$ Product of slopes$= m_5 \times m_6 = (1)(-1) = -1$

$\Rightarrow \overline{AC} \parallel \overline{BD}$ no interior angle is 90^o.

Hence, it is clear that $ABCD$ is rhombus.

Q#15) the vertices of a triangle ABC are $A(-2,3), B(-4,1)$ and $C(3,5)$. Find the Centre of the circumcenter of the triangle.

Sol: Circumcenter

Circumcenter of a triangle is the point of concurrency of right bisectors of its sides.

Let $\overline{PQ}$ and $\overline{RS}$ be the right bisectors $\overline{BC}$ and $\overline{AC}$ respectively.

Slope of $\overline{AC} = m_1 = \frac{5-3}{3+2} = \frac{2}{5}$

Slope of $\overline{AC} = m_2 = \frac{5-1}{3+4} = \frac{4}{7}$

Mid point of $\overline{BC} = D\left(\frac{-4+3}{2}, \frac{1+5}{2}\right) = L(-\frac{1}{2}, 3)$

Mid point of $\overline{AC} = E\left(\frac{-2+3}{2}, \frac{5+3}{2}\right) = M(\frac{1}{2}, 4)$

Equation of the Altitude $\overline{RS}$

Slope of $\overline{RS} = -\frac{1}{m_1} = -\frac{5}{2}$

Now, $y - y_1 = m(x - x_1) \Rightarrow y - 4 = -\frac{5}{2}\left(x - \frac{1}{2}\right)$

using the point $M\left(\frac{1}{2}, 4\right)$ on $\overline{RS}$.

$2y - 8 = -5x + \frac{5}{2} \Rightarrow 4y - 16 = -10x + 5$

$\Rightarrow 10x + 4y - 21 = 0$…… (1)

Equation of the Bisector $\overline{PQ}$

Slope of $\overline{PQ} = -\frac{1}{m_2} = -\frac{7}{4}$

Now, $y - y_1 = m(x - x_1) \Rightarrow y - 3 = -\frac{7}{4}\left(x + \frac{1}{2}\right)$

using the point $L\left(-\frac{1}{2}, 3\right)$ on $\overline{PQ}$.

$4y - 12 = -7x - \frac{7}{2} \Rightarrow 14x + 8y - 17 = 0$…… (2)

From eq. (2)

$14x + 8y - 17 = 0 \Rightarrow x = \frac{-8y+17}{14}$

put in Eq. (1)

$10x + 4y - 21 = 0 \Rightarrow 10(\frac{-8y+17}{14} + 4y - 21 = 0$

$\Rightarrow -80y + 170 + 56y - 294 = 0$

$\Rightarrow -24y - 124 = 0 \Rightarrow y = \frac{124}{-24} = -\frac{31}{6}$ put in Eq. (2)

$x = \frac{-8y+17}{14} \Rightarrow x = \frac{-8(-\frac{31}{6}+17}{14} \Rightarrow x = \frac{25}{6}$

Hence, the Circumcenter is $\frac{25}{6}, -\frac{31}{6}$).

Q#16) Express the given system of equations in matrix form. Find in each case whether the lines are concurrent or not.

(a) $x + 3y - 2 = 0, 2x - y + 14 = 0$ and $x - 11y + 14 = 0$

Sol:

$x + 3y - 2 = 0$

$2x - y + 14 = 0$

$x - 11y + 14 = 0$

In matrix form

$$\begin{pmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{pmatrix}\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix} = 1(-14+44) - 3(28-4) - 2(-22+1$$

$$= 30 - 72 + 42 = 0$$

Hence, the given lines are not concurrent.

(b) $2x + 3y + 4 = 0, x - 2y - 3 = 0$ and $3x + 1y - 8 = 0$

Sol:

$2x + 3y + 4 = 0$

$x - 2y - 3 = 0$

$3x + 1y - 8 = 0$

In matrix form

$$\begin{pmatrix} 2 & 3 & 4 \\ 1 & -2 & -3 \\ 3 & 1 & -8 \end{pmatrix}\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider

$$\begin{vmatrix} 2 & 3 & 4 \\ 1 & -2 & -3 \\ 3 & 1 & -8 \end{vmatrix} = 2(16+3) - 3(-8+9) + 4(1+6)$$

$$= 38 - 3 + 28 = 63 \neq 0$$

Hence, the given lines are not concurrent.

(c) $3 - 4y - 2 = 0, x + 2y - 4 = 0$ and $3x - 2y + 5 = 0$

Sol:

$3 - 4y - 2 = 0$

$x + 2y - 4 = 0$

$3x - 2y + 5 = 0$

In matrix form

$$\begin{pmatrix} 3 & -4 & -2 \\ 1 & 2 & -4 \\ 3 & -2 & 5 \end{pmatrix}\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider

$$\begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & -4 \\ 3 & -2 & 5 \end{vmatrix} = 3(10-8) + 4(5+12) - 2(-2 - 6$$

$$= 6 + 68 + 16 = 90 \neq 0$$

Hence, the given lines are not concurrent

Q#17) Find a system of linear equations corresponding to the given matrix form. Check whether the lines represented by the system of concurrent.

(a)

Sol:

$$\begin{pmatrix}1 & 0 & -1\\ 2 & 0 & 1\\ 0 & -1 & 2\end{pmatrix}\begin{bmatrix}x\\ y\\ 0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}x+0y-1\\ 2x+0y+1\\ 0x-y+2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

System of linear equations are

$x+0y-1=0$

$2x+0y+1=0$

$0x-y+2=0$

Consider

$$\begin{vmatrix}1 & 0 & -1\\ 2 & 0 & 1\\ 0 & -1 & 2\end{vmatrix}=1(0+1)-0(4-0)-1(-2-0)$$

$$=1-4+3=0$$

Hence, the given lines are concurrent.

(b)

Sol:

$$\begin{pmatrix}1 & 1 & 2\\ 2 & 4 & -3\\ 3 & 6 & -5\end{pmatrix}\begin{bmatrix}x\\ y\\ 0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}x+y+2\\ 2x+4y-3\\ 3x+6y-5\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

System of linear equations are

$x+y+2=0$

$2x+4y-3=0$

$3x+6y-5=0$

Consider

$$\begin{vmatrix}1 & 1 & 2\\ 2 & 4 & -3\\ 3 & 6 & -5\end{vmatrix}=1(-20+18)-1(-10+9)+2(12-12)$$

$$=-2+1+0=-1\neq 0$$

Hence, the given lines are not concurrent.

Homogenous Equation of the second degree in two variables:

Suppose two straight lines

$a_1x+b_1y+c=0\rightarrow(i)$ *and* $a_2x+b_2y+c_2=0\rightarrow(ii)$

so by (i)*and* (ii)

$$(a_1x+b_1y+c)(a_2x+b_2y+c_2)=0\rightarrow(iii)$$

It is second degree equation in

x *and* y *Eq.*(iii)*is called joint*

Equation of the pair of the lines (i)*and* (ii)

General Homogenous Equation:

$\boldsymbol{ax^2+2hxy+by^2=0}$*where* a,h,b *non* $-$ *zero*)

Is called general homogenous quadratic equation.

Note:

Let $y=m_1x$ *and* $y=m_2x$ *be two lines passing through origin. their joint equation is*

$$(y-m_1x)(m_2x)=0$$

Or $y^2-m_2xy-m_1xy+m_1m_2x^2=0$

⇨ $y^2-(m_1+m_2)xy+m_1m_2x^2=0$

⇨ This is special types of second degree homogenous equation.

Homogenous Equation:

Let $f(x,y)=0\rightarrow(i)$*be any equation in variables* x *and* y *is called homogenous equation of degree* n $(a+ve$ *integer*$)$*if*

$$f(kx,ky)=k^nf(x,y)\text{for } k\in R$$

For example

$$y^2-(m_1m_2)xy+m_1m_2y^2=0$$

Replacing x,y *by* kx *and* ky

⇨ $ky)^2-(m_1+m_2)(kx)(ky)+m_1m_2(ky)^2=0$

⇨ $k^2(y^2-(m_1+m_2)xy+m_1m_2y^2)=0$

⇨ $k^2f(x,y)=0$

thus it is Homogenous equation of degree 2

A general second homogenous equation can be written as $ax^2+2hxy+by^2=0$

Where a,h,b *are simultaneously not zero.*

Theorem:

Every homogenous equation of second degree $\boldsymbol{ax^2+2hxy+by^2=}$ $\boldsymbol{0}$ ***represents a pair of lines***

Through the origin the lines are

i. **Real and distinct if $\boldsymbol{h^2>ab}$**
ii. **Real and distinct ,if $\boldsymbol{h^2=ab}$**
iii. **Imaginary, if $\boldsymbol{h^2<ab}$**

Proof:

$$\because ax^2+2hhxy+ax^2=0$$

$$(equadratic\ eq.in\ y)$$

Using quadratic formula

$$y=\frac{-2hx\pm\sqrt{(2hx)^2-4(b)(ax^2}}{2b}$$

$$y=\frac{-2hx\pm\sqrt{4h^2x^2-4bax^2}}{2b}$$

$$y=\frac{-2hx\pm\sqrt{4x^2(h^2-ab}}{2b}$$

$$y=\frac{-2hx\pm 2x\sqrt{(h^2-ab}}{2b}$$

$$y=\frac{2(-hx\pm x)\sqrt{(h^2-ab)}}{2b}$$

$$y=\left(\frac{-h\pm\sqrt{(h^2-ab}}{b}\right)x$$

Clearly this represents a pair of lines through origin the lines are

i. Real and distinct if $h^2>ab$
ii. Real and coincident if $h^2=ab$
iii. Imaginary *if* $h^2<ab$

To find measure of the angle between the lines represented by $\boldsymbol{ax^2+2hxy+by^2=0}$

We know that every homogenous equation a pair of lines through origin is

$$y=\frac{-h\pm\sqrt{(h^2-ab}}{b}\Big)x$$

$y = \frac{-h+\sqrt{(h^2-ab}}{b}\Big) x$ and $y = \frac{-h-\sqrt{(h^2-ab}}{b}\Big) x$

Slope of $l_1 = m_1 = y = \frac{-h+\sqrt{(h^2-ab}}{b} x$

Slope of $l_2 = m_2 = y = \frac{-h-\sqrt{(h^2-ab}}{b}$

$$\because m_1 + m_2 = \left(\frac{-h+\sqrt{(h^2-ab}}{b}\right) + \left(\frac{-h-\sqrt{(h^2-ab}}{b}\right)$$

$$= \frac{-h+\sqrt{(h^2-ab)} - h - \sqrt{(h^2-ab}}{b}$$

$$m_1 + m_2 = -\frac{2h}{b}$$

And $m_1 m_2 = \frac{-h+\sqrt{(h^2-ab}}{b} \quad \frac{-h-\sqrt{(h^2-ab}}{b}$

$$m_1 m_2 = \frac{-h)^2 - \left(\sqrt{(h^2-ab)}\right)^2}{b^2} = \frac{h^2-(h^2-ab)}{b^2}$$

$$m_1 m_2 = \frac{h^2-h^2+ab}{b^2}$$

⇨ $m_1 m_2 = \frac{a}{b}$

If θ *is measured from* l_1 *to* l_2 *so*

$$Tan\theta = \frac{m_2 - m_1}{1+m_2 m_1}$$

$$Tan\theta = \frac{\sqrt{m_1+m_2)^2 - 4m_1m_2}}{4m_2m_1}$$

$$(\because (a+b)^2 - (a-b)^2) = 4ab$$

⇨ $\sqrt{(a+b)^2 - 4ab} = a - b$

⇨ $Tan\theta = \frac{\sqrt{\left(-\frac{2h}{b}\right)^2 - \frac{4a}{b}}}{1+\frac{a}{b}}$

⇨ $= \frac{\sqrt{\left(-\frac{2h}{b}\right)^2 - \frac{4a}{b}}}{\frac{b+a}{b}}$

⇨ $\frac{\sqrt{\frac{4h^2-4a}{b^2}}}{\frac{b+a}{b}} = \frac{\sqrt{\frac{4(h^2-ab)}{b}}}{\frac{a+b}{b}}$

$Tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$

Note:

The two lines are parallel $if\ \theta = 0\ so\ Tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$

⇨ $if\ \theta = 0\ so\ Tan(0) = \frac{2\sqrt{h^2-ab}}{a+b}$

⇨ $0 = \frac{2\sqrt{h^2-ab}}{a+b}$

⇨ $2\sqrt{h^2-ab} = 0$

⇨ $h^2 - ab = 0$

⇨ $h^2 = ab$

Thus lines will be parallel if $h^2 = ab$

Two lines are perpendicular $if\ \theta = 90^0\ so$

$$Tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$$

$Tan90^0 = \frac{2\sqrt{h^2-ab}}{a+b}$

⇨ $\frac{1}{0} = \frac{2\sqrt{h^2-ab}}{a+b}$

⇨ $a + b = (0)(2\sqrt{h^2-ab})$

⇨ $a + b = 0$ *thus lines will be perpendicual*

If $a + b = 0$

Exercise 4.5

Find the lines represented by each of the following and also find measure of the angle between them **(Problems 1-6):**

Q#1)

$10x^2 - 23xy - 5y^2 = 0$

$10x^2 - 25xy + 2xy - 5y^2 = 0$

$5x(2x-5y) + y(2x-5y) = 0$

$(2x-5y)(5x+y) = 0$

Hence $(2x-5y) = 0$ *and* $(5x+y) = 0$ are the required lines.

For angle

$10x^2 - 23xy - 5y^2 = 0$

Comparing it with $ax^2 + 2hxy + by^2 = 0$, we have

$a = 10, b = -5, 2h = -23 \Rightarrow h = -\frac{23}{2}$

As $tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$

$= \frac{2\sqrt{\left(-\frac{23}{2}\right)^2 - 10)(-5}}{(10)+(-5)} = \frac{2\sqrt{\frac{529}{4}+50}}{5}$

$= \frac{2\sqrt{\frac{529+200}{4}}}{5} = \frac{2\sqrt{\frac{729}{4}}}{5}$

$= \frac{2(\frac{27}{2}}{5} = \frac{27}{5}$

$tan\theta = \frac{27}{5}$

$\theta = tan^{-1}\left(\frac{27}{5}\right) = 79.51^o$

Q#2)

$3x^2 + 7xy + 2y^2 = 0$

$3x^2 + 6xy + xy + 2y^2 = 0$

$3x(x+2y) + y(x+2y) = 0$

$(x+2y)(3x+y) = 0$

Hence$(x+2y) = 0$ *and* $(5x+y) = 0$ are the required lines.

For angle

$3x^2 + 7xy + 2y^2 = 0$

Comparing it with $ax^2 + 2hxy + by^2 = 0$, we have

$a = 3, b = 2, 2h = 7 \Rightarrow h = \frac{7}{2}$

As $tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$

$= \frac{2\sqrt{\left(\frac{7}{2}\right)^2 - 3)(2}}{(3)+(2)} = \frac{2\sqrt{\frac{49}{4}-6}}{5}$

$= \frac{2\sqrt{\frac{49-24}{4}}}{5} = \frac{2\sqrt{\frac{25}{4}}}{5}$

$= \frac{2(\frac{5}{2}}{5} = \frac{5}{5} = 1$

$tan\theta = 1$

$\theta = tan^{-1}(1) = 45^o$

Q#3)

$9x^2 + 24xy + 16y^2 = 0$
$9x^2 + 12xy + 12xy + 16y^2 = 0$
$3x(3x + 4y) + 4y(3x + 4y) = 0$
$(3x + 4y)(3x + 4y) = 0$
Hence $(3x + 4y) = 0$ *and* $(3x + 4y) = 0$ are the required lines.

For angle
$9x^2 + 24xy + 16y^2 = 0$
Comparing it with $ax^2 + 2hxy + by^2 = 0$, we have
$a = 9, b = 16, 2h = 24 \Rightarrow h = \frac{24}{2} = 12$

As $tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$
$= \frac{2\sqrt{(12)^2-(9)(16)}}{(9)+(16)} = \frac{2\sqrt{144-144}}{25}$
$= \frac{2\sqrt{0}}{25} = 0$
$tan\theta = 0$
$\theta = tan^{-1}(0) = 0^o$
Both lines are parallel.

Q#4)
$2x^2 + 3xy - 5y^2 = 0$
$2x^2 - 2xy + 5xy - 5y^2 = 0$
$2x(x - y) + 5y(x - y) = 0$
$(2x + 5y)(x - y) = 0$
Hence$(2x + 5y) = 0$ *and* $(x - y) = 0$ are the required lines.

For angle
$2x^2 + 3xy - 5y^2 = 0$
Comparing it with $ax^2 + 2hxy + by^2 = 0$, we have
$a = 2, b = -5, 2h = 3 \quad \Rightarrow h = \frac{3}{2}$

As $tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$
$= \frac{2\sqrt{\left(\frac{3}{2}\right)^2 - 2)(-5}}{(2)+(-5)} = \frac{2\sqrt{\frac{9}{4}+10}}{-3}$
$= \frac{2\sqrt{\frac{9+40}{4}}}{-3} = \frac{2\sqrt{\frac{49}{4}}}{-3}$
$= \frac{2(\frac{7}{2}}{-3} = \frac{7}{-3}$
$tan\theta = \frac{7}{-3}$
$\theta = tan^{-1}\left(\frac{7}{-3}\right) = 180^o - tan^{-1}\left(\frac{7}{3}\right) =$
$180^o - 66.8^o$
$\theta = 113.2^o$

Q#5)
$6x^2 - 19xy + 15y^2 = 0$
$6x^2 - 10xy - 9xy + 15y^2 = 0$
$2x(3x - 5y) - 3y(3x - 5y) = 0$
$(3x - 5y)(2x - 3y) = 0$
Hence $(3x - 5y) = 0$ *and* $(2x - 3y) = 0$ are the required lines.

For angle
$6x^2 - 19xy + 15y^2 = 0$
Comparing it with $ax^2 + 2hxy + by^2 = 0$, we have
$a = 6,\ b = 15, 2h = -19 \quad \Rightarrow h = \frac{-19}{2}$

As $tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$
$= \frac{2\sqrt{\left(\frac{-19}{2}\right)^2 - 6)(15}}{(6)+(15)} = \frac{2\sqrt{\frac{361}{4}-90}}{21}$
$= \frac{2\sqrt{\frac{361-360}{4}}}{21} = \frac{2\sqrt{\frac{1}{4}}}{21}$
$= \frac{2(\frac{1}{2}}{21} = \frac{1}{21}$
$tan\theta = \frac{1}{21}$
$\theta = tan^{-1}\left(\frac{1}{21}\right)$
$\theta = 2.73^o$

Q#6)
$x^2 + 2xy\ sec\alpha + y^2 = 0$
$y^2 + (2x\ sec\alpha)y + x^2 = 0$
This is quadratic equation is y
$a = 1, b = 2x\ sec\alpha, c = x^2$
$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$= \frac{-\ 2x\ sec\alpha) \pm \sqrt{(2x\ sec\alpha)^2 - 4(1)(x^2}}{2(1)}$
$= \frac{-(2x\ sec\alpha) \pm \sqrt{4x^2(\ sec^2\alpha - 1\ )}}{2(1)}$
$= \frac{-\ 2x\ sec\alpha) \pm 2x\sqrt{(\ tan^2\alpha}}{2(1)}$
$= (-sec\alpha \pm \tan\alpha)x$
$= \frac{-1}{cos\alpha} \pm \frac{\sin\alpha}{cos\alpha}\Big) x$
$y = \frac{-1 \pm \sin\alpha}{cos\alpha}\Big) x$
$cos\alpha y = (-1 \pm \sin\alpha)x$
$cos\alpha y = (-1 + \sin\alpha)x$
$cos\alpha y = (-x - \sin\alpha)x$
$(1 - \sin\alpha)x + cos\alpha y = 0$
$(1 + \sin\alpha)x + cos\alpha y = 0$
Hence$(1 - \sin\alpha)x + cos\alpha y = 0$
and $(1 + \sin\alpha)x + cos\alpha y = 0$ are the required lines.

For angle
$x^2 + 2xy\ sec\alpha + y^2 = 0$
Comparing it with $ax^2 + 2hxy + by^2 = 0$, we have
$a = 1,\ b = 1, 2h = 2\ sec\alpha \quad \Rightarrow h = sec\alpha$

As $tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$
$= \frac{2\sqrt{(sec\alpha)^2-(1\ (1)}}{(1)+(1)} = \frac{2\sqrt{sec^2\alpha-1}}{2}$
$= \frac{2\sqrt{tan^2\alpha}}{2} = \frac{2\tan\alpha}{2}$
$= tan\alpha$
$tan\theta = tan\alpha$
$\theta = \alpha$

Q#7) Find a joint equation of the lines through the origin and perpendicular to the lines:
$\boldsymbol{x^2 - 2xytan\alpha - y^2 = 0}$
Sol:
$x^2 - 2xytan\alpha - y^2 = 0$
Comparing it with $ax^2 + 2hxy + by^2 = 0$, we have

$a = 1,\ b = -1, 2h = -2\ tan\alpha \quad \Rightarrow h = -tan\alpha$

Suppose m_1 and m_2 are slopes of given lines, then

$$m_1 + m_2 = -\frac{2h}{b}$$
$$= -\frac{2(-tan\alpha}{-1}$$

$m_1 + m_2 = -2tan\alpha$

Also, $m_1.m_2 = \frac{a}{b} = \frac{1}{-1} = -1$

Now, Slopes perpendicular to the given slopes are given by $\frac{-1}{m_1}$ and $\frac{-1}{m_2}$, their corresponding equations are as

$y = \frac{-1}{m_1}x$ and $y = \frac{-1}{m_2}x$

$\Rightarrow \quad m_1y = -x$ and $m_2y = -x$

$\Rightarrow\ m_1y + x = 0$ and $m_2y + x = 0$

Joint equation form

$m_1y + x)(m_2y + x) = 0$

$m_1m_2y^2 + m_1xy + m_2xy + x^2 = 0$

$m_1m_2)y^2 + (m_1 + m_2)xy + x^2 = 0$

Putting values of $m_1 + m_2$ and $m_1.m_2$ in above

$-1)y^2 + (-2tan\alpha)xy + x^2 = 0$

$x^2 - 2tan\alpha xy\ - y^2 = 0\ \ Req. joint\ equation.$

Q#8) Find a joint equation of the lines through the origin and perpendicular to the lines:
$\boldsymbol{ax^2 + 2hxy + by^2 = 0}$

Sol:

$ax^2 + 2hxy + by^2 = 0$

Comparing it with $ax^2 + 2hxy + by^2 = 0$, we have

$a = a,\ b = b, 2h = 2h \quad \Rightarrow h = h$

Suppose m_1 and m_2 are slopes of given lines, then

$$m_1 + m_2 = -\frac{2h}{b}$$

Also, $m_1.m_2 = \frac{a}{b}$

Now, Slopes perpendicular to the given slopes are given by $\frac{-1}{m_1}$ and $\frac{-1}{m_2}$, their corresponding equations are as

$y = \frac{-1}{m_1}x$ and $y = \frac{-1}{m_2}x$

$\Rightarrow \quad m_1y = -x$ and $m_2y = -x$

$\Rightarrow\ m_1y + x = 0$ and $m_2y + x = 0$

Joint equation form

$m_1y + x)(m_2y + x) = 0$

$m_1m_2y^2 + m_1xy + m_2xy + x^2 = 0$

$m_1m_2)y^2 + (m_1 + m_2)xy + x^2 = 0$

Putting values of $m_1 + m_2$ and $m_1.m_2$ in above

$$\frac{a}{b})y^2 + (-\frac{2h}{b})xy + x^2 = 0$$

Multiplying by b, we get

$bx^2 - 2hxy + ay^2 = 0\ \ req. joint\ equation.$

Q#9) Find the area of the region bounded by:
$\boldsymbol{10x^2 - xy - 21y^2 = 0}$ **and** $\boldsymbol{x + y + 1 = 0}$

Sol:

$10x^2 - xy - 21y^2 = 0$

$10x^2 - 15xy + 14xy - 21y^2 = 0$

$5x(2x - 3y) + 7y(2x - 3y) = 0$

$(2x - 3y)(5x + 7y) = 0$

Hence, $x + y + 1 = 0$.....(1) $(2x - 3y) = 0$ ....(2) and $(5x + 7y) = 0$.........(3) are the lines, that bounded the area. We solve them and find the point if intersection.

From Eq. (1) and (2)

$x + y + 1 = 0\ \Rightarrow x = -y - 1$ put in Eq. (2)

$2x - 3y = 0\ \Rightarrow\ 2(-y - 1) - 3y = 0\ \Rightarrow -2y - 2 - 3y = 0$

$\Rightarrow\ -5y - 2 = 0 \Rightarrow y = -\frac{2}{5}$ put in Eq. (1)

$x = -y - 1 \Rightarrow x = -\left(-\frac{2}{5}\right) - 1\ \Rightarrow x = \frac{2-5}{5}\ \Rightarrow x = \frac{-3}{5}$

Hence point of intersection of Eq. (1) and (2) is $A(-\frac{3}{5}, -\frac{2}{5})$.

From Eq. (1) and (3)

$x + y + 1 = 0\ \Rightarrow x = -y - 1$ put in Eq. (3)

$5x + 7y = 0\ \Rightarrow\ 5(-y - 1) + 7y = 0\ \Rightarrow -5y - 5 + 7y = 0$

$\Rightarrow\ 2y - 5 = 0 \Rightarrow y = \frac{5}{2}$ put in Eq. (1)

$x = -y - 1 \Rightarrow x = -\left(\frac{5}{2}\right) - 1\ \Rightarrow x = \frac{-5-2}{2}\ \Rightarrow x = -\frac{7}{2}$

Hence point of intersection of Eq. (1) and (3) is $B(-\frac{7}{2}, \frac{5}{2})$.

From Eq. (2) and (3)

$2x - 3y = 0\ \Rightarrow x = -\frac{3y}{2}$ put in Eq. (3)

$5x + 7y = 0\ \Rightarrow\ 5\left(-\frac{3y}{2}\right) + 7y = 0\ \Rightarrow -\frac{15y}{2} + 7y = 0$

$\Rightarrow\ -15y + 14y = 0 \Rightarrow y = 0$ put in Eq. (2)

$x = -\frac{3y}{2} \Rightarrow x = -\frac{3(0)}{2} = 0$

Hence point of intersection of Eq. (2) and (3) is $C(0,0)$.

Now Area of triangular region $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} =$

$$\frac{1}{2}\begin{vmatrix} \frac{-3}{5} & \frac{-2}{5} & 1 \\ \frac{-7}{2} & \frac{5}{2} & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding by R_3

$$= \frac{1}{2}[-0 + 0 - 1\left(-\frac{3}{5} \times \frac{5}{2}\ -\ -\frac{2}{5} \times -\frac{7}{2}\right)]$$
$$= \frac{-1}{2}[\left(-\frac{15}{10}\ -\ \frac{14}{10}\right)]$$
$$= \frac{-1}{2}\left(\frac{-15-14}{10}\right) = \frac{29}{20} \text{ Square Units}$$