Integration:

The technique or method to find such a function whose derivative is given involves the inverse process of differentiation, called anti derivative or integration.

Differential of variable:

Let f be a differentiable function defined as

$$y = f(x) \Rightarrow y + \delta y = f(x + \delta x)$$
$$\Rightarrow \delta x = f(x + \delta x) - y \Rightarrow \delta y = f(x + \delta x) - f(x)$$
$$Now \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$
$$\Rightarrow \frac{dy}{dx} = f'(x)$$

$\because$ *before the limit is reached.* $\frac{\delta y}{\delta x}$ *differs from* $f'(x)$ *by small real number* $\in$. *i.e* $\frac{\delta y}{\delta x} = f'(x) + \in$

$\Rightarrow f'(x)$ *is called differntial of dependent varibale y we denoted differential of y as dy.*

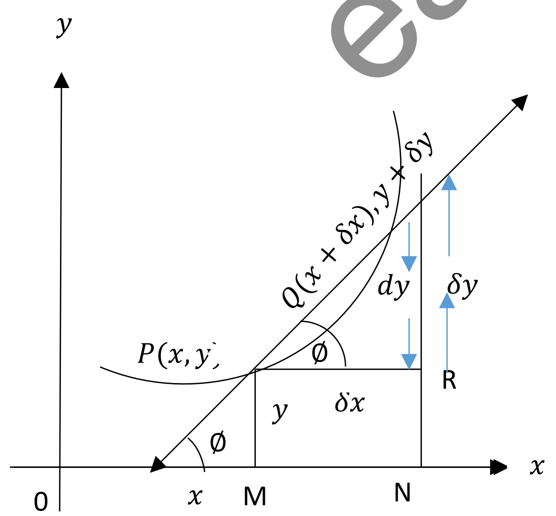
so $dy = f'(x)\delta x \Rightarrow dx = \delta y = f'(x)dx$

Note: 1.The differential of x is denoted by dx and defined as $dx = \delta x$

$$\boldsymbol{i.e\ for\ y = x \Rightarrow dy = \frac{d}{dx}(x)\delta x}$$
$$\boldsymbol{\Rightarrow dy = 1.\delta x \Rightarrow dx = \delta x \quad \because y = x}$$

2. $f'(x)$ is used differential coefficeient.

Distinguishing between dy and δ



Let us draw the graph of curve of a function $y = f(x)$ *Let* $P(x, y)$ *and* $Q(x + \delta x, y + \delta y)$ *be two neighbouring points on the curve at point* $P(x, y)$ *s.that it makes an angle* $\emptyset$ *with* x *− axis. Also*

Draw $\perp$ *PM and QN on x − axix also draw* $\perp$ PR *on QN on x − axis. in fig.* $|PR| = dx$

$$|QR| = |QT| + |TR|$$
$$\Rightarrow \delta y = |QT| + |TR| \to (i)$$
$$In\ \triangle TPR, \quad tan\emptyset dx = \frac{|TR|}{|PR|} = \frac{|TR|}{dx}$$
$$\Rightarrow |TR| = tan\emptyset dx$$

So $(i) \Rightarrow \delta y = tan\emptyset dx + |QT|$

$$\Rightarrow \delta y = \left(\frac{dy}{dx}\right) dx + |QT| \quad \because \frac{dy}{dx} = tan\emptyset$$
$$\delta y = dy + |QT| \quad \because |QT|\ is\ very\ small$$

So *by neglecting* $|QT|$

$$\Rightarrow \delta y \approx dy$$

Example:

Find δy and dy of the function defined as

$$\boldsymbol{f(x) = x^2\ when\ x = 2\ and\ dx = 0.01}$$

Solution:

Let $y = f(x) \quad dy = ?$

⇨ $y = x^2$

⇨ $\frac{dy}{dx} = 2x \Rightarrow dy = 2dx$

Take $x = 2$ *and* $dx = 0.01$

$$dy = 2(2)(0.01) = 0.04$$

Now we find $\delta y, y + \delta y = (x + \delta x)^2$

⇨ $\delta y = (x + \delta x)^2 - y \quad , y = (x)^2 = (2)^2 = 4$

$= (2 + 0.01)^2 - 4 \quad \because dx = \delta x = 0.01$

$\delta y = 4.041 - 4 = 0.0401$

Example:

Use differentials find $\frac{dy}{dx}$ when $\frac{y}{x} - lnx = lnc$

Solution:

$$\boldsymbol{\frac{y}{x} - lnx = lnc}$$

⇨ $\boldsymbol{d\left(\frac{y}{x} - lnx\right) = d(lnc)}$

⇨ $\boldsymbol{d\left(\frac{y}{x}\right) - d(lnx) = 0}$

⇨ $\boldsymbol{\frac{xdy - ydx}{x^2} - \frac{1}{x}dx = 0}$

⇨ $\boldsymbol{\frac{xdy - ydx}{x^2} = \frac{1}{x}dx}$

⇨ $\boldsymbol{xdy - ydx = xdx}$

⇨ $\boldsymbol{xdy = xdx + ydx}$

⇨ $\boldsymbol{dy = \frac{x+y}{x}dx}$

⇨ $\boldsymbol{\frac{dy}{dx} = \frac{x+y}{x}}$

Exercise 3.1

Q.1: Find δy and dy in the following cases:

i) $\boldsymbol{y = x^2 - 1}$

when x changes from 3 to 3.02

SOLUTION:

$y = x^2 - 1$ *As x changes from 3 to 3.02, so*

$y = x^2 - 1$

$d(y) = d(x^2 - 1)$

$dy = 2x\,dx - 0 = 2x\,dx$

Put value of x and dx

$dy = 2(3)(0.02)$

$dy = 0.12$

Now

$y + \delta y = (x + \delta x)^2 - 1$

$\delta y = (x + \delta x)^2 - 1 - y$

Put value of y

$\delta y = (x + \delta x)^2 - 1 - (x^2 - 1)$

$\delta y = (x + \delta x)^2 - 1 - x^2 + 1$

$\delta y = (x + \delta x)^2 - x^2$

$x = 3$

$\delta x = dx = 3.02 - 3 = 0.02$

Put value of x and δx

$\delta y = (3 + 0.02)^2 - (3)^2$

$\delta y = 0.1204$

ii) $\boldsymbol{y = x^2 + 2x}$

when x changes from 2 to 1.8

SOLUTION:

$y = x^2 + 2x$

Now

$y = x^2 + 2x$

$d(y) = d(x^2 + 2x)$

$dy = 2x\,dx + 2dx$

Put value of x and dx

$dy = 2(2)(-0.2) + 2(-0.2)$

$dy = -1.2$

Now

$y + \delta y = (x + \delta x)^2 + 2(x + \delta x)$

$\delta y = (x + \delta x)^2 + 2x + 2\delta x - y$

Put value of y

$\delta y = (x + \delta x)^2 + 2x + 2\delta x - (x^2 + 2x)$

$\delta y = (x + \delta x)^2 + 2x + 2\delta x - x^2 - 2x$

$\delta y = (x + \delta x)^2 + 2\delta x - x^2$

$x = 2$,

$\delta x = dx = 1.8 - 2 = -0.2$

Put value of x and δx

$\delta y = (2 - 0.2)^2 + 2(-0.2) - (2)^2$

$\delta y = -1.16$

iii) $\boldsymbol{y = \sqrt{x}}$

when x changes from 4 to 4.01

SOLUTION:

$y = \sqrt{x}$

Now

$y = \sqrt{x}$

$d(y) = d(\sqrt{x})$

$dy = \frac{1}{2\sqrt{x}}\,dx$

Put value of x and dx

$dy = \frac{1}{2\sqrt{4}}\,(0.41)$

$dy = 0.1025$

Now.

$y + \delta y = \sqrt{x + \delta x}$

$\delta y = \sqrt{x + \delta x} - y$

Put value of y

$\delta y = \sqrt{x + \delta x} - \sqrt{x}$

$x = 4$,

$\delta x = dx = 4.41 - 4 = 0.41$

Put value of x and δx

$\delta y = \sqrt{4 + 0.41} - \sqrt{4}$

$\delta y = 0.1$

Q.2: Using differentials find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in the following equations.

i) $\boldsymbol{xy + x = 4}$

Taking differentials on both sides

$d(xy + x) = d(4)$

$d(xy) + d(x) = 0$

$x\,dy + y\,dx + dx = 0$

$x\,dy + (y + 1)dx = 0$

$x\,dy = -(y + 1)dx$

$\frac{dy}{dx} = -\frac{y+1}{x}$ *and*

$\frac{dx}{dy} = -\frac{x}{y+1}$

ii) $\boldsymbol{x^2 + 2y^2 = 16}$

Taking differentials on both sides

$d(x^2 + 2y^2) = d(16)$

$d(x^2) + 2d(y^2) = 0$

$2x\,dx + 2.2y^{2-1}.dy = 0$

$2x\,dx + 4y\,dy = 0$

$4y\,dy = -2x\,dx$

$\frac{dy}{dx} = -\frac{2x}{4y} = -\frac{x}{2y}$ *and*

$\frac{dx}{dy} = -\frac{2y}{x}$

iii) $\boldsymbol{x^4 + y^2 = xy^2}$

Taking differentials on both sides

$d(x^4 + y^2) = d(xy^2)$

$d(x^4) + d(y^2) = (x)'(y^2) + (y^2)'x$

$4x^3\,dx + 2y\,dy = dx.y^2 + (2y\,dy)x$

$4x^3\,dx + 2y\,dy = y^2\,dx + 2xy\,dy$

$2y\,dy - 2xy\,dy = y^2\,dx - 4x^3\,dx$

$(2y - 2xy)\,dy = (y^2 - 4x^3)dx$

$\frac{Dy}{dx} = \frac{y^2 - 4x^3}{2y - 2xy}$ *taking reciprocal*

$\frac{dx}{dy} = \frac{2y - 2xy}{y^2 - 4x^3}$

iv) $\boldsymbol{xy - \ln x = c}$

Taking differentials on both sides

$d(xy - \ln x) = d(c)$

$d(xy) - d(\ln x) = 0$

$x\,dy + y\,dx - \frac{1}{x}dx = 0$

$x\,dy = -y\,dx + \frac{1}{x}dx$

$x\,dy = -\left(y - \frac{1}{x}\right)dx$

$x\,dy = -\left(\frac{xy-1}{x}\right)dx$

$\frac{dy}{dx} = \frac{1-xy}{x^2}$ and

$\frac{dx}{dy} = \frac{x^2}{1-xy}$

Q.3: Use differentials to approximate the values of:

i) $\sqrt[4]{17}$

SOLUTION:

Let $y = \sqrt[4]{x} = x^{\frac{1}{4}}$

We take $x = 16$,

$\delta x = dx = 17 - 16 = 1$

$y = (16)^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = 2$

Now $y = x^{\frac{1}{4}}$

$d(y) = d\left(x^{\frac{1}{4}}\right)$

$dy = \frac{1}{4}x^{\frac{1}{4}-1}\,dx$

$dy = \frac{1}{4}x^{-\frac{3}{4}}\,dx$

Put $x = 16$, $dx = 1$

$dy = \frac{1}{4}(16)^{-\frac{3}{4}}\,(1) = \frac{1}{4}(2^4)^{-\frac{3}{4}}$

$dy = \frac{1}{4}(2)^{-3} = \frac{1}{4}.\frac{1}{8} = \frac{1}{32}$

$dy = 0.03125$

Thus $\sqrt[4]{17} \approx y + dy$

$= 2 + 0.03125$

$= 2.03125$

ii) $(31)^{\frac{1}{5}}$

SOLUTION:

Let $y = x^{\frac{1}{5}}$

We take $x = 32$,

$\delta x = dx = 31 - 32 = -1$

$y = (32)^{\frac{1}{5}} = (2^5)^{\frac{1}{5}} = 2$

Now $y = x^{\frac{1}{5}}$

$d(y) = d\left(x^{\frac{1}{5}}\right)$

$dy = \frac{1}{5}x^{\frac{1}{5}-1}\,dx$

$dy = \frac{1}{5}x^{-\frac{4}{5}}\,dx$

Put $x = 32$, $dx = -1$

$dy = \frac{1}{5}(32)^{-\frac{4}{5}}\,(-1) = -\frac{1}{5}(2^5)^{-\frac{4}{5}}$

$dy = \frac{1}{5}(2)^{-4} = \frac{1}{5}.\frac{1}{16} = \frac{1}{80}$

$dy = -0.0125$

Thus $(31)^{\frac{1}{5}} \approx y + dy$

$= 2 - 0.0125$

$= 1.9875$

iii) $\cos 29°$

SOLUTION:

Let $y = \cos x$

We take $x = 30°$,

$\delta x = dx = 29° - 30° = -1° = -0.01745$

$y = \cos 30° = 0.866$

Now $y = \cos x$

$d(y) = d(\cos x)$

$dy = -\sin x\,\,dx$

$dy = -\sin 30°\,(-0.01745)$

$dy = -(0.5)\,\,(-0.01745)$

$dy = 0.0087$

Thus $\cos 29° \approx y + dy$

$= 0.866 + 0.0087$

$= 0.8747$

iv) $\sin 61°$

SOLUTION:

Let $y = \sin x$

We take $x = 60°$,

$\delta x = dx = 61° - 60° = 1° = 0.01745$

$y = \sin 60° = 0.866$

Now $y = \sin x$

$d(y) = d(\sin x)$

$dy = \cos x\,\,dx$

$dy = \cos 60°\,(0.01745)$

$dy = (0.5)\,(0.01745)$

$dy = 0.0087$

Thus $\sin 61° \approx y + dy$

$= 0.866 + 0.0087$

$= 0.8747$

Q.4: Find the approximate increase in the volume of a cube if the lenght of each edge changes from 5 to 5.02.

SOLUTION:

Lenght of each edge of cube $= x$ unit

Volume of a cube $= L.W.H$

$V = x.x.x$

$V = x^3$

$d(V) = (x^3)$

$dV = 3x^2\,dx$

when x changes from 5 to 5.02, so

$x = 5$, $dx = 5.02 - 5 = 0.02$

$dV = 3(5)^2\,(0.02) = 1.5$ cubic units

Q.5: Find the approximate increase in the area of a circular disc if its diameter is increased from 44 cm to 44.4 cm.

SOLUTION:

Let radius of circular disc $= x$ cm

Area of a disc $= \pi r^2$

$A = \pi x^2$

$d(A) = d(\pi x^2)$

$dA = \pi.2x\,dx$

As diameter changes from 44 to 44.4,

so radius changes from 22 to 22.4, so

$x = 22$, $dx = 22.2 - 22 = 0.2$

$dA = \pi(2)(22)(0.2)$

$dA = 27.646\ \ cm^2$

Integration as anti-derivative (inverse of derivative)

$\boldsymbol{Integration}$: $\boldsymbol{v.v.v.important\ defination}(***)$

The inverse process of differentiation is called anti − differentiation or integration.

Consider $F(x)$ is antidervative of a function if

$$F'(x) = f(x)\ \ then \int f(x)dx = \int F'^{(x)}dx$$
$$= \int \frac{d}{dx}F(x)dx$$
$$\int f(x)dx = F(x) + c$$

$\because \frac{d}{dx}$ and $\int dx$ are inverse operations of each other.

*The symbol

$\int \ldots dx$ indicates that integrand is two integrated w.r.t "x"

*The anti- derivative of a function is also called integrated is called integrand of the integral.

*The function which is to be integrated is called integrand of the integral.

Some standard formulae for Anti-derivatives

$$\int 1dx = x + c\ , \int x^n dx = \frac{x^{n+1}}{n+1} + c(n \neq -1)$$

$$\int sindx = -cosx + c\ , \int cosxdx = sinx + c$$

$$\int \sec^2 x\,dx = tanx + c, \quad \int cosec^2 dx = -cotx + c$$

$$\int secxtanxdx = secx + c, \int cosecxcotxdx$$
$$= -cosecx$$

$$\int e^x dx = e^x + c \quad , \int a^x dx = \frac{1}{lna}.a^x + c$$

$$\int \frac{1}{x}dx = ln|x| + c, x \neq 0\ \ , \int tanxdx$$
$$= lnsec|x| + c = -ln|cosx| + c$$

$$\int cotxdx = ln|sinx| + c$$

$$\int secxdx = \ln|secx + tanx| + c$$

$$\int cosecxdx = \ln|cosecx - cots| + c$$

Here c is constant of integration. These formulae can be verified by showing that the derivatives of right hand side of each w.r.t "x" is equal to the corresponding integral

Examples:

1. $\int x^5 dx \qquad \because \int x^n dx = \frac{x^{(n+1)}}{\boldsymbol{n+1}} + c$

$$= \frac{\boldsymbol{x^{5+1}}}{\boldsymbol{5+1}} + \boldsymbol{c} = \frac{\boldsymbol{x^6}}{\boldsymbol{6}} + \boldsymbol{c}$$

2. $\int \frac{1}{\sqrt{x^3}}dx$

$$= \int \frac{1}{x^{\left(\frac{2}{3}\right)}}dx = \int x^{-\frac{3}{2}}dx = \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + c$$
$$= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c = -\frac{2}{\sqrt{x}} + c$$

3. $\int \frac{1}{(2x+3)^4}dx =$

$$\int (2x+3)^{-4}dx = \frac{1}{2}.\frac{(2x+3)^{-4+1}}{-4+1} + c$$
$$= -\frac{1}{6(2x+3)^3} + c$$

4. $\int cos2xdx \qquad \because \int cosaxdx = \frac{sinax}{a} + c$

$$= \frac{sin2x}{2} + c$$

5. $\int sin3xdx \qquad \because \int sinaxdx = -\frac{cosax}{a} + c$

$$= -\frac{cos3x}{3} + c$$

6. $\int cosec^2 xdx$

$= -cotx + c$

7. $\int sec5xtan5xdx$

$$= \frac{sec5x}{5} + c \qquad \because \int secaxtanaxdx = \frac{secax}{a} + c$$

$= \frac{se5x}{5} + c$

8. $\int e^{ax+b}dx \qquad \because \int e^{ax}dx = \frac{e^{ax}}{a} + c$

$\frac{e^{ax+b}}{a} + c$

9. $\int 3^{\lambda x}dx = \quad \because \int e^{ax}dx = \frac{e^{ax}}{a} + c$

$= \frac{3^{\lambda x}}{\lambda ln3}$

10. $\int \frac{1}{ax+b}dx$

$$\int (ax+b)^{-1}dx = \frac{1}{a}\ln(ax+b) + c$$

1. $\int af(x)dx = a\int f(x)dx$
2. $\int [f_1(x) \pm f_2(x)]dx = \int f_1(x)dx + \int f_2(x)dx$

Prove that

$$\int [\boldsymbol{f(x)}]^{\boldsymbol{n}}\boldsymbol{f'(x)dx} = \frac{[\boldsymbol{f(x)}]^{\boldsymbol{n+1}}}{\boldsymbol{n+1}} + \boldsymbol{c}, (\boldsymbol{n} \neq -\boldsymbol{1}$$

Proof:

We know that $\frac{d}{dx}\left(f^{n+1}(x)\right)$

$= (n+1)f^n(x).\frac{d}{dx}f(x)$

⇨ $\frac{d}{dx}(f^n(n+1) = (n+1)f^n(x).f'(x)dx$

Taking integration

$$\int \frac{d}{dx} f^{(n+1)}(x)dx = (n+1)\int f^n(x).f'(x)dx$$

⇨ $f^{n+1}(x) = (n+1)\int f^n(x)f'(x)dx$

⇨ $\int f^n(x)\,f'(x)dx = \frac{f^{n+1}(x)}{n+1} + c \quad by\ def.$

Hence proved.

Prove that $\int \frac{f'(x)}{f(x)} dx = lnf(x) + c$

Proof:

We know that

$$\frac{d}{dx}[lnf(x)] = \frac{1}{f(x)}.f'(x)$$

Taking integration both sides

$$\int \frac{d}{dx}[lnf(x)]dx = \int \frac{1}{f(x)}.f'(x)dx$$

⇨ $lnf(x) = \int \frac{f'(x)}{f(x)}\,dx$

⇨ $\int \frac{f'(x)}{f(x)}\,dx = lnf(x) + c$ ***by defination***

$(\int f(x)dx = F(x) + c)$

Hence proved.

Exercise 3.2

Q. 1: Evaluate the following indefinate integrals:

$i)\ \int(3x^2 - 2x + 1)\ dx$

SOLUTION:

$= \int 3x^2\,dx - \int 2x\,dx + \int 1\,dx$

$= 3\int x^2\,dx - 2\int x\,dx + \int 1\,dx$

$= 3.\frac{x^{2+1}}{2+1} - 2.\frac{x^{1+1}}{1+1} + x + c$

$= 3.\frac{x^3}{3} - 2.\frac{x^2}{2} + x + c$

$= x^3 - x^2 + x + c$

$ii)\ \int(\sqrt{x} + \frac{1}{\sqrt{x}})\ dx$

SOLUTION:

$= \int \sqrt{x}\,dx + \int \frac{1}{\sqrt{x}}dx$

$= \int x^{\frac{1}{2}}\,dx + \int x^{-\frac{1}{2}}dx$

$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + x + c$

$iii)\ \int x(\sqrt{x} + 1)\ dx$

SOLUTION:

$= \int x(\sqrt{x} + 1)\ dx$

$= \int x\sqrt{x}\,dx + \int x\,dx$

$= \int x^{1+\frac{1}{2}}\,dx + \int x\,dx$

$= \int x^{\frac{3}{2}}\,dx + \int x\,dx$

$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{x^{1+1}}{1+1} + c$

$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^2}{2} + c$

$= \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{2}x^2 + c$

$iv)\ \int(2x + 3)^{\frac{1}{2}}\ dx$

SOLUTION:

$= \int(2x+3)^{\frac{1}{2}}\ dx$

$\times$ *and* $\div$ *by 2 to make derivative*

$= \frac{1}{2}\int(2x+3)^{\frac{1}{2}}.2\,dx$

$= \frac{1}{2}\frac{(2x+3)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= \frac{1}{2}\frac{(2x+3)^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{1}{2}.\frac{2}{3}\ (2x+3)^{\frac{3}{2}} + c$

$= \frac{1}{3}\ (2x+3)^{\frac{3}{2}} + c$

$v)\ \int(\sqrt{x} + 1)^2\ dx$

SOLUTION:

$= \int(\sqrt{x} + 1)^2\ dx$

$= \int((\sqrt{x})^2 + 2\sqrt{x}.1 + (1)^2)\ dx$

$= \int[x + 2\sqrt{x} + 1]$

$= \int x\,dx + 2\int x^{\frac{1}{2}}\,dx + \int 1\,dx$

$= \frac{x^{1+1}}{1+1} + 2.\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + x + c$

$= \frac{x^2}{2} + 2.\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + x + c$

$= \frac{1}{2}x^2 + 2.\frac{2}{3}x^{\frac{3}{2}} + x + c$

$= \frac{1}{2}x^2 + \frac{4}{3}x^{\frac{3}{2}} + x + c$

$vi)\ \int(\sqrt{x} - \frac{1}{\sqrt{x}})^2\ dx$

SOLUTION:

$= \int(\sqrt{x} - \frac{1}{\sqrt{x}})^2\ dx$

$= \int\left[\left(\sqrt{x}\right)^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2\sqrt{x}.\frac{1}{\sqrt{x}}\right]\ dx$

$= \int\left[x + \frac{1}{x} - 2\right]\ dx$

$= \int x\,dx + \int \frac{1}{x}dx - 2\int 1\,dx$

$= \frac{x^{1+1}}{1+1} + \ln x - 2\,x + c$

$= \frac{1}{2}x^2 + \ln x - 2\,x + c$

NOTE: FOR Q. (vi)

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FOR EXAMPLE:

$\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} = \frac{1}{0} = \infty$

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vii) $\int \frac{3x+2}{\sqrt{x}}\,dx$

SOLUTION:

$\int \frac{3x+2}{\sqrt{x}}\,dx$

$= \int \left[\frac{3x}{\sqrt{x}} + \frac{2}{\sqrt{x}}\right] dx$

$= \int \left[\frac{3\sqrt{x}\sqrt{x}}{\sqrt{x}} + \frac{2}{\sqrt{x}}\right] dx \quad \because x = \sqrt{x}.\sqrt{x}$

$= \int \left[3\sqrt{x} + \frac{2}{\sqrt{x}}\right] dx$

$= \int \left[3x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}\right] dx$

$= 3\int x^{\frac{1}{2}}\,dx + 2\int x^{-\frac{1}{2}}\,dx$

$= 3\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 2\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= 3\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 2\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 3\frac{2}{3}\,x^{\frac{3}{2}} + 2\,.2x^{\frac{1}{2}} + c$

$= 2x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + c$

viii) $\int \frac{\sqrt{y}(y+1)}{y}\,dx$

SOLUTION:

$\int \frac{\sqrt{y}(y+1)}{y}\,dx$

$= \int \frac{\sqrt{y}(y+1)}{\sqrt{y}\,\sqrt{y}}\,dx$

$= \int \frac{y+1}{\sqrt{y}}\,dx$

$= \int \left[\frac{y}{\sqrt{y}}dx + \frac{1}{\sqrt{y}}\,dx\right]$

$= \int \left[\sqrt{y}dx + \frac{1}{\sqrt{y}}\,dx\right]$

$= \int \left[y^{\frac{1}{2}}\,dx + y^{-\frac{1}{2}}\,dx\right]$

$= \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \frac{y^{\frac{3}{2}}}{\frac{3}{2}} + \frac{y^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}y^{\frac{3}{2}} + 2y^{\frac{1}{2}} + x + c$

ix) $\int \frac{(\sqrt{\theta}-1)^2}{\sqrt{\theta}}\,d\theta$

SOLUTION:

$\int \frac{(\sqrt{\theta}-1)^2}{\sqrt{\theta}}\,d\theta$

$= \int \frac{(\sqrt{\theta})^2+(1)^2-2\sqrt{\theta}}{\sqrt{\theta}}\,d\theta$

$= \int \frac{\theta+1-2\sqrt{\theta}}{\sqrt{\theta}}\,d\theta$

$= \int \left[\frac{\theta}{\sqrt{\theta}} + \frac{1}{\sqrt{\theta}} - \frac{2\sqrt{\theta}}{\sqrt{\theta}}\right] d\theta$

$= \int \left[\sqrt{\theta} + \frac{1}{\sqrt{\theta}} - 2\right] d\theta$

$= \int \theta^{\frac{1}{2}}\,d\theta + \int \theta^{-\frac{1}{2}}\,d\theta - 2\int 1\,d\theta$

$= \frac{\theta^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{\theta^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - 2\theta + c$

$= \frac{\theta^{\frac{3}{2}}}{\frac{3}{2}} + \frac{\theta^{\frac{1}{2}}}{\frac{1}{2}} - 2\theta + c$

$= \frac{2}{3}\theta^{\frac{3}{2}} + 2\theta^{\frac{1}{2}} - 2\theta + c$

x) $\int \frac{(1-\sqrt{x})^2}{\sqrt{x}}\,dx$

SOLUTION:

$\int \frac{(1-\sqrt{x})^2}{\sqrt{x}}\,dx$

$= \int \frac{(\sqrt{x})^2+(1)^2-2\sqrt{x}}{\sqrt{x}}\,dx$

$= \int \frac{x+1-2\sqrt{x}}{\sqrt{x}}\,dx$

$= \int \left[\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}} - \frac{2\sqrt{x}}{\sqrt{x}}\right] dx$

$= \int \left[\sqrt{x} + \frac{1}{\sqrt{x}} - 2\right] dx$

$= \int x^{\frac{1}{2}}\,dx + \int x^{-\frac{1}{2}}\,dx - 2\int 1\,dx$

$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - 2x + c$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} - 2x + c$

$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 2x + c$

xi) $\int \frac{e^{2x}+e^x}{e^x}\,dx$

SOLUTION:

$\int \frac{e^{2x}+e^x}{e^x}\,dx$

$= \int \left[\frac{e^{2x}}{e^x} + \frac{e^x}{e^x}\right] dx$

$= \int [e^x + 1]\,dx$

$= \int e^x\,dx + \int 1\,dx$

$= \frac{e^x}{1} + x + c$

$= e^x + x + c$

NOTE: DERIVATION M

EXPONENTIAL FUNCTION KA JAB DERIVATIVE LATY H T FUNCTION AS IT AUR POWER KA DERIVATIVE MULTIPLY KARTY H. LAKIN INTEGRATION M DIVIDE KARE GAI.

Q. 2: *Evaluate*:

i) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

SOLUTION:

$\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

$= \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \cdot \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}\,dx$

$= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{(\sqrt{x+a})^2-(\sqrt{x+b})^2}\,dx = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{x+a-x-b}\,dx$

$= \frac{1}{a-b}\int\left(\sqrt{x+a} - \sqrt{x+b}\right)dx$

$= \frac{1}{a-b}\left\{\int (x+a)^{\frac{1}{2}}\,dx + \int (x+b)^{\frac{1}{2}}\,dx\right\}$

using $\int [f(x)]^n.f'(x) = \frac{[f(x)]^{n+1}}{n+1} + c$

$$= \frac{1}{a-b}\left\{\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{(x+b)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right\} + c$$
$$= \frac{1}{a-b}\left\{\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right\} + c$$
$$= \frac{1}{a-b}\left\{\frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(x+b)^{\frac{3}{2}}\right\} + c$$
$$= \frac{2}{3(a-b)}\left\{(x+a)^{\frac{3}{2}} + (x+b)^{\frac{3}{2}}\right\} + c$$

ii) $\int \frac{1-x^2}{1+x^2}\ dx$

SOLUTION:

$$\int \frac{1-x^2}{1+x^2}\ dx$$
$$= \int \frac{2-1-x^2}{1+x^2}\ dx$$
$$= \int \frac{2-(1+x^2)}{1+x^2}\ dx$$
$$= \int \frac{2}{1+x^2}\ dx - \int \frac{1+x^2}{1+x^2}\ dx$$
$$= 2\int \frac{1}{1+x^2}\ dx - \int 1\ dx$$
$$= 2\tan^{-1}x - x + c$$

iii) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$

SOLUTION:

$$\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$$
$$= \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\ \frac{\sqrt{x+a}-\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}\ dx$$
$$= \int \frac{\sqrt{x+a}-\sqrt{x}}{(\sqrt{x+a})^2-(\sqrt{x})^2}\ dx = \int \frac{\sqrt{x+a}-\sqrt{x}}{x+a-x}\ dx$$
$$= \frac{1}{a}\int\left(\sqrt{x+a} - \sqrt{x+}\right)dx$$
$$= \frac{1}{a}\left\{\int (x+a)^{\frac{1}{2}}\ dx + \int (x)^{\frac{1}{2}}\ dx\right\}$$

using $\int [f(x)]^n . f'(x) = \frac{[f(x)]^{n+1}}{n+1} + c$

$$= \frac{1}{a}\left\{\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{(x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right\} + c$$
$$= \frac{1}{a}\left\{\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right\} + c$$
$$= \frac{1}{a}\left\{\frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(x)^{\frac{3}{2}}\right\} + c$$
$$= \frac{2}{3a}\left\{(x+a)^{\frac{3}{2}} + (x)^{\frac{3}{2}}\right\} + c$$

iv) $\int (a-2x)^{\frac{3}{2}}\ dx$

SOLUTION:

$$\int (a-2x)^{\frac{3}{2}}\ dx$$

$\times$ *and* $\div$ *by* 2

$$= \frac{1}{-2}\int (a-2x)^{\frac{3}{2}}.(-2)\ dx$$
$$= -\frac{1}{2}\frac{(a-2x)^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$$
$$= -\frac{1}{2}\frac{(a-2x)^{\frac{5}{2}}}{\frac{5}{2}} + c$$
$$= -\frac{1}{2}.\frac{2}{5}(a-2x)^{\frac{5}{2}} + c$$
$$= -\frac{1}{5}(a-2x)^{\frac{5}{2}} + c$$

FUNCTION AS IT AUR POWER KE DERIVATIVE S DIVIDE KARNA H.

$\int e^x\ dx = \frac{e^x}{1} + c = e^x + c$

v) $\int \frac{(1+e^x)^3}{e^x}\ dx$

SOLUTION:

$$\int \frac{(1+e^x)^3}{e^x}\ dx$$

$\because (a+b)^3 = a^3 + b^3 + 3ab(a+b)$

$$= \int \frac{1^3+(e^x)^3+3(1)(e^x)(1+e^x)}{e^x}\ dx$$
$$= \int \frac{1+e^{3x}+3e^x(1+e^x)}{e^x}\ dx$$
$$= \int \left[\frac{1}{e^x} + \frac{e^{3x}}{e^x} + \frac{3e^x(1+e^x)}{e^x}\right]\ dx$$
$$= \int [e^{-x} + e^{2x} + 3 + 3e^x]\ dx$$
$$= \int e^{-x}\ dx + \int e^{2x}\ dx + 3\int 1\ dx + 3\int e^x\ dx$$
$$= \frac{e^{-x}}{-1} + \frac{e^{2x}}{2} + 3x + 3\frac{e^x}{1} + c$$
$$= -e^{-x} + \frac{1}{2}e^{2x} + 3x + 3e^x + c$$

vi) $\int \sin(a+b)x\ dx$

SOLUTION:

$$\int \sin(a+b)x\ dx$$
$$= \frac{-\cos(a+b)x}{a+b} + c$$
$$= -\frac{1}{a+b}\cos(a+b)x + c$$

DERIVATION M FUNCTION KA DERIVATIVE LENA HOTA H AUR SATH ANGLE KE DERIVATIVE KO MULTIPLY KARTY H. BUT INTEGRATION M ANGLE KE DERIVATIVE K DIVIDE KARE GAI.

vii) $\int \sqrt{1-\cos 2x}\ dx$

SOLUTION:

$$\int \sqrt{1-\cos 2x}\ dx$$

As $\sin^2 x = \frac{1-\cos 2x}{2}$

So $1 - \cos 2x = 2\sin^2 x$

$$= \int \sqrt{2\sin^2 x}\ dx$$
$$= \int \sqrt{2}\sqrt{\sin^2 x}\ dx$$
$$= \sqrt{2}\int \sin x\ dx$$
$$= \sqrt{2}(-\cos x) + c$$
$$= -\sqrt{2}\cos x + c$$

viii) $\int \ln x\ \frac{1}{x}\ dx$

SOLUTION:

$$\int \ln x\ .\ \frac{1}{x}\ dx$$

As $f(x) = \ln x$

And $f'(x) = \frac{1}{x}$, ***so***

using $\int [f(x)]^n = \frac{[f(x)]^{n+1}}{n+1}$

$$= \frac{(\ln x)^{1+1}}{1+1} + c$$
$$= \frac{(\ln x)^2}{2} + c$$

ix) $\int \sin^2 x\ dx$

SOLUTION:

$$\int \sin^2 x\ dx$$

As $\sin^2 x = \frac{1-\cos 2x}{2}$

$$= \int \frac{1-\cos 2x}{2}\ dx$$

$=\frac{1}{2}\int(1-\cos 2x)dx$

$=\frac{1}{2}\left[x-\frac{\sin 2x}{2}\right]+c$

$=\frac{1}{2}x-\frac{1}{4}\sin 2x+c$

$$x)\ \int\frac{1}{1+\cos x}\,dx$$

SOLUTION:

$\int\frac{1}{1+\cos x}\,dx$

$As\ \cos^2\frac{x}{2}=\frac{1+\cos x}{2}$

$So\ 1+\cos x=2\cos^2\frac{x}{2}$

$=\int\frac{1}{2\cos^2\frac{x}{2}}\,dx$

$=\frac{1}{2}\int\sec^2\frac{x}{2}\,dx$

$=\frac{1}{2}\frac{\tan\frac{x}{2}}{\frac{1}{2}}+c=\tan\frac{x}{2}+c$

$\sin^2 x$, $\cos^2 x$, $\tan^2 x$, $\cot^2 x$ *in functions k derivative exist ni karty jab b ye function a jay t ap ye formula use kare.*

$\sin^2 x=\frac{1-\cos 2x}{2}$

$\cos^2 x=\frac{1+\cos 2x}{2}$

$\tan^2 x=\sec^2 x-1$

$\cot^2 x=\csc^2 x-1$

FUNCTIONS K DERIVATIVES K JO ANSWER H UN KI INTEGRTION HOTI H IS K ILAWA FUNCTIONS KI INTEGRATION NI H HOTI. E.G.

$\sin^2 x$, $\cos^2 x$, $\tan^2 x$, $\cot^2 x$ *IN KI INTEGRATION NI HOTI.*

$(\sin x)'=\cos x$

$(\cos x)'=-\sin x$

$(\tan x)'=\sec^2 x$

$(\cot x)'=-\csc^2 x$

$(\sec x)'=\sec x\tan x$

$(\operatorname{cosec} x)'=-\csc x\cot x$

FUNCTIONS K DERIVATIVES K JO ANSWER H UN KI INTEGRTION HOTI H IS K ILAWA FUNCTIONS KI INTEGRATION NI H HOTI. E.G.

$\sin^2 x$, $\cos^2 x$, $\tan^2 x$, $\cot^2 x$ *IN KI INTEGRATION NI HOTI.*

$xi)\ \int\frac{ax+b}{ax^2+2bx+c}\,dx$

SOLUTION:

$\int\frac{ax+b}{ax^2+2bx+c}\,dx$

$\times$ & $\div$ *by 2 to make derivative uper*

$=\frac{1}{2}\int\frac{2(ax+b)}{ax^2+2bx+c}\,dx$

$=\frac{1}{2}\int\frac{2ax+2b}{ax^2+2bx+c}\,dx$

$Using\ \int\frac{f'(x)}{[f(x)]}=\ln[f(x)]$

$=\frac{1}{2}\ln(ax^2+2bx+c)+c$

$xii)\ \int\cos 3x\sin 2x\,dx$

SOLUTION:

$\int\cos 3x\sin 2x\,dx$

$\times$ & $\div$ *by 2 to make formula*

$=\frac{1}{2}\int 2\cos 3x\sin 2x\,dx$

As $2\cos\alpha\sin\beta=\sin(\alpha+\beta)-\sin(\alpha-\beta)$

$=\frac{1}{2}\int[\sin(3x+2x)-\sin(3x-2x)]\,dx$

$=\frac{1}{2}\int[\sin(5x)-\sin(x)]\,dx$

$=\frac{1}{2}\{\int\sin 5x\,dx-\int\sin x\,dx\}$

$=\frac{1}{2}\left\{\frac{-\cos 5x}{5}-\frac{-\cos x}{1}\right\}+c$

$=-\frac{1}{2}\left\{\frac{\cos 5x}{5}-\cos x\right\}+c$

$xiii)\ \int\frac{\cos 2x-1}{1+\cos 2x}\,dx$

SOLUTION:

$=\int\frac{\cos 2x-1}{1+\cos 2x}\,dx$

$=-\int\frac{1-\cos 2x}{1+\cos 2x}\,dx$

$\because \sin^2 x=\frac{1-\cos 2x}{2}\Rightarrow 2\sin^2 x=1-\cos 2x$

$\because \cos^2 x=\frac{1+\cos 2x}{2}\Rightarrow 2\cos^2 x=1+\cos 2x$

$=-\int\frac{2\sin^2 x}{2\cos^2 x}\,dx=-\int\tan^2 x\,dx$

$=-\int(\sec^2 x-1)\,dx \quad \because 1+\tan^2 x=\sec^2 x$

$=-\int\sec^2 x\,dx+\int 1\,dx$

$=-\tan x+x+c$

$xiv)\ \int\tan^2 x\,dx$

SOLUTION:

$\int\tan^2 x\,dx$

$=\int(\sec^2 x-1)\,dx \quad \because 1+\tan^2\theta=\sec^2\theta$

$=\int\sec^2 x\,dx-\int 1\,dx=\tan x-x+c$

Integration by method of substitution

Sometimes it is possible to convert an integral into standard form by a suitable change of a variable. This is called substitution method.

i.e Evaluate $\int f(x)dx$ *by method of subsitution*

$$Let\ x=\phi(t)\Rightarrow dx=\phi''(t)dt$$

So $\int f(x)dx=\int f(\phi(t))\phi'(t)dt$

Some useful substitutions:

1. $\sqrt{a^2-x^2}$ $\quad$ $put\ x=a\sin\theta$ $\quad$ $(\because 1-\sin^2\theta=\cos^2\theta)$
2. $\sqrt{x^2-a^2}$ $\quad$ $put\ x=a\sec\theta$ $\quad$ $(\because \sec^2\theta-1=\tan^2\theta$
3. $\sqrt{a^2+x^2}$ $\quad$ $put\ x=a\tan\theta$ $\quad$ $(\because \sec^2\theta=1+\tan^2\theta$
4. $\sqrt{x+a}(0r)\sqrt{x-a}$ $\quad$ $put\ \sqrt{x+a}=t$ $\quad$ $or(\sqrt{x-a})=t$
5. $\sqrt{2ax-x^2}$ $\quad$ $put\ x-a=a\sin\theta$
6. $\sqrt{2ax+x^2}$ $\quad$ $put\ x+a=a\sec\theta$

Exercise 3.2

Evaluate the following integrals:

Q.1: $\int \frac{-2x}{\sqrt{4-x^2}}\ dx$

SOLUTION:

$\int \frac{-2x}{\sqrt{4-x^2}}\ dx$

$= \int (4-x^2)^{-\frac{1}{2}}\ (-2x)\ \ dx$

Here $f(x) = 4 - x^2$

$f'(x) = -2x$

$= \frac{(4-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$

$= \frac{(4-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \ 2\sqrt{4-x^2} + c \quad \because t = 4 - x^2$

Q.2: $\int \frac{dx}{x^2+4x+13}$

SOLUTION:

By completing square

$= \int \frac{dx}{x^2+4x+4-4+13}$

$= \int \frac{dx}{(x+2)^2+9}$

$= \int \frac{1}{(x+2)^2+(3)^2}\ dx$

$\because \int \frac{1}{a^2+x^2}\ dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$= \frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + c$

Q.3: $\int \frac{x^2}{4+x^2}\ dx$

SOLUTION:

$(+)$ and $(-)$ 4

$= \int \frac{4+x^2-4}{4+x^2}\ dx$

$= \int \left(\frac{4+x^2}{4+x^2} - \frac{4}{4+x^2}\right)\ dx$

$= \int 1\ dx - \int \frac{4}{4+x^2}\ dx$

$= \int 1\ dx - 4\int \frac{1}{2^2+x^2}\ dx$

$= x - 4.\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

$= x - 2\tan^{-1}\left(\frac{x}{2}\right) + c$

Q.4: $\int \frac{1}{x\ln x}\ dx$

SOLUTION:

$\int \frac{1}{\ln x}\cdot\frac{1}{x}\ dx$

As $f(x) = \ln x$

And $f'(x) = \frac{1}{x}$, so

using $\int \frac{f'(x)}{[f(x)]} = \ln[f(x)]$

$= \ln[\ln x] + c$

Q.5: $\int \frac{e^x}{e^x+3}\ dx$

SOLUTION:

$\int \frac{e^x}{e^x+3}\ dx$

Here $f(x) = e^x$

And $f'(x) = e^x$, so

using $\int \frac{f'(x)}{[f(x)]} = \ln[f(x)] + c$

$= \ln(e^x + 3) + c$

Q.6: $\int \frac{x+b}{(x^2+2bx+c)^{\frac{1}{2}}}\ dx$

SOLUTION:

$\int \frac{x+b}{(x^2+2bx+c)^{\frac{1}{2}}}\ dx$

$\int (x^2+2bx+c)^{-\frac{1}{2}}.(x+b)\ dx$

Here $f(x) = x^2 + 2bx + c$

Here $f'(x) = 2x + 2b = 2(x+b)$

$\times$ and $\div$ by 2

$= \frac{1}{2}\int (x^2+2bx+c)^{-\frac{1}{2}}.2(x+b)\ dx$

$= \frac{1}{2}\frac{(x^2+2bx+c)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \frac{1}{2}\frac{(x^2+2bx+c)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \sqrt{x^2+2bx+c} + c$

Q.7: $\int \frac{\sec^2 x}{\sqrt{\tan x}}\ dx$

SOLUTION:

$\int \frac{\sec^2 x}{\sqrt{\tan x}}\ dx$

$= \int (\tan x)^{-\frac{1}{2}}\ \sec^2 x$

Here $f(x) = \tan x$

Here $f'(x) = \sec^2 x$

$= \frac{(\tan x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \frac{(\tan x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 2\sqrt{\tan x} + c$

Q.8: (a) Show that

$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x + \sqrt{x^2-a^2}\right) + c$

SOLUTION:

$L.H.S = \int \frac{dx}{\sqrt{x^2-a^2}}$

Put $x = a\sec\theta \implies dx = a\sec\theta\ \tan\theta\ d\theta$

$= \int \frac{a\sec\theta\ \tan\theta\ d\theta}{\sqrt{(a\sec\theta)^2-a^2}} = \int \frac{a\sec\theta\ \tan\theta}{\sqrt{a^2\sec^2\theta-a^2}}\ d\theta$

$= \int \frac{a\sec\theta\ \tan\theta}{\sqrt{a^2(\sec^2\theta-1)}}\ d\theta = \int \frac{\sec\theta\ \tan\theta}{\sqrt{\tan^2\theta}}\ d\theta$

$= \int \frac{\sec\theta\ \tan\theta}{\tan\theta}\ d\theta = \int \sec\theta\ d\theta$

$= \ln|\sec\theta + \tan\theta| + c_1$

Then back substitution:

$x = a\sec\theta \quad \implies \quad \frac{x}{a} = \sec\theta$

And $1 + \tan^2 = \sec^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$\tan\theta = \sqrt{\sec^2\theta - 1}$

$\tan\theta = \sqrt{\left(\frac{x}{a}\right)^2 - 1}$

$\tan\theta = \sqrt{\frac{x^2-a^2}{a^2}}$

$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$

Now put values

$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + c_1$

$= ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c_1$

Using $\ln\frac{A}{B} = lnA - lnB$

$= ln|\ x + \sqrt{x^2-a^2}| - lna + c_1$

Where $c = -lna + c_1$

$= ln|\ x + \sqrt{x^2-a^2}| + c$

Q.9: $\int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$

SOLUTION:

$\int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$

Put $x = \tan\theta$

$\Rightarrow \quad d(x) = \sec^2\theta\, d\theta$

$= \int \frac{\sec^2\theta}{(1+\tan^2\theta)^{\frac{3}{2}}}\ d\theta$

$= \int \frac{\sec^2\theta}{(\sec^2\theta)^{\frac{3}{2}}}\ d\theta$

$= \int \frac{\sec^2\theta}{\sec^3\theta}\ d\theta$

$= \int \frac{1}{\sec\theta}\ d\theta$

$= \int \cos\theta$

$= \frac{\sin\theta}{1} + c$

$= \frac{\sin\theta}{\cos\theta}.\cos\theta + c$

$= \tan\theta.\cos\theta + c$

$= \frac{\tan\theta}{\sec\theta} + c$

$= \frac{\tan\theta}{\sqrt{\sec^2\theta}} + c$

$= \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} + c$

Put $\tan\theta = x$

$= \frac{x}{\sqrt{1+x^2}} + c$

Q.10: $\int \frac{1}{(1+x^2)Tan^{-1}x}\ dx$

SOLUTION:

$\int \frac{1}{(1+x^2)Tan^{-1}x}\ dx$

$\int \frac{1}{Tan^{-1}x}\cdot\frac{1}{(1+x^2)}\ dx$

Here $f(x) = Tan^{-1}x$

Here $f'(x) = \frac{1}{(1+x^2)}$

using $\int \frac{f'(x)}{[f(x)]} = ln[f(x)] + c$

$= ln|Tan^{-1}x| + c$

Q.11: $\int \sqrt{\frac{1+x}{1-x}}\ dx$

SOLUTION:

By rationalizing

$= \int \sqrt{\frac{1+x}{1-x}} \times \sqrt{\frac{1+x}{1+x}}\ dx$

$= \int \sqrt{\frac{1+x}{1-x}\times\frac{1+x}{1+x}}\ dx$

$= \int \sqrt{\frac{(1+x)^2}{1-x^2}}\ dx$

$= \int \frac{1+x}{\sqrt{1-x^2}}\ dx$

$= \int \frac{1}{\sqrt{1-x^2}}\ dx + \int \frac{x}{\sqrt{1-x^2}}\ dx$

$= \sin^{-1}x + \frac{1}{-2}\int(1-x^2)^{-\frac{1}{2}}(-2x)\ dx$

$= \sin^{-1}x - \frac{1}{2}.\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \sin^{-1}x - \frac{1}{2}.\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \sin^{-1}x - \sqrt{1-x^2} + c$

Q.12: $\int \frac{\sin\theta}{1+\cos^2\theta}\ d\theta$

SOLUTION:

$\int \frac{\sin\theta}{1+\cos^2\theta}\ d\theta = \int \frac{1}{1+\cos^2\theta}\ \sin\theta\ d\theta$

Put $\cos\theta = t \Rightarrow -\sin\theta\ d\theta = dt$

$\int \frac{1}{1+t^2}\ .-dt = -\tan^{-1}t + c$

Put $t = \cos\theta$

$= -\tan^{-1}(\cos\theta) + c$

Q.13: $\int \frac{ax}{\sqrt{a^2-x^4}}\ dx$

SOLUTION:

$\int \frac{ax}{\sqrt{a^2-x^4}}\ dx = a\int \frac{x}{\sqrt{a^2-(x^2)^2}}\ dx$

Put $x^2 = t \quad \Rightarrow 2x\,dx = dt \quad \Rightarrow x\,dx = \frac{1}{2}\,dt$

$= \frac{a}{2}\int \frac{1}{\sqrt{a^2-t^2}}\ dt = \frac{a}{2}\ \sin^{-1}\frac{t}{a} + c$

using $\int \frac{1}{\sqrt{a^2-x^2}}\ dx = \sin^{-1}\frac{x}{a}$

$= \frac{a}{2}\sin^{-1}\frac{x^2}{a} + c \qquad \because\ x^2 = t$

Q.14: $\int \frac{dx}{\sqrt{7-6x-x^2}}$

SOLUTION:

$\int \frac{dx}{\sqrt{7-6x-x^2}}$

By completing square

$= \int \frac{dx}{\sqrt{7-x^2-6x-9+9}}$

$= \int \frac{dx}{\sqrt{7-(x^2+6x+9)+9}}$

$= \int \frac{dx}{\sqrt{16-(x+3)^2}}$

Using $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + c$

$= \sin^{-1}\frac{x+3}{a} + c$

Q.15: $\int \frac{\cos x}{\sin x\, ln\sin x}\ dx$

SOLUTION:

$\int \frac{1}{ln\sin x}\ \frac{\cos x}{\sin x}\ dx$

Here $f(x) = ln\sin x$

And $f'(x) = \frac{\cos x}{\sin x}$ *, so*

using $\int \frac{f'(x)}{[f(x)]} = ln[f(x)] + c$

$= ln[ln\sin x] + c$

Q.16: $\int \cos x\frac{ln\sin x}{\sin x}\ dx$

SOLUTION:

$\int ln\sin x.\frac{\cos x}{\sin x}\ dx$

Here $f(x) = ln\sin x$

And $f'(x) = \frac{\cos x}{\sin x}$ *, so*

$= \frac{[ln\sin x]^{1+1}}{1+1} + c$

$= \frac{1}{2}[ln\sin x]^2 + c$

$Q.17$: $\int \frac{x\,dx}{4+2x+x^2}$

SOLUTION:

$\int \frac{x\,dx}{4+2x+x^2}$

$= \frac{1}{2}\int \frac{2x}{4+2x+x^2}\ dx$

$= \frac{1}{2}\int \frac{2x+2-2}{4+2x+x^2}\ dx$

$= \frac{1}{2}\left\{\int \frac{2x+2}{4+2x+x^2}dx - \int \frac{2}{4+2x+x^2}\ dx\right\}$

$= \frac{1}{2}\left\{\ln(4+2x+x^2) - \int \frac{2}{x^2+2x+1^2+4-1^2}\ dx\right\}$

$= \frac{1}{2}\ln(4+2x+x^2) - \frac{1}{2}\int \frac{2}{(x+1)^2+(\sqrt{3})^2}\ dx$

using $\int \frac{1}{a^2+x^2}\ dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

$= \frac{1}{2}\ln(x^2+2x+4) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{(x+1)}{\sqrt{3}} + c$

$Q.18$: $\int \frac{x}{x^4+2x^2+5}\ dx$

SOLUTION:

$= \int \frac{x}{(x^2)^2+2x^2+5}\ dx$

Put $x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{1}{2}\,dt$

$= \int \frac{\frac{1}{2}}{t^2+2t+5}\ dt$

$= \frac{1}{2}\int \frac{1}{t^2+2t+1+5-1}\ dt$

$= \frac{1}{2}\int \frac{1}{(t+1)^2+2^2}\ dt$

$= \frac{1}{2}\cdot\frac{1}{2}\tan^{-1}\frac{t+1}{2} + c$

Put $x^2 = t$

$= \frac{1}{4}\ \tan^{-1}\frac{x^2+1}{2} + c$

$Q.19$: $\int \left[\cos\left(\sqrt{x} - \frac{x}{2}\right) \times \left(\frac{1}{\sqrt{x}} - 1\right)\right]\ dx$

SOLUTION:

$\int \left[\cos\left(\sqrt{x} - \frac{x}{2}\right) \times \left(\frac{1}{\sqrt{x}} - 1\right)\right]\ dx$

Put $\sqrt{x} - \frac{x}{2} = t$

$\Longrightarrow\ d\left(\sqrt{x} - \frac{x}{2}\right) = d(t)$

$\frac{1}{2\sqrt{x}} - \frac{1}{2} = dt$

$\frac{1}{2}\left(\frac{1}{\sqrt{x}} - 1\right) = dt$

$\left(\frac{1}{\sqrt{x}} - 1\right) = 2dt$

$= \int[\cos t \times 2\,dt]$

$= 2\int[\cos t\ dt]$

$= 2\ \frac{\sin t}{1} + c$

Put value of t

$= 2\ \sin\left(\sqrt{x} - \frac{x}{2}\right) + c$

$Q.20$: $\int \frac{x+2}{\sqrt{x+3}}\ dx$

[$Q.19$: ***solve on page*** 9]

SOLUTION:

$\int \frac{x+2}{\sqrt{x+3}}\ dx = \int \frac{x+2+1-1}{\sqrt{x+3}}\ dx = \int \frac{x+3}{\sqrt{x+3}}\ dx - \int \frac{1}{\sqrt{x+3}}\ dx = \int \sqrt{x+3}\ dx - \int \frac{1}{\sqrt{x+3}}\ dx = \int (x+3)^{\frac{1}{2}}.1\ dx - \int (x+3)^{-\frac{1}{2}}.1\ dx$

Now integrate

$\frac{(x+3)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{(x+3)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x+3)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}(x+3)^{\frac{3}{2}} + 2\sqrt{x+3} + c$

$Q.21$: $\int \frac{\sqrt{2}}{\sin x + \cos x}\ dx$

SOLUTION:

$\int \frac{1}{\frac{1}{\sqrt{2}}(\cos x + \sin x)}\ dx$

$= \int \frac{1}{\cos x\,.\frac{1}{\sqrt{2}} + \sin x\ .\frac{1}{\sqrt{2}}}\ dx$

$= \int \frac{1}{\cos x\ \cos 45° + \sin x\ \sin 45°}\ dx$

using $\cos(\alpha - \beta) = \cos\alpha\ \sin\beta + \sin\alpha\ \sin\beta$

$= \int \frac{1}{\cos(x-45°)}\ dx$

$= \int \sec(x-45°)\ dx$

using $\int \sec x\ \ dx = ln|\sec x + \tan x| + c$

$ln|\sec(x-45°) + \tan(x-45°)| + c$

$Q.22$: $\int \frac{dx}{\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x}$

SOLUTION:

$= \int \frac{1}{\sin x\,.\frac{1}{2} + \cos x\,.\frac{\sqrt{3}}{2}}\ dx$

$= \int \frac{1}{\sin x\,.\ \cos 60° + \cos x\,.\ \sin 60°}\ dx$

using $\sin(\alpha + \beta) = \sin\alpha\ \cos\beta + \cos\alpha\ \sin\beta$

$= \int \frac{1}{\sin(x+60°)}\ dx$

$= \int\ \mathrm{cosec}(x+60°)\ dx$

using $\int\ \mathrm{cosec}\,x\ \ dx = ln|\mathrm{cosec}\,x - \cot x| + c$

$= ln|\mathrm{cosec}(x-60°) + \cot(x-60°)| + c$

Integration by parts.

We know that for two functions f *and* g

$$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x)$$

$$\Rightarrow\ f(x)g'(x) = \frac{d}{dx}\big(\big(f(x)g(x)\big) = f'(x)g(x)$$

Taking integrations w.r.t x*we get*

$$\int f(x)g'(x)dx = \int\left[\frac{d}{dx}f(x)g(x) - f'(x)g(x)\right]dx$$

$$= \int \Big(\frac{d}{dx}\big(f(x)g(x)\big) - \int f'(x)g(x)dx$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Or $\int f(x)g'(x)dx = f(x)\int g'(x)\,dx - \int (g'(x)dx)f'(x)dx$

In other words.

$$\int (1st\ function\,)(2nd\ function)dx$$

$$= (1st\ funct).\int (2nd\ funct.)\,dx$$

$$- \int (integrated\ funct.)\frac{d}{dx}(istfunction)dx$$

This is called “integrations by parts”

Some basic rules for Integration by parts.

*some the function as 2nd function whose integration is known or possible.
*if integration of both given functions are known but one of the given function is polynomial functions then whose polynomial function as first function.
*if integration of both given function are known but no one is polynomial function. Then we may choose any function as 1st.
*if we are given only one function whose integration is unknown or cannot be easily find.

$$i.e, \sin^{-1}x, \cos^{-1}x, \sqrt{a^2-x^2}, \frac{1}{\sqrt{x^2-a^2}} e.t.c$$

Then we take 1 as 2nd function.

"Review above Rules"

	1st function	2nd function
$\int x^n cosdx$	x^n	$cosx$
$\int x^n sinxdx$	x^n	$sinx$
$\int x^n \sin^{-1}xdx$	$\sin^{-1}x$	x^n
$\int x^n \tan^{-1}xdx$	$\tan^{-1}x$	x^n
$\int e^x sinxdx$	e^x or $sinx$	$sinx$ e^x
$\int lnxx^n dx$	x^n	knx
$\int \tan^{-1}xdx$	$\tan^{-1}x$	1
$\int \sqrt{a^2+x^2}dx$	$\sqrt{a^2+x^2}$	1

You may remember the word "ILATE"
I=inverse function
L=logarithmic function
A=algebraic function
T=trigonometric functions
E=exponential functions.

Remember useful formulas

1. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^1\left(\frac{x}{a}\right) + c$
2. $\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left|x+\sqrt{x^2-a^2}\right| + c$
3. $\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left|x+\sqrt{x^2+a^2}+c\right|$

Prove that $\int e^x\big(f(x)+f'(x)\big)dx + e^x f(x) + c$

Prove: $\int e^x f(x)dx = f(x)e^x - \int e^x f'(x)dx$

⇨ $\int e^x f(x)dx + \int e^x f'(x)dx = e^x f(x)$

⇨ $\int e^x\big(f(x)+f'(x)\big)dx = e^x f(x) + c$

Hence proved.

Exercise 3.4

i) $\int x \sin x \, dx$

SOLUTION:

$\int x \sin x \, dx$

$Here \ U = x \ , \ V = \sin x$

$Using \ \int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= x.\int \sin x\,dx - \int[(x)'.\int \sin x \,dx]\,dx$

$= x.(-\cos x) - \int[1.(-\cos x)] \,dx$

$= -x \cos x - \int[-\cos x] \,dx$

$= -x \cos x + \int[\cos x] \,dx$

$= -x \cos x + \sin x + c$

$= \sin x - x \cos x + c$

ii) $\int \ln x \, dx$

SOLUTION:

$\int \ln x.\ 1\,dx$

$Here \ U = \ln x \ , \ V = 1$

$Using \ \int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= \ln x.\int 1\,dx - \int[(\ln x)'.\int 1\,dx]\,dx$

$= \ln x.x - \int\left[\frac{1}{x}.x\right]dx$

$= \ln x.x - \int 1\,dx$

$= x \ln x - x + c$

iii) $\int x \ln x \, dx$

SOLUTION:

$\int x \ln x \, dx$

$Here \ U = \ln x \ , \ V = x$

$Using \ \int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= \ln x.\int x\,dx - \int[(\ln x)'.\int x\,dx]\,dx$

$= \ln x.\frac{x^2}{2} - \int\left[\frac{1}{x}.\frac{x^2}{2}\right]dx$

$= \ln x.\frac{x^2}{2} - \frac{1}{2}\int x\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}.\frac{x^2}{2} + c$

$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$

iv) $\int x^2 \ln x \, dx$

SOLUTION:

$\int x^2 \ln x \, dx$

$Here \ U = \ln x \ , \ V = x^2$

$Using \ \int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= \ln x.\int x^2\,dx - \int[(\ln x)'.\int x^2\,dx]\,dx$

$= \ln x.\frac{x^3}{3} - \int\left[\frac{1}{x}.\frac{x^3}{3}\right]dx$

$= \ln x.\frac{x^3}{3} - \frac{1}{3}\int x^2\,dx$

$= \frac{x^3}{3}\ln x - \frac{1}{3}.\frac{x^3}{3} + c$

$= \frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + c$

v) $\int x^3 \ln x \, dx$

SOLUTION:

$\int x^2 \ln x \, dx$

$Here \ U = \ln x \ , \ V = x^3$

$Using \ \int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= \ln x.\int x^3\,dx - \int[(\ln x)'.\int x^3\,dx]\,dx$

$= \ln x.\frac{x^4}{4} - \int\left[\frac{1}{x}.\frac{x^4}{4}\right]dx$

$= \ln x.\frac{x^4}{4} - \frac{1}{4}\int x^3\,dx$

$= \frac{x^4}{4} \ln x \;-\frac{1}{4} . \frac{x^4}{4} + c$

$= \frac{x^4}{4}\left(\ln x \;-\frac{1}{4}\right) + c$

***vi*) $\int x^4 \ln x \, dx$**

***SOLUTION*:**

$\int x^4 \;\ln x \; dx$

Here $U = \ln x \;\;, V = x^4$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$

$= \ln x \;. \int x^4\,dx - \int [(\ln x)' . \int x^4\,dx]\,dx$

$= \ln x \;. \frac{x^5}{5} - \int \left[\frac{1}{x} . \frac{x^5}{5}\right]\,dx$

$= \ln x \;. \frac{x^5}{5} - \frac{1}{5}\int x^4\,dx$

$= \frac{x^5}{5} \ln x \;-\frac{1}{5} . \frac{x^5}{5} + c$

$= \frac{x^5}{5}\left(\ln x \;-\frac{1}{5}\right) + c$

***vii*) $\int \tan^{-1} x \, dx$**

***SOLUTION*:**

$\int 1 . \tan^{-1} x \; dx$

Here $U = \tan^{-1} x \;\;, V = 1$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$

$= \tan^{-1} x \;. \int 1\,dx - \int [(\tan^{-1} x)' . \int 1\,dx]\,dx$

$= \tan^{-1} x \;. x \;- \int \left[\frac{1}{1+x^2} . x\right]\,dx$

$= x \tan^{-1} x \;-\frac{1}{2}\int \frac{2x}{1+x^2}\,dx$

$= x \tan^{-1} x \;-\frac{1}{2}\ln|1 + x^2| +$

***viii*) $\int x^2 \sin x \, dx$**

***SOLUTION*:**

$\int x^2 \;\sin x \; dxn$

Here $U = x^2 \;\;, V = \sin x$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$

$= x^2 . \int \sin x\,dx - \int [(x^2)' . \int \sin x \;dx]\,dx$

$= x^2 . (-\cos x) - \int [2x . (-\cos x)]\;dx$

$= -x^2 \;\cos x + 2\int [x . \cos x]\;dx$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$

$= -x^2 \;\cos x + 2\{x . \int \cos x\,dx -$
$\int [(x)' . \int \cos x \;dx]\,dx\}$

$= -x^2 \;\cos x + 2\{x . \sin x - \int [1 . \sin x]\}$

$= -x^2 \;\cos x + 2x . \sin x - 2\int \sin x \;dx$

$= -x^2 \;\cos x + 2x . \sin x - 2(-\cos x) + c$

$= -x^2 \;\cos x + 2x \;\sin x + 2\cos x + c$

***ix*) $\int x^2 \tan^{-1} x \, dx$**

***SOLUTION*:**

$\int x^2 . \tan^{-1} x \; dx$

Here $U = \tan^{-1} x \;\;, V = x^2$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$

$= \tan^{-1} x . \int x^2\,dx - \int [(\tan^{-1} x)' . \int x^2\,dx]\,dx$

$= \tan^{-1} x . \frac{x^3}{3} - \int \left[\frac{1}{1+x^2} . \frac{x^3}{3}\right]\,dx$

$= \frac{x^3}{3}\tan^{-1} x \;-\frac{1}{3}\int \frac{x^3}{1+x^2}\,dx \quad ,$

$= \frac{x^3}{3}\tan^{-1} x \;-\frac{1}{3}\int \left(x - \frac{x}{1+x^2}\right)\,dx$

$= \frac{x^3}{3}\tan^{-1} x \;-\frac{1}{3}\int x\,dx + \frac{1}{3.2}\int \frac{2x}{1+x^2}\,dx$

$= \frac{x^3}{3}\tan^{-1} x \;-\frac{1}{3}\frac{x^2}{2} + \frac{1}{6}\ln|1 + x^2| + c$

$$\begin{array}{r} x \\ 1+x^2\overline{)\,x^3} \\ \underline{\pm x^3 \pm 1} \\ -1 \end{array}$$

$\because \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

***x*) $\int x \tan^{-1} x \, dx$**

***SOLUTION*:**

$\int x . \tan^{-1} x \; dx$

Here $U = \tan^{-1} x \;\;, V = x$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$

$= \tan^{-1} x . \int x\,dx - \int [(\tan^{-1} x)' . \int x\,dx]\,dx$

$= \tan^{-1} x . \frac{x^2}{2} \;- \int \left[\frac{1}{1+x^2} . \frac{x^2}{2}\right]\,dx$

$= \frac{x^2}{2}\tan^{-1} x \;-\frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$

$= \frac{x^2}{2}\tan^{-1} x \;-\frac{1}{2}\int \left(\frac{1+x^2-1}{1+x^2}\right)\,dx$

$= \frac{x^2}{2}\tan^{-1} x \;-\frac{1}{2}\int \frac{1+x^2}{1+x^2}\,dx - \frac{1}{2}\int \frac{1}{1+x^2}\,dx$

$= \frac{x^2}{2}\tan^{-1} x \;-\frac{1}{2}\int 1\,dx - \frac{1}{2}\;\tan^{-1} x$

$= \frac{x^2}{2}\tan^{-1} x \;-\frac{1}{2}\,x - \frac{1}{2}\;\tan^{-1} x + c$

$= \left(\frac{1}{2}\;\tan^{-1} x\right)(x^2 + 1) - \frac{1}{2}x + c$

$$\begin{array}{r} 1 \\ 1+x^2\overline{)\,x^2} \\ \underline{\pm x^2 \pm 1} \\ -1 \end{array}$$

***xi*) $\int x^3 \tan^{-1} x \, dx$**

***SOLUTION*:**

$\int x^3 . \tan^{-1} x \; dx$

Here $U = \tan^{-1} x \;\;, V = x^3$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$

$= \tan^{-1} x . \int x^3\,dx - \int [(\tan^{-1} x)' . \int x^3\,dx]\,dx$

$= \tan^{-1} x . \frac{x^4}{4} \;- \int \left[\frac{1}{1+x^2} . \frac{x^4}{4}\right]\,dx$

$= \frac{x^4}{4}\tan^{-1} x \;-\frac{1}{4}\int \frac{x^4}{1+x^2}\,dx$

$= \frac{x^4}{4}\tan^{-1} x \;-\frac{1}{4}\int \left(x^2 - 1 + \frac{1}{1+x^2}\right)\,dx$

$= \frac{x^4}{4}\tan^{-1} x \;-\frac{1}{4}\int x^2\,dx + \frac{1}{4}\int 1 - \frac{1}{4}\int \frac{1}{1+x^2}\,dx$

$= \frac{x^4}{4}\tan^{-1} x \;-\frac{1}{4}\frac{x^3}{3} + \frac{1}{4}x - \frac{1}{4}\;\tan^{-1} x + c$

$= \frac{1}{4}\left[x^4\;\tan^{-1} x \;-\frac{x^3}{3} + x - \;\tan^{-1} x\right] + c$

$= \frac{1}{4}\left[(x^4 - 1)\;\tan^{-1} x \;-\frac{x^3}{3} + x\right] + c$

$= \frac{1}{4}\left[x^4 \tan^{-1} x - \frac{x^3}{3} + x - \tan^{-1} x\right] + c$

$$\begin{array}{r} x^2 - 1 \\ 1+x^2\overline{)\,x^4} \\ \underline{\pm x^4 \pm x^2} \\ -x^2 \\ \underline{\mp x^2 \mp 1} \\ 1 \end{array}$$

***xii*) $\int x^3 \cos x \, dx$**

***SOLUTION*:**

$\int x^3 \;\cos x \; dxn$

Here $U = x^3 \;\;, V = \cos x$

Using $\int U.V = U . \int V\,dx - \int [U' . \int V\,dx]\,dx$

$= x^3 . \int \cos x\,dx - \int [(x^3)' . \int \cos x \;dx]\,dx$

$= x^3 . (\sin x) - \int [3x^2 . (\sin x)]\;dx$

$= x^3 \;\sin x - 3\int [x^2 \;\sin x]\;dx$

Using $\int U.V = U \;\int V\,dx - \int [U' \int V\,dx]\,dx$

$= x^3 \;\sin x - 3\{x^2 . \int \sin x\,dx -$
$\int [(x^2)' . \int \sin x \;dx]\,dx\}$

$= x^3 \;\sin x - 3\{x^2 . (-\;\cos x) - \int [2x\,(-\;\cos x)\,]\}$

$= x^3 \;\sin x + 3x^2 \;\cos x - 6\int \;x \cos x \;dx$

$= x^3 \;\sin x + 3x^2 \;\cos x - 6\{x \int \cos x \;dx -$
$\int [(x)' \int \cos x\,dx]\,dx\}$

$= x^3 \;\sin x + 3x^2 \;\cos x - 6\{x \;\sin x - \int [1. \sin x]\,dx\}$

$= x^3 \sin x + 3x^2 \cos x - 6\{x \sin x - \int \sin x \, dx\}$
$= x^3 \sin x + 3x^2 \cos x - 6\{x \sin x - (-\cos x)\} + c$
$= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x + c$
$= (x^3 - 6x)\sin x + (3x^2 - 6)\cos x + c$

$$xiii) \int \sin^{-1} x \, dx$$

SOLUTION:
$\int 1 . \sin^{-1} x \, dx$
Here $U = \sin^{-1} x$, $V = 1$
Using $\int U.V = U . \int V \, dx - \int [U' . \int V \, dx] \, dx$
$= \sin^{-1} x . \int 1 \, dx - \int \left[(\sin^{-1} x)' . \int 1 \, dx\right] dx$
$= \sin^{-1} x . x - \int \left[\frac{1}{\sqrt{1-x^2}} . x\right] dx$ *skip*
$= x \sin^{-1} x - \frac{1}{-2}\int (1-x^2)^{-\frac{1}{2}}(-2x) \, dx$
$= x \sin^{-1} x + \frac{1}{2} \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$
$= x \sin^{-1} x + \frac{1}{2} \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= x \sin^{-1} x + \sqrt{1-x^2} + c$

$$xiv) \int x \sin^{-1} x \, dx$$

SOLUTION:
$\int x . \sin^{-1} x \, dx$
Here $U = \sin^{-1} x$, $V = x$
Using $\int U.V = U . \int V \, dx - \int [U' . \int V \, dx] \, dx$
$= \sin^{-1} x . \int x \, dx - \int \left[(\sin^{-1} x)' . \int x \, dx\right] dx$
$= \sin^{-1} x . \frac{x^2}{2} - \int \left[\frac{1}{\sqrt{1-x^2}} . \frac{x^2}{2}\right] dx$
$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}} \, dx$
$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \frac{-x^2}{\sqrt{1-x^2}} \, dx + c$
$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}} \, dx + c$
$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \frac{1-x^2}{\sqrt{1-x^2}} \, dx - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}} \, dx + c$
$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \sqrt{1-x^2} \, dx - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}} \, dx + c$
Using $\sqrt{a^2 - x^2} dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + c$
$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\left\{\frac{1}{2}\sin^{-1} x + \frac{x}{2}\sqrt{1-x^2}\right\} - \frac{1}{2}\sin^{-1} x + c$
$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{4}\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1} x + c$
$= \frac{x^2}{2}\sin^{-1} x + \left(\frac{1}{4} - \frac{1}{2}\right)\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + c$
$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + c$

$$xv) \int e^x \sin x \cos x \, dx$$

SOLUTION:
Let $I = \int e^x \sin x \cos x \, dx$
Multiply and divide by 2
$I = \frac{1}{2}\int e^x 2\sin x \cos x \, dx$
$I = \frac{1}{2}\int e^x \sin 2x \, dx$
Here $U = \sin 2x$, $V = e^x$
Using $\int U.V = U . \int V \, dx - \int [U' . \int V \, dx] \, dx$
$I = \sin 2x \int e^x \, dx - \int [(\sin 2x)' . \int e^x \, dx] \, dx$
$I = \sin 2x \, e^x - \int [\cos 2x . 2 . e^x] \, dx$
$I = \sin 2x \, e^x - 2\int \cos 2x \, e^x \, dx$
Using $\int U.V = U . \int V \, dx - \int [U' . \int V \, dx] \, dx$
$I = \sin 2x \, e^x - 2\{\cos 2x \int e^x \, dx - \int [(\cos 2x)' . \int e^x \, dx] \, dx \}$
$I = \sin 2x \, e^x - 2\{\cos 2x \, e^x - \int (-\sin 2x) \, e^x \, dx \}$
$I = e^x \sin 2x - 2\cos 2x \, e^x - 2\int 2 \sin x \cos x \, e^x \, dx$
$I = e^x \sin 2x - 2\cos 2x \, e^x - 4\int \sin x \cos x \, e^x \, dx$
Put $I = \int e^x \sin x \cos x \, dx$
$I = e^x \sin 2x - 2\cos 2x \, e^x - 4I$
$5I = e^x(\sin 2x - 2\cos 2x)$
$I = \frac{e^x}{5}(\sin 2x - 2\cos 2x)$

$$xvi) \int x \sin x \cos x \, dx$$

SOLUTION:
$\int x \sin x \cos x \, dx = \frac{1}{2}\int x . 2\sin x \cos x \, dx = \frac{1}{2}\int x \sin 2x \, dx$
Here $U = x$, $V = \sin 2x$
Using $\int U.V = U . \int V \, dx - \int [U' . \int V \, dx] \, dx$
$= \frac{1}{2}[x . \int \sin 2x \, dx - \int [(x)' . \int \sin 2x \, dx] \, dx]$
$= \frac{1}{2}x . \left(-\frac{\cos 2x}{2}\right) - \frac{1}{2}\int \left[1\left(-\frac{\cos 2x}{2}\right)\right] dx = -\frac{1}{4}x \cos 2x + \frac{1}{4}\int \cos 2x \, dx$
$= -\frac{1}{4}x \cos 2x + \frac{1}{4}\frac{\sin 2x}{2} + c = \frac{1}{4}\left[-x \cos 2x + \frac{\sin 2x}{2}\right] + c = \frac{1}{4}\left[-x \cos 2x + \frac{2\sin x \cos x}{2}\right] + c$
$= \frac{1}{4}[-x \cos 2x + \sin x \cos x] + c = \frac{1}{4}[\sin x \cos x - x \cos 2x] + c$

$$xvii) \int x \cos^2 x \, dx$$

SOLUTION:
$\int x \cos^2 x \, dx = \int x . \frac{1+\cos 2x}{2} \, dx$ ***As*** $\boldsymbol{\cos^2 x = \frac{1+\cos 2x}{2}}$
$= \frac{1}{2}\int x . (1 + \cos 2x) \, dx = \frac{1}{2}\int (x + x\cos 2x) \, dx$
$= \frac{1}{2}\int x \, dx + \frac{1}{2}\int x \cos 2x \, dx = \frac{1}{2}\frac{x^2}{2} + \frac{1}{2}\int x \cos 2x \, dx$
Here $U = x$, $V = \cos 2x$
Using $\int U.V = U . \int V \, dx - \int [U' . \int V \, dx] \, dx$
$= \frac{x^2}{4} + \frac{1}{2}[x . \int \cos 2x \, dx - \int [x' . \int \cos 2x \, dx] \, dx]$
$= \frac{x^2}{4} + \frac{1}{2}\left[x . \frac{\sin 2x}{2} - \int \left[1 . \frac{\sin 2x}{2}\right] dx\right]$
$= \frac{x^2}{4} + \frac{1}{2}x . \frac{\sin 2x}{2} - \frac{1}{4}\int \sin 2x \, dx$
$= \frac{x^2}{4} + \frac{x \sin 2x}{4} - \frac{1}{4}\frac{-\cos 2x}{2}$
$= \frac{1}{4}\left(x^2 + x\sin 2x + \frac{1}{2}\cos 2x\right) + c$

$$xviii) \int x \sin^2 x \, dx$$

SOLUTION:

$\int x\ \sin^2 x\ dx = \int x\,.\frac{1-\cos 2x}{2}\ dx \quad As\ \sin^2 x = \frac{1-\cos 2x}{2}$

$= \frac{1}{2}\int x\,.(1-\cos 2x)\ dx = \frac{1}{2}\int\ (x - x\cos 2x)\ dx$

$= \frac{1}{2}\int x\ dx - \frac{1}{2}\int\ x\cos 2x\ dx = \frac{1}{2}\ \frac{x^2}{2} - \frac{1}{2}\int\ x\cos 2x\ dx$

Here $U = x\ \ ,\ V = \cos 2x$

Using $\int U.V = U\,.\int V\ dx - \int[U'.\int V\ dx]\ dx$

$= \frac{x^2}{4} - \frac{1}{2}\ [x\,.\int \cos 2x\ dx - \int[x'.\int \cos 2x\ dx]\ dx]$

$= \frac{x^2}{4} - \frac{1}{2}\ \left[x\,.\frac{\sin 2x}{2} - \int\left[1.\frac{\sin 2x}{2}\right]dx\right]$

$= \frac{x^2}{4} - \frac{1}{2}x\,.\frac{\sin 2x}{2} + \frac{1}{4}\int \sin 2x\,dx$

$= \frac{x^2}{4} - \frac{x\sin 2x}{4} + \frac{1}{4}\left(\frac{-\cos 2x}{2}\right) + c$

$= \frac{1}{4}\left(x^2 - \ x\sin 2x - \frac{1}{2}\cos 2x\right) + c$

xix) $\int(\ln x)^2\ dx$

SOLUTION:

$\int(\ln x)^2\,.1\,dx$

Here $U = (\ln x)^2\ \ ,\ V = 1$

Using $\int U.V = U\,.\int V\ dx - \int[U'.\int V\ dx]\ dx$

$= (\ln x)^2\,.\int 1\ dx - \int[((\ln x)^2)'.\int 1\ dx]\ dx$

$= (\ln x)^2\,.x\ - \int\left[2(\ln x)\frac{1}{x}.x\right]\ dx$

$= x(\ln x)^2\ \ - 2\int \ln x\ dx$

$= x(\ln x)^2\ \ - 2[\int(\ln x)\,.1\ dx]$

Here $U = \ln x\ \ ,\ V = 1$

Using $\int U.V = U\,.\int V\ dx - \int[U'.\int V\ dx]\ dx$

$= x(\ln x)^2\ \ - 2[\ln x\ .\int 1\,dx - \int[(\ln x)'.\int 1\ dx]\ dx]$

$= x(\ln x)^2\ \ - 2\left[\ln x\ .x\ - \int\left[\frac{1}{x}.x\right]\ dx\right]$

$= x(\ln x)^2\ \ - 2[\ln x\ .x\ - \int 1\,dx]$

$= x(\ln x)^2\ \ - 2[x\ \ln x\ \ - x] + c$

$= x(\ln x)^2\ \ - 2x\ \ln x\ + 2x + c$

$= x\ln x\ (\ln x - 2\,) + 2x + c$

xx) $\int \ln(\tan x)\ \sec^2 x\ dx$

SOLUTION:

$\int \ln(\tan x)\ \sec^2 x\ dx$

Here $U = \ln(\tan x)\ \ \ ,\ V = \sec^2 x$

Using $\int U.V = U\,.\int V\ dx - \int[U'.\int V\ dx]\ dx$

$= \ln(\tan x)\ \ .\int \sec^2 x\ dx - \int[[\ln(\tan x)\,]'.\int \sec^2 x\ dx]\,dx$

$= \ln(\tan x)\,.\ \tan x - \int\left[\frac{\sec^2 x}{\tan x}.\tan x\right]\ dx$

$= \tan x\,.\ln(\tan x)\ \ - \int \sec^2 x\ dx$

$= \tan x\,.\ln(\tan x)\ \ - \tan x + c$

xxi) $\int \frac{x\,.\ \sin^{-1} x}{\sqrt{1-x^2}}\ dx$

SOLUTION:

$\int \frac{x\,.\ \sin^{-1}x}{\sqrt{1-x^2}}\ dx = \frac{1}{-2}\int \sin^{-1}x\left[(1-x^2)^{-\frac{1}{2}}(-2x)\right]$

Here $U = \sin^{-1} x\ \ \ ,\ V = (1-x^2)^{-\frac{1}{2}}(-2x)$

Using $\int U.V = U\,.\int V\ dx - \int[U'.\int V\ dx]\ dx$

$= -\frac{1}{2}\left\{\sin^{-1}x\int(1-x^2)^{-\frac{1}{2}}(-2x)\ dx - \int\left[(\sin^{-1}x)'.\int(1-x^2)^{-\frac{1}{2}}(-2x)\ dx\right]\ dx\right\}$

$= -\frac{1}{2}\left\{\sin^{-1}x\,\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - \int\left[\frac{1}{\sqrt{1-x^2}}\,.\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]\ dx\right\} = -\frac{1}{2}\left\{\sin^{-1}x\,\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} - \int\left[\frac{1}{\sqrt{1-x^2}}\,.\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}\right]\ dx\right\}$

$= -\frac{1}{2}\{2\ \sin^{-1}x\,\sqrt{1-x^2} - \int[2]\ dx\} = -\frac{1}{2}\{2\ \sin^{-1}x\,\sqrt{1-x^2} - 2\int 1\ dx\} = -\ \sin^{-1}x\,\sqrt{1-x^2} + x + c$

$= x - \sqrt{1-x^2}\ \sin^{-1}x + c$

Q. 2: ***Evaluate the following integrals:***

i) $\int \tan^4 x\ dx$

SOLUTION:

$\int \tan^4 x\ dx$

$= \int \tan^2 x\ .\tan^2 x\ dx$

$= \int \tan^2 x\ (\sec^2 x - 1)\ dx$

$= \int \tan^2 x\ \sec^2 x\,dx - \int(\sec^2 x - 1)\ \ dx$

$= \frac{\tan^3 x}{3} - \int\ \sec^2 x\,dx + \int 1\ \ dx$

$= \frac{\tan^3 x}{3} - \tan x + x + c$

ii) $\int \sec^4 x\ dx$

SOLUTION:

$\int \sec^4 x\ dx$

$= \int \sec^2 x\ .\sec^2 x\ dx$

$= \int \sec^2 x\ \ (1 + \ \tan^2 x)\ dx$

$= \int \sec^2 x\ \ dx + \int\ \sec^2 x\ \tan^2 x\ \ dx$

$= \tan x + \frac{\tan^3 x}{3} + c$

$= \tan x + \frac{1}{3}\tan^3 x + c$

iv) $\int \tan^3 x \sec x\ dx$

SOLUTION:

$\int \tan^3 x\sec x\ dx$

$\int \tan^2 x\ \ \tan x\sec x\ dx$

$= \int(\sec^2 x - 1)\ \tan x\sec x\ dx$

$= \int \sec^2 x\sec x\tan x\ dx - \int \sec x\tan x\ \ dx$

$= \frac{1}{3}\sec^3 x - \sec x + c$

v) $\int x^3\,e^{5x}\,dx$

SOLUTION:

$\int x^3\,e^{5x}\ dx$

Here $U = x^3\ \ ,\ V = e^{5x}$

Using $\int U.V = U\,.\int V\ dx - \int[U'.\int V\ dx]\ dx$

$= x^3\ \int e^{5x}\ dx - \int[(x^3)'.\int e^{5x}\ dx]\ dx$

$= x^3\ \frac{e^{5x}}{5} - \int 3x^2\ \frac{e^{5x}}{5}\ \ dx$

$= x^3\ \frac{e^{5x}}{5} - \frac{3}{5}\{\int x^2\ \ e^{5x}\ \ dx\}$

Again integrating by parts

$= x^3 \frac{e^{5x}}{5} - \frac{3}{5}\{x^2 \int e^{5x}\, dx - \int[(x^2)'.\int e^{5x}\, dx]\, dx\}$

$$= x^3 \frac{e^{5x}}{5} - \frac{3}{5}\left\{x^2 \frac{e^{5x}}{5} - \int 2x \frac{e^{5x}}{5}\, dx\right\}$$

$= x^3 \frac{e^{5x}}{5} - \frac{3}{25}x^2 e^{5x} + \frac{6}{25}\int x\, e^{5x}\, dx$

Again integrating by parts

$= x^3 \frac{e^{5x}}{5} - \frac{3}{25}x^2 e^{5x} + \frac{6}{25}\{x \int e^{5x}\, dx -$
$\int[(x)'.\int e^{5x}\, dx]\, dx\}$

$= x^3 \frac{e^{5x}}{5} - \frac{3}{25}x^2 e^{5x} + \frac{6}{25}\left\{x \frac{e^{5x}}{5} - \int 1.\frac{e^{5x}}{5}\, dx\right\}$

$= x^3 \frac{e^{5x}}{5} - \frac{3}{25}x^2 e^{5x} + \frac{6}{125}x\, e^{5x} - \frac{6}{125}\int e^{5x}\, dx$

$= x^3 \frac{e^{5x}}{5} - \frac{3}{25}x^2 e^{5x} + \frac{6}{125}x\, e^{5x} - \frac{6}{125}\frac{e^{5x}}{5} + c$

$= \frac{e^{5x}}{5}\left(x^3 - \frac{3}{5}x^2 + \frac{6}{25}x - \frac{6}{125}\right) + c$

vi) $\int e^{-x} \sin 2x\, dx$

SOLUTION:

Let $I = \int \sin 2x\, e^{-x}\, dx$

Here $U = \sin 2x \;,\; V = e^{-x}$

Using $\int U.V = U.\int V\, dx - \int[U'.\int V\, dx]\, dx$

$I = \sin 2x \int e^{-x}\, dx - \int[(\sin 2x)' \int e^{-x}\, dx]\, dx$

$I = \sin 2x \frac{e^{-x}}{-1} - \int\left[(\cos 2x.2)\frac{e^{-x}}{-1}\right] dx$

$I = -e^{-x}\sin 2x + 2\int e^{-x}\cos 2x\, dx$

Again integrating by parts

$I = -e^{-x}\sin 2x + 2\{\cos 2x \int e^{-x}\, dx -$
$\int[(\cos 2x)' \int e^{-x}\, dx]\, dx\}$

$I = -e^{-x}\sin 2x + 2\left\{\cos 2x \frac{e^{-x}}{-1} -\right.$
$\left.\int\left[(-\sin 2x.2)\frac{e^{-x}}{-1}\right] dx\right\}$

$I = -e^{-x}\sin 2x - 2\cos 2x\, e^{-x} -$
$4\int e^{-x}\sin 2x\, dx$

$I = -e^{-x}\sin 2x - 2\cos 2x\, e^{-x} - 4I + c_1$

$5I = -e^{-x}\sin 2x - 2\cos 2x\, e^{-x} + c_1$

$I = -\frac{2}{5}\cos 2x\, e^{-x} - \frac{1}{5}e^{-x}\sin 2x + \frac{c_1}{5}$

$I = -\frac{2}{5}e^{-x}(\cos 2x + \frac{1}{2}e^{-x}\sin 2x) + c$ *where* $c = \frac{c_1}{5}$

vii) $\int e^{2x} \cos 3x\, dx$

SOLUTION:

Let $I = \int e^{2x}\cos 3x\, dx$

Here $U = \cos 3x \;,\; V = e^{2x}$

Using $\int U.V = U.\int V\, dx - \int[U'.\int V\, dx]\, dx$

$I = \cos 3x \int e^{2x}\, dx - \int[(\cos 3x)' \int e^{2x}\, dx]\, dx$

$I = \cos 3x \frac{e^{2x}}{2} - \int\left[(-\sin 3x.3)\frac{e^{2x}}{2}\right] dx$

$I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{2}\int[\sin 3x\, e^{2x}]\, dx$

Again integrating by parts

$I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{2}\{\sin 3x \int e^{2x}\, dx -$
$\int[(\sin 3x)' \int e^{2x}\, dx]\, dx\}$

$I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{2}\left\{\sin 3x \frac{e^{2x}}{2} -\right.$
$\left.\int\left[\cos 3x.3.\frac{e^{2x}}{2}\right] dx\right\}$

$I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{4}\sin 3x\, e^{2x} - \frac{9}{4}\int \cos 3x\, e^{2x}\, dx$

$I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{4}\sin 3x\, e^{2x} - \frac{9}{4}I + c_1$

$I + \frac{9}{4}I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{4}\sin 3x\, e^{2x} + c_1$

$\frac{13}{4}I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{4}\sin 3x\, e^{2x} + c_1$

$I = \frac{4}{13}\frac{e^{2x}}{2}(\cos 3x + \frac{3}{2}\sin 3x) + \frac{4}{13}c_1$

$I = \frac{2}{13}e^{2x}\left(\cos 3x + \frac{3}{2}\sin 3x\right) +$
c *where* $\frac{4}{13}c_1 = c$

$I = \frac{3}{13}e^{2x}(\sin 3x + \frac{2}{3}\cos 3x) + c$

viii) $\int \mathbf{cosec^3\, x\, dx}$

SOLUTION:

Let $I = \int \text{cosec}^2 x\, \text{cosec}\, x\, dx$

Here $U = \text{cosec}\, x \;,\; V = \text{cosec}^2 x$

Using $\int U.V = U.\int V\, dx - \int[U'.\int V\, dx]\, dx$

$= \text{cosec}\, x \int \text{cosec}^2 x\, dx -$
$\int[(\text{cosec}\, x)'.\int \text{cosec}^2 x\, dx]\, dx$

$I = \text{cosec}\, x\,(-\cot x) - \int[(-\text{cosec}\, x \cot x)(-\cot x)]\, dx$

$I = -\text{cosec}\, x \cot x - \int \text{cosec}\, x \cot^2 x\, dx$

$I = -\text{cosec}\, x \cot x - \int \text{cosec}\, x\,(\text{cosec}^2 x - 1)\, dx$

$I = -\text{cosec}\, x \cot x - \int \text{cosec}^3 x\, dx + \int \text{cosec}\, x\, dx$

$I = -\text{cosec}\, x \cot x - I + \int \text{cosec}\, x\, dx$

$2I = -\text{cosec}\, x \cot x + \ln|\text{cosec}\, x - \cot x| + c_1$

$I = -\frac{1}{2}[\cot x\, \text{cosec}\, x - \ln|\text{cosec}\, x - \cot x|] + \frac{1}{2}c_1$

$I = -\frac{1}{2}[\cot x\, \text{cosec}\, x - \ln|\text{cosec}\, x - \cot x|] + c$

TIT BIT:

Jab pure quadratic equation h aur us ka derivative b majood na h t substituition s solve karty h aur substitution m trigonometry functions hi let karty lakin j c s start hu w let nai karny nai t book answer ni aye ga ut jin pure quadratic equation walay questions ki power $\frac{1}{2}$ h t un k ap by parts integration k method s b kar saktay h.

Q.3: ***Show that*** $\int e^{ax} \sin bx\, dx$

$$= \frac{1}{\sqrt{a^2 + b^2}} e^{ax} \sin\left(bx - \tan^{-1}\left(\frac{b}{a}\right)\right) + c$$

SOLUTION:

Let $I = \int e^{ax} \sin bx\, dx$

Here $U = \sin bx \;,\; V = e^{ax}$

Using $\int U.V = U.\int V\, dx - \int[U'.\int V\, dx]\, dx$

$I = \sin bx \int e^{ax}\, dx - \int[(\sin bx)'.\int e^{ax}\, dx]\, dx$

$I = \sin bx \frac{e^{ax}}{a} - \int\left[\cos bx\, .\, b.\frac{e^{ax}}{a}\right] dx$

$I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a}\int \cos bx\, e^{ax}\, dx$

Again integrating by parts

$I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a}\{\cos bx \int e^{ax}\, dx -$
$\int[(\cos bx)'.\int e^{ax}\, dx]\, dx\}$

$I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a}\left\{\cos bx \frac{e^{ax}}{a} - \int\left[-\sin bx\, .b.\frac{e^{ax}}{a}\right] dx\right\}$

$I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a^2}\cos bx\, e^{ax} - \frac{b^2}{a^2}\int \sin bx\, e^{ax}\, dx$

$I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a^2}\cos bx\, e^{ax} - \frac{b^2}{a^2}I + c_1$

$I + \frac{b^2}{a^2} I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a^2}\cos bx\, e^{ax} + c_1$

$\left(\frac{a^2+b^2}{a^2}\right) I = e^{ax}\left(\sin bx \frac{1}{a} - \frac{b}{a^2}\cos bx\right) + c_1$

$I = \frac{a^2}{a^2+b^2} e^{ax}\left(\sin bx \frac{1}{a} - \frac{b}{a^2}\cos bx\right) + \frac{a^2}{a^2+b^2}c_1$

$I = \frac{1}{a^2+b^2} e^{ax}(a \sin bx - b\cos bx) + c$ (A)   where $\frac{a^2}{a^2+b^2}c_1 = c$

Let $a = r\cos\theta$ (1) , $b = r\sin\theta$ (2)

Squaring and adding (1) and (2)

dividing (1) and (2)

$a^2+b^2 = r^2\cos^2\theta + r^2\sin^2\theta$ $\frac{r\sin\theta}{r\cos\theta} = \frac{b}{a}$

$a^2+b^2 = r^2(\cos^2\theta+\sin^2\theta)$ $\tan\theta = \frac{b}{a}$

$a^2+b^2 = r^2 \Rightarrow r = \sqrt{a^2+b^2}$ $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Now Put values in (A)

$I = \frac{1}{a^2+b^2}e^{ax}(r\cos\theta \sin bx - r\sin\theta\cos bx) + c$

$I = \frac{r}{a^2+b^2}e^{ax}(\cos\theta \sin bx - \sin\theta\cos bx) + c$ (Take r common)

$I = \frac{\sqrt{a^2+b^2}}{a^2+b^2}e^{ax}(\sin bx\cos\theta - \cos bx\sin\theta) + c$ (Put value r)

$I = \frac{1}{\sqrt{a^2+b^2}}e^{ax}(\sin(bx-\theta))c$

Using $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$I = \frac{1}{\sqrt{a^2+b^2}}e^{ax}\sin\left(bx - \tan^{-1}\left(\frac{b}{a}\right)\right) + c$

Put $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ have proved.

Q.4: Evaluate the following indefinate integrals:

i) $\int \sqrt{a^2 - x^2}\, dx$

SOLUTION:

Let $I = \int \sqrt{a^2-x^2}\,.1\,dx$

Here $U = \sqrt{a^2-x^2}$, $V = 1$

Using $\int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$I = \sqrt{a^2-x^2}\int 1\,dx - \int\left[\left(\sqrt{a^2-x^2}\right)'.\int 1\,dx\right]dx$

$I = \sqrt{a^2-x^2}\,.x - \int\left[\frac{-2x}{2\sqrt{a^2-x^2}}.x\right]dx$

$I = x\sqrt{a^2-x^2} - \int\frac{-x^2}{\sqrt{a^2-x^2}}\,dx$

$I = x\sqrt{a^2-x^2} - \int\frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\,dx$

$I = x\sqrt{a^2-x^2} - \int\frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx + \int\frac{a^2}{\sqrt{a^2-x^2}}\,dx$

$I = x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,dx + a^2\int\frac{1}{\sqrt{a^2-x^2}}\,dx$

Using $\int\frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c$

$I = x\sqrt{a^2-x^2} - I + a^2\sin^{-1}\frac{x}{a} + c_1$

$I + I = x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a} + c_1$

$2I = x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a} + c_1$

$I = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{c_1}{2}$

$I = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$ where $\frac{c_1}{2} = c$

ii) $\int\sqrt{x^2-a^2}\,dx$

SOLUTION:

Let $I = \int\sqrt{x^2-a^2}\,.1\,dx$

Here $U = \sqrt{x^2-a^2}$, $V = 1$

Using $\int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$I = \sqrt{x^2-a^2}\int 1\,dx - \int\left[\left(\sqrt{x^2-a^2}\right)'.\int 1\,dx\right]dx$

$I = \sqrt{x^2-a^2}\,.x - \int\left[\frac{2x}{2\sqrt{x^2-a^2}}.x\right]dx$

$I = x\sqrt{x^2-a^2} - \int\frac{x^2}{\sqrt{x^2-a^2}}\,dx$

$I = x\sqrt{x^2-a^2} - \int\frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}\,dx$

$I = x\sqrt{x^2-a^2} - \int\frac{x^2-a^2}{\sqrt{x^2-a^2}}\,dx - \int\frac{a^2}{\sqrt{x^2-a^2}}\,dx$

$I = x\sqrt{x^2-a^2} - \int\sqrt{x^2-a^2}\,dx - a^2\int\frac{1}{\sqrt{x^2-a^2}}\,dx$

Using $\int\frac{1}{\sqrt{x^2-a^2}}\,dx = ln\left|x+\sqrt{x^2-a^2}\right| + c$

$I = x\sqrt{x^2-a^2} - I - a^2 ln\left|x+\sqrt{x^2-a^2}\right| + c_1$

$I + I = x\sqrt{x^2-a^2} - a^2 ln\left|x+\sqrt{x^2-a^2}\right| + c_1$

$2I = x\sqrt{x^2-a^2} - a^2 ln\left|x+\sqrt{x^2-a^2}\right| + c_1$

$I = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}ln\left|x+\sqrt{x^2-a^2}\right| + \frac{c_1}{2}$

$I = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}ln\left|x+\sqrt{x^2-a^2}\right| + c$

where $\frac{c_1}{2} = c$

iii) $\int\sqrt{4-5x^2}\,dx$

SOLUTION:

Let $I = \int\sqrt{4-5x^2}\,.1\,dx$

Here $U = \sqrt{4-5x^2}$, $V = 1$

Using $\int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$I = \sqrt{4-5x^2}\int 1\,dx - \int\left[\left(\sqrt{4-5x^2}\right)'.\int 1\,dx\right]dx$

$I = \sqrt{4-5x^2}\,.x - \int\left[\frac{-10x}{2\sqrt{4-5x^2}}.x\right]dx$

$I = x\sqrt{4-5x^2} - \int\frac{-5x^2}{\sqrt{4-5x^2}}\,dx$

$I = x\sqrt{4-5x^2} - \int\frac{4-5x^2-4}{\sqrt{4-5x^2}}\,dx$

$I = x\sqrt{4-5x^2} - \int\frac{4-5x^2}{\sqrt{4-5x^2}}\,dx + \int\frac{4}{\sqrt{4-5x^2}}\,dx$

$I = x\sqrt{4-5x^2} - \int\sqrt{4-5x^2}\,dx + 4\int\frac{1}{\sqrt{5\left(\frac{4}{5}-x^2\right)}}\,dx$

$I = x\sqrt{4-5x^2} - \int\sqrt{4-5x^2}\,dx + \frac{4}{\sqrt5}\int\frac{1}{\sqrt{\left(\frac{2}{\sqrt5}\right)^2-x^2}}\,dx$

Using $\int\frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c$

$I = x\sqrt{4-5x^2} - I + \frac{4}{\sqrt5}\sin^{-1}\left(\frac{x}{\frac{2}{\sqrt5}}\right) + c_1$

$2I = x\sqrt{4-5x^2} + \frac{4}{\sqrt5}\sin^{-1}\left(\frac{\sqrt5 x}{2}\right) + c_1$

$I = \frac{x}{2}\sqrt{4-5x^2} + \frac{2}{\sqrt5}\sin^{-1}\left(\frac{\sqrt5 x}{2}\right) + \frac{c_1}{2}$

$I = \frac{x}{2}\sqrt{4-5x^2} + \frac{2}{\sqrt5}\sin^{-1}\left(\frac{\sqrt5 x}{2}\right) + c$ where $\frac{c_1}{2} = c$

iv) $\int\sqrt{3-4x^2}\,dx$

SOLUTION:

Let $I = \int\sqrt{3-4x^2}\,.1\,dx$

Here $U = \sqrt{3-4x^2}$, $V = 1$

$Using\ \int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$I = \sqrt{3-4x^2}\ \int 1\,dx - \int\left[\left(\sqrt{3-4x^2}\right)'.\int 1\,dx\right]\,dx$

$I = \sqrt{3-4x^2}\,.x - \int\left[\frac{-8x}{2\sqrt{3-4x^2}}.x\right]\,dx$

$I = x\sqrt{3-4x^2} - \int\frac{-4x^2}{\sqrt{3-4x^2}}\,dx$

$I = x\sqrt{3-4x^2} - \int\frac{3-4x^2-3}{\sqrt{3-4x^2}}\,dx$

$I = x\sqrt{3-4x^2} - \int\frac{3-4x^2}{\sqrt{3-4x^2}}\,dx + \int\frac{3}{\sqrt{3-4x^2}}\,dx$

$I = x\sqrt{3-4x^2} - \int\sqrt{3-4x^2}\,dx + 3\int\frac{1}{\sqrt{4\left(\frac{3}{4}-x^2\right)}}\,dx$

$I = x\sqrt{3-4x^2} - \int\sqrt{3-4x^2}\,dx + \frac{3}{2}\int\frac{1}{\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2-x^2}}\,dx$

$Using\quad \int\frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c$

$I = x\sqrt{3-4x^2} - I + \frac{3}{2}\sin^{-1}\left(\frac{x}{\frac{\sqrt{3}}{2}}\right) + c_1$

$2I = x\sqrt{3-4x^2} + \frac{3}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + c_1$

$I = \frac{x}{2}\sqrt{3-4x^2} + \frac{3}{4}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + \frac{c_1}{2}$

$I = \frac{x}{2}\sqrt{3-4x^2} + \frac{3}{4}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + c \qquad where\ \ \frac{c_1}{2} =$

v) $\int\sqrt{x^2+4}\,dx$

SOLUTION:

$Let\ \ I = \int\sqrt{x^2+4}\,.1\,dx$

$Here\ U = \sqrt{x^2+4}\ ,\ V = 1$

$Using\ \int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$I = \sqrt{x^2+4}\ \int 1\,dx - \int\left[\left(\sqrt{x^2+4}\right)'.\int 1\,dx\right]\,dx$

$I = \sqrt{x^2+4}\,.x - \int\left[\frac{2x}{2\sqrt{x^2+4}}.x\right]\,dx$

$I = x\sqrt{x^2+4} - \int\frac{x^2}{\sqrt{x^2+4}}\,dx$

$I = x\ \sqrt{x^2+4} - \int\frac{x^2+4-4}{\sqrt{x^2+4}}\,dx$

$I = x\ \sqrt{x^2+4} - \int\frac{x^2+4}{\sqrt{x^2+4}}\,dx + \int\frac{4}{\sqrt{x^2+4}}\,dx$

$I = x\ \sqrt{x^2+4} - \int\sqrt{x^2+4}\,dx + 4\int\frac{1}{\sqrt{x^2+4}}\,dx$

$Using\quad \int\frac{1}{\sqrt{x^2+a^2}}\,dx = ln\left|x+\sqrt{x^2+a^2}\right| + c$

$I = x\sqrt{x^2+4} - I + 4ln\left|x+\sqrt{x^2+2^2}\right| + c_1$

$I + I = x\sqrt{x^2+4} + 4ln\left|x+\sqrt{x^2+2^2}\right| + c_1$

$2I = x\sqrt{x^2+4} + 4ln\left|x+\sqrt{x^2+2^2}\right| + c_1$

$I = \frac{x}{2}\sqrt{x^2+4} + \frac{4}{2}ln\left|x+\sqrt{x^2+2^2}\right| + \frac{c_1}{2}$

$I = \frac{x}{2}\sqrt{x^2+4} + 2\ ln\left|x+\sqrt{x^2+2^2}\right| + c$

$where\ \ \frac{c_1}{2} = c$

vi) $\int x^2 e^{ax}\,dx$

SOLUTION:

$\int x^2 e^{ax}\,dx$

$Here\ U = x^2\ ,\ V = e^{ax}$

$Using\ \int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= x^2\int e^{ax}\,dx - \int[(x^2)'.\int e^{ax}\,dx]\,dx$

$= x^2\frac{e^{ax}}{a} - \int 2x\frac{e^{ax}}{a}\,dx$

$= x^2\frac{e^{ax}}{a} - \frac{2}{a}\{\int x\ e^{ax}\ dx\}$

$Again\ integrating\ by\ parts$

$= x^2\frac{e^{ax}}{a} - \frac{2}{a}\left\{x\frac{e^{ax}}{a} - \int 1.\frac{e^{ax}}{a}\,dx\right\}$

$= x^2\frac{e^{ax}}{a} - \frac{2}{a^2}x\,e^{ax} + \frac{2}{a^2}\int e^{ax}\,dx$

$= x^2\frac{e^{ax}}{a} - \frac{2}{a^2}x\,e^{ax} + \frac{2}{a^2}\frac{e^{ax}}{a} + c$

$Take\ common\ \frac{e^{ax}}{a}$

$= \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right) + c$

Q. 5: *Evaluate the following integrals*:

i) $\int e^x\left(\frac{1}{x} + \ln x\right)dx$

SOLUTION:

$= \int e^{1\cdot x}\left(1.\ln x + \frac{1}{x}\right)dx$

$\because\ \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{1.x}\ \ln x + c$

$= e^x\ \ln x + c$

ii) $\int e^x(\cos x + \sin x)dx$

SOLUTION:

$= \int e^{1\cdot x}(1.\sin x + \cos x)dx$

$\because\ \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{1.x}\sin x + c$

$= e^x\sin x + c$

iii) $\int e^{ax}\left(a\sec^{-1}x + \frac{1}{x\sqrt{x^2-1}}\right)dx$

SOLUTION:

$\int e^{ax}\left(a\sec^{-1}x + \frac{1}{x\sqrt{x^2-1}}\right)dx$

$\because\ \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{ax}\sec^{-1}x + c$

iv) $\int e^{3x}\left(\frac{3\sin x - \cos x}{\sin^2 x}\right)dx$

SOLUTION:

$= \int e^{3x}\left(\frac{3\sin x}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right)dx$

$= \int e^{3x}\left(\frac{3}{\sin x} - \frac{\cos x}{\sin x.\sin x}\right)dx$

$= \int e^{3x}(3\,\mathrm{cosec}\,x - \cot x\,\mathrm{cosec}\,x)dx$

$\because\ \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{3x}\,\mathrm{cosec}\,x + c$

v) $\int e^{2x}(-\sin x + 2\cos x)dx$

SOLUTION:

$= \int e^{2x}(2\cos x - \sin x)dx$

$= \int e^{2x}(2\cos x + (-\sin x))dx$

$\because\ \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{2x}\cos x + c$

vi) $\int\frac{x\,e^x}{(1+x)^2}dx$

SOLUTION:

$= \int e^x\left[\frac{1+x-1}{(1+x)^2}\right]dx$

$= \int e^x\left[\frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2}\right]dx$

$= \int e^x\left[\frac{1}{1+x} - \frac{1}{(1+x)^2}\right]dx$

$= e^x.\frac{1}{1+x} + c$

vii) $\int e^{-x}(\cos x - \sin x)dx$

SOLUTION:

$= \int e^{-x}(-\sin x + \cos x)dx$

$= \int e^{-1.\,x}(-1.\sin x + \cos x)dx$

$\because \int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^{-1.x}\sin x + c$

$= e^{-x}\sin x + c$

viii) $\int \frac{e^{m\tan^{-1}x}}{1+x^2}dx$

SOLUTION:

$= \int e^{m\tan^{-1}x}\frac{1}{1+x^2}dx$

Put $y = \tan^{-1}x$

$dy = \frac{1}{1+x^2}dx$

$= \int e^{my}dy = \frac{e^{my}}{m} + c = \frac{e^{m\tan^{-1}x}}{m} + c$

Put $y = \tan^{-1}x$

ix) $\int \frac{2x}{1-\sin x}dx$

SOLUTION:

$\int \frac{2x}{1-\sin x} \times \frac{1+\sin x}{1+\sin x}dx = \int \frac{2x(1+\sin x)}{1-\sin^2 x}dx =$

$\int \frac{2x(1-\sin x)}{\cos^2 x}dx = \int 2x\left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos x\cos x}\right)dx$

$= \int 2x(\sec^2 x + \tan x\sec x)dx = \int 2x\ \sec^2 x\ dx -$

$\int 2x\ \tan x\sec x\ dx$

Here $U = 2x$, $V = \sec^2 x$ and $U = 2x$,

$V = \tan x\sec x$

Using $\int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= [2x\ \tan x - \int 2(1)\tan x\ dx] + [2x\ \sec x -$

$\int 2(1)\sec x\ dx]$

$= 2x.\tan x - 2\ln|\sec x| + 2x.\sec x - 2\,ln|\sec x +$

$\tan x| + c$

x) $\int \frac{e^x(1+x)}{(2+x)^2}dx$

SOLUTION:

$= \int e^x\left[\frac{2-1+x}{(2+x)^2}\right]dx$

$= \int e^x\left[\frac{(2+x)-1}{(2+x)^2}\right]dx$

$= \int e^x\left[\frac{2+x}{(2+x)^2} - \frac{1}{(2+x)^2}\right]dx$

$= \int e^x\left[\frac{1}{2+x} - \frac{1}{(2+x)^2}\right]dx$

$\int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^x.\frac{1}{2+x} + c$

xi) $\int\left(\frac{1-\sin x}{1-\cos x}\right)e^x dx$

SOLUTION:

$= \int e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$

$= \int e^x\left(\frac{1}{2\sin^2\frac{x}{2}} - \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$

$= \int e^x\left(\frac{1}{2}\text{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right)dx$

$= \int e^x\left(-\cot\frac{x}{2} + \frac{1}{2}\text{cosec}^2\frac{x}{2}\right)dx$

$\int e^{ax}[a\,f(x) + f'(x)]dx = e^{ax}f(x) + c$

$= e^x\left(-\cot\frac{x}{2}\right) + c = -e^x\cot\frac{x}{2} + c$

Integration involving Partial Fraction

If $P(x), Q(x)$ are two polynomial function and $Q(x) \neq 0$

In rational fraction

$\frac{P(x)}{Q(x)}$ *can be factorized into linear and*

Quadratic (irreducible) factors then the rational function is written as as a sum of simpler rational functions, each of which can be integrated by methods already known.

Here we discuss examples of the three cases of partial fraction and then apply integrated.

Case1.

when $Q(x)$ contain non − repeated linear factors. e.g;

$$\frac{P(x)}{(x-a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

Or $\frac{-x+6}{(x-2)(x-3)(x-4)} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{x-4}$ *e.tc*

Case2.

when $Q(x)$ contain non repeated and repeates linear factors.

$$\frac{P(x)}{(x-a)(x+b)^2} = \frac{A}{x-a} + \frac{B}{x+b} + \frac{C}{(x+b)^2}$$

$$\frac{2x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}\ e.t.c$$

Case3.

When $Q(x)$ contain non repeated irreducible quadratic factors.

$$\frac{P(x)}{(x+b)(x^2+c)} = \frac{A}{x+b} + \frac{Bx+C}{x^2+c}$$

$$\frac{1}{(x-1)(x^2+1+2x)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1+2x}$$

Exercise 3.5

Evaluate the following integrals.

Q1. $\int \frac{3x+1}{x^2-x-6}dx$

Solution: $\int \frac{3x+1}{x^2-x-6}dx$

Now

$\frac{3x+1}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$

$$\Rightarrow 3x+1 = A(x+2) + B(x-3) \rightarrow (i)$$

$$Put\ x-3 = 0 \Rightarrow x = 3\ in(i)$$

$$3(3)+1 = A(3+2) + B(0) \Rightarrow 5A = 10 \Rightarrow A = 2$$

$$Put x+2 = 0 \Rightarrow x = -2\ in(i)$$

$$3(-2)+1 = A(0) + B(-2-3) \Rightarrow -5B = -6+1$$

$\Rightarrow -5B = -5 \Rightarrow B = 1$

$$so\ \frac{3x+1}{x^2-x-6} = \frac{2}{x-3} + \frac{1}{x+2}$$

$$\Rightarrow \int \frac{3x+1}{x^2-x-6}dx = 2\int \frac{1}{x-3} + \int \frac{1}{x+2}dx$$

$$= 2\ln|x-3| + \ln|x+2| + c$$

Q2. $\int \frac{5x+8}{(x+3)(2x-1)}dx$

Solution: $\int \frac{5x+8}{(x+3)(2x-1)}dx$

Now.

$$\frac{5x+8}{(x+3)(2x-1)} = \frac{A}{x+3} + \frac{B}{2x-1}$$

⇨ $5x+8 = A(2x-1) + B(x+3) \to (i)$

$$Put\ 2x-1 = 0 \Rightarrow x = \frac{1}{2}\ in(i)$$

⇨ $5\left(\frac{1}{2}\right) + 8 = A(0) + B\left(\frac{1}{2}+3\right)$

⇨ $\frac{5+16}{2} = B\left(\frac{1+6}{2}\right) \Rightarrow 7B = 21 \Rightarrow B = 3$

Put $x+3 = 0 \Rightarrow x = -3\ in\ (i)$

⇨ $5(-3) + 8 = A(2(-3)-1) + B(0)$

⇨ $-15 + 8 = -7A \Rightarrow -7 = -7A \Rightarrow A = 1$

So $\frac{5x+8}{(x+3)(2x-1)} = \frac{1}{x+3} + \frac{3}{2x-1}$

$$\int \frac{5x+8}{(x+3)(2x-1)}dx$$

$$= \int \frac{1}{x+3}dx + 3\int \frac{1}{2x-1}dx$$

$$= \ln|x+3| + \frac{3}{2}\int \frac{2}{2x-1}dx$$

$$\int \frac{5x+8}{(x+3)(2x-1)}dx$$

$$= \ln|x+3| + \frac{3}{2}\ln|2x-1| + c$$

Q3. $\int \frac{x^2+3x-34}{x^2+2x-15}dx$

Solution: $\int \frac{x^2+3x-34}{x^2+2x-15}dx$

So $\int \left(1 + \frac{x-19}{x^2+2x-15}\right)dx$

$= \int 1dx + \int \frac{x-19}{x^2+2x-15}$

$$\begin{array}{r|l} & 1 \\ x^2+2x-15 & \overline{x^2+3x-34} \\ & \underline{\pm x^2 \pm 2x \mp 15} \\ & x-19 \end{array}$$

$Now\ \ \frac{x-19}{x^2+2x-15} = \frac{A}{x-3} + \frac{B}{x+5} \to (i)$

$\Rightarrow x - 19 = A(x+5) + B(x-3) \to (ii)$

$\because x^2+2x-15$
$= x^2+5x-3x-15$
$x(x+5) - 3(x+5)$
$(x-3)(x+5)$

$put\ x-3 = 0 \Rightarrow x = 3\ in(ii)$

$\Rightarrow 3 - 19 = A(3+5) + B(0)$

$\Rightarrow -16 = 8A \Rightarrow A = -2$

$put\ x+5 = 0 \Rightarrow x = -5\ in(ii)$

$\Rightarrow -5 - 19 = A(0) + B(-5-3) \Rightarrow -24$
$= -8B$

$\Rightarrow B = 3$

$$(i) \Rightarrow \frac{x-19}{x^2+2x-15} = -\frac{2}{x-3} + \frac{3}{x+5}$$

Thus,

$.\int \frac{x^2+3x-34}{x^2+2x-15}dx = \int 1dx + \int \frac{-2}{x-3}dx + \int \frac{3}{x+5}dx$

$$= x - 2\ln|x-3| + 3\ln|x+5| + c$$

Q4. $\int \frac{(a-b)x}{(x-a)(x-b)}dx, (a > b)$

Solution: $\int \frac{(a-b)x}{(x-a)(x-b)}dx$

Now

$$\frac{(a-b)x}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

$$\Rightarrow (a-b)x = A(x-b) + B(x-a) \to (i)$$

$$Put\ x-a = 0 \Rightarrow x = a\ in\ (i)$$

$$\Rightarrow (a-b).a = A(a-b) + B(a-a)$$

$$(a-b).b = A(0) + B(a-b) \Rightarrow A = a$$

$$Put\ x-b = 0 \Rightarrow x = b\ in\ (i)$$

$$\Rightarrow (a-b).b = A(0) + B(b-a)$$

$$(a-b).b = -B(a-b)$$

$$b = -B$$

$$B = -b$$

Thus

$$\frac{(a-b)x}{(x-a)(x-b)} = \frac{a}{x-a} + \frac{-b}{x-b}$$

$$\int \frac{(a-b)x}{(x-a)(x-b)}dx$$

$$= \int \frac{a}{x-a}dx - \int \frac{b}{x-b}dx$$

$$= a\ln|x-a| - b\ln|x-b| + c$$

Q5. $\int \frac{3-x}{1-x-6x^2}dx$

Solution: $\int \frac{3-x}{1-x-6x^2}dx$

$\because 1-x-6x^2$
$= -6x^2 - x + 1$
$-3x(2x+1) + 1(2x+1)$
$(2x+1)(1-3x)$

Now

$\frac{3-x}{1-x-6x^2} = \frac{A}{2x+1} + \frac{B}{1-3x}$

$\Rightarrow 3 - x = A(1-3x) + B(2x+1) \to (i)$

$$Put\ 2x+1 = 0 \Rightarrow x = -\frac{1}{2}in\ (i)$$

$\Rightarrow 3 - \left(-\frac{1}{2}\right) = A\left(1 - 3\left(-\frac{1}{2}\right) + B(0)\right)$

$\Rightarrow 3 + \frac{1}{2} = A\left(1 + \frac{3}{2}\right) \Rightarrow \frac{7}{2} = A\left(\frac{5}{2}\right)$

$\Rightarrow A = \frac{7}{2}$

$Put\ 1-3x = 0 \Rightarrow 1 = 3x \Rightarrow x = \frac{1}{3}in\ (i)$

$$\Rightarrow 3 - \frac{1}{3} = A(0) + B\left(2\left(\frac{1}{3}\right) + 1\right)$$

$$\frac{9-1}{3} = B\left(\frac{2+3}{3}\right) \Rightarrow 8 = 5B \Rightarrow \frac{8}{5}$$

So

$$\frac{3-x}{1-x-6x^2} = \frac{7/5}{2x+1} + \frac{8/5}{1-3x}$$

$: \int \frac{3-x}{1-x-6x^2}dx = \frac{7}{5}\int \frac{1}{2x+1}dx + \frac{8}{5}\int \frac{1}{1-3x}dx$

$$= \frac{7}{10}\int \frac{2}{2x+1}dx - \frac{8}{15}\int \frac{-3}{1-3x}dx$$

$$= \frac{7}{10}\ln|2x+1| - \frac{8}{5}\ln|1-3x| + C$$

Q.6 $\int \frac{2x}{x^2-a^2}\,dx$

Solution: $\int \frac{2x}{x^2-a^2}\,dx$

Now

$$\frac{2x}{x^2-a^2} = \frac{A}{x-a} + \frac{B}{x+a} \qquad \because x^2-a^2 = (x-a)(x+a)$$

$$\Rightarrow 2x = A(x+a) + B(x-a) \rightarrow (i)$$

Put $x - a = 0 \Rightarrow x = a \; in \; (i)$

$\Rightarrow 2a = A(a+a) + B(0) \Rightarrow 2a = 2A \Rightarrow A = 1$

Put $x + a = 0 \Rightarrow x = -a \; in \; (i)$

$\Rightarrow 2(-a) = A(0) + B(-a-a) \Rightarrow -2a = -2aB$

$$\Rightarrow B = 1$$

So $\frac{2x}{x^2-a^2} = \frac{1}{x-a} + \frac{1}{x+a}$

$$\int \frac{2x}{x^2-a^2}\,dx = \int \frac{1}{x-a}\,dx + \int \frac{1}{x+a}\,dx$$

$$= \ln|x-a| + \ln|x+a| + c$$

$$= \ln|(x-a)(x+a)| + c$$

$$= \ln|x^2-a^2| + c$$

Q.7 $\int \frac{1}{6x^2+5x-4}\,dx$

Solution: $\int \frac{1}{6x^2+5x-4}\,dx$

Now

$$\because 6x^2+5x-6 = 6x^2-3x+8x-4 \qquad 3x(2x-1)+4(2x-1) \qquad (2x-1)(3x+4)$$

$$\frac{1}{(2x-1)(3x+4)} = \frac{A}{2x-1} + \frac{B}{3x+4}$$

$\Rightarrow 1 = A(3x+4) + B(2x-1) \rightarrow (i)$

Put $2x - 1 = 0 \Rightarrow x = \frac{1}{2} in (i)$

$\Rightarrow 1 = A\left(3\left(\frac{1}{2}\right)+4\right) + B(0) \Rightarrow 1 = A\left(\frac{3+8}{2}\right)$

$\Rightarrow 3 = -11B \Rightarrow B = -\frac{3}{11}$

$$Put \; 3x + 4 = 0 \Rightarrow x = -\frac{4}{3} in \; (i)$$

$\Rightarrow 1 = A(0) + B\left(2\left(-\frac{4}{3}\right)-1\right) \Rightarrow 1 = B\left(\frac{-8-3}{3}\right)$

$$\Rightarrow 3 = -11B \Rightarrow B = -\frac{3}{11}$$

So

$$\frac{1}{(2x-1)(3x+4)} = \frac{A}{2x-1} + \frac{B}{3x+4}$$

$$1 = A(3x+4) + B(2x-1) \rightarrow (i)$$

$$Put \; 2x - 1 = 0 \Rightarrow x = \frac{1}{2} \; put \; in \; (i)$$

$$\Rightarrow 1 = A\left(3\left(\frac{1}{2}\right)+4\right) \Rightarrow 1 = A\left(\frac{3}{2}+4\right)$$

$$\Rightarrow 1 = A\left(\frac{3+8}{2}\right) \Rightarrow 1 = A\left(\frac{11}{2}\right) \Rightarrow A = \frac{2}{11}$$

Put $3x + 4 = 0 \Rightarrow x = -\frac{4}{3}$ put in (i)

$$\Rightarrow 1 = A(0) + B\left(2\left(-\frac{4}{3}\right)-1\right) \Rightarrow 1$$

$$= B\left(\frac{-8-3}{3}\right)$$

$$\Rightarrow 3 = -11B \Rightarrow B = -\frac{3}{11}$$

So, $\frac{1}{6x^2+5x-4} = \frac{\frac{2}{11}}{2x-1} + \frac{-3/11}{3x+4}$

$\Rightarrow \int \frac{1}{6x^2+5x-4}\,dx = \frac{1}{11}\int \frac{2}{2x-1}\,dx - \frac{1}{11}\int \frac{3}{3x+4}\,dx$

$$= \frac{1}{11}\ln|2x-1| - \frac{1}{11} ln|3x+4| + c$$

$$= \frac{1}{11}\ln\left|\frac{2x-1}{3x+4}\right| + c$$

Q.8 $\int \frac{2x^2-3x^2-x-7}{2x^2-3x-2}\,dx$

$$\begin{array}{r} x \\ 2x^2-3x-2\overline{\big)\,2x^3-3x^2-x-7} \\ \underline{\pm 2x^3 \pm 3x^2 \mp 2x} \\ x-7 \end{array}$$

$$\int \frac{2x^2-3x^2-x-7}{2x^2-3x-2}\,dx = \int \left(x + \frac{x-7}{2x^2-3x-2}\right)dx$$

$$= \int x dx + \int \frac{x-7}{2x^2-3x-2}\,dx$$

NOw

$$\frac{x-7}{(x-2)(2x+1)} = \frac{A}{x-2} + \frac{B}{2x+1} \qquad \because 2x^2-3x-2 = 2x^2-4x+x-2 \qquad 3x(x-2)+1(x-2)) \qquad (x-2)(3x+1)$$

$x - 7 = A(2x+1) + B(x-2) \rightarrow (i)$

$\Rightarrow Put \; x - 2 = 0 \Rightarrow x = 2 \; in \; (i)$

$\Rightarrow 2 - 7 = A(2(2)+1) + B(0) \Rightarrow -5 = 5A \Rightarrow A = -\frac{5}{5} = -1$

$$A = -1$$

$$Put \; 2x + 1 = 0 \Rightarrow x = -\frac{1}{2} \; in \; (i)$$

$\Rightarrow -\frac{1}{2} - 7 = A(0) + B\left(-\frac{1}{2}-2\right) \Rightarrow \frac{-1-14}{2} = B\left(\frac{-1-4}{2}\right)$

$\Rightarrow -15 = -5B \Rightarrow B = 3$

So

$$\frac{x-7}{2x^2-3x-2} = \frac{-1}{x-2} + \frac{3}{2x+1}$$

Thus $\int \frac{2x^2-3x^2-x-7}{2x^2-3x-2}\,dx = \int xdx = \int \frac{1}{x-2}\,dx + 3\int \frac{1}{2x+!}\,dx$

$$= \frac{x^2}{2} - \ln|x-2| + \frac{3}{2}\int \frac{2}{2x+1}\,dx$$

$$= \frac{x^2}{2} - \ln|x-2| + \frac{3}{2}\ln|2x+1| + c$$

Q.9 $\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}\,dx$

Solution: $\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}\,dx$

Now

$$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$
$$3x^2-12x+11=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)\rightarrow(i)$$
Put $x-1=0\Rightarrow x=1\ in\ (i)$
$$\Rightarrow 3(1)^2-12(1)+11=A(1-2)(1-3)+B(0)+C(0)$$
$$=3-12+11=A(-1)(-2)$$
$$\Rightarrow 2=2A\Rightarrow A=1$$
$$Put\ x-2=0\Rightarrow x=2in(i)$$
$$\Rightarrow 3(2)^2-12(2)+11=A(0)+B(2-1)(2-3)+C(0)$$
$$\Rightarrow 12-24+11=-B$$
$$\Rightarrow -1=-B\Rightarrow B=1$$
$$Put\ x-3=0\Rightarrow x=3\ in(i)$$
$$\Rightarrow 3(3)^2-12(3)+11=A(0)+B(0)+C(3-1)(3-2)$$
$$\Rightarrow 3(9)-36+11=C(2)(1)$$
$$\Rightarrow 27-36+11=2C$$
$$\Rightarrow 2=2C\Rightarrow C=1$$
So
$$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}=\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}$$
$$\int\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx=\int\frac{1}{x-1}dx+\int\frac{1}{x-2}dx+\int\frac{1}{x-3}dx$$
$$=\ln|x-1|+\ln|x-2|+\ln|x-3|+c$$

Q10. $\int\frac{2x-1}{x(x-1)(x-3)}dx$

Solution: $\int\frac{2x-1}{x(x-1)(x-3)}dx$

Now
$$\frac{2x-1}{x(x-1)(x-3)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-3}$$
$$2x-1=A(x-1)(x-3)+B(x)(x-3)+C(x)(x-1)\rightarrow(i)$$
$$Put x=0\ in\ (i)$$
$$2(0)-1=A(0-1)(0-3)+B(0)(C(0)$$
$$\Rightarrow -1=A(-1)(-3)\Rightarrow -1=3A\Rightarrow A=-\frac{1}{3}$$
Put $x-1=0\Rightarrow x=1\ in\ (i)$
$$\Rightarrow 2(1)-1=A(0)+B(1)(1-3)+C(0)$$
$$\Rightarrow 1=B(-2)\Rightarrow B=-\frac{1}{2}$$
Put $x-3=0\Rightarrow x=3\ in\ (i)$
$$\Rightarrow 2(3)-1=A(0)+B(0)+C(3)(3-1)$$
$$\Rightarrow 5=6C\Rightarrow C=\frac{5}{6}$$
So
$$\frac{2x-1}{x(x-1)(x-3)}=\frac{-1}{3x}+\frac{-1}{2(x-1)}+\frac{5}{6(x-3)}$$
$$\frac{2x-1}{x(x-1)(x-3)}=-\frac{1}{3x}-\frac{1}{2(x-1)}+\frac{5}{6(x-3)}$$
$$\int\frac{x-1}{x(x-1)(x-3)}dx=-\frac{1}{3}\int\frac{1}{x}dx-\frac{1}{2}\int\frac{1}{x-1}+\frac{5}{6}\int\frac{1}{x-3}$$
$$=-\frac{1}{3}\ln|x|+\frac{1}{2}\ln|x-1|+\frac{5}{6}\ln|x-3|+c$$

Q.11 $\int\frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx$

Solution: $\int\frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx$

Now
$$\frac{5x^2+9x+6}{(x^2-1)(2x+3)}=\frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{2x+3}$$
$$5x^2+9x+6=A(x+1)(2x+3)+B(x-1)(2x+3)+C(x+1)(x-1)$$
Put $x+1=0\Rightarrow x=-1\ in(i)$
$$5(-1)^2+9(-1)+6=A(0)+B(-1-1)(2(-1)+3)+C(0)$$
$$\Rightarrow 5-9+6=B(-2)(1)$$
$$\Rightarrow 2=-2B\Rightarrow B=-1$$
Put $x-1=0\Rightarrow x=1\ in(i)$
$$\Rightarrow 5(1)^2+9(1)+6=A(1+1)(2(1)+3)+B(0)+C(0)$$
$$\Rightarrow 5+9+6=A(2)(5)\Rightarrow 20=A10\Rightarrow A=2`$$
Put $2x+3=0\Rightarrow x=-\frac{3}{2}\ in\ (i)$
$$\Rightarrow 5\left(-\frac{3}{2}\right)^2+9\left(-\frac{3}{2}\right)+6=A(0)+B(0)+C\left(-\frac{3}{2}+1\right)\left(-\frac{3}{2}-1\right)$$
$$5\left(\frac{9}{4}\right)+\left(-\frac{27}{2}\right)+6=C\left(\frac{-3+2}{2}\right)\left(\frac{-3-2}{2}\right)$$
$$\frac{45}{4}-\frac{27}{2}+6=C\left(-\frac{1}{2}\right)\left(-\frac{5}{2}\right)$$
$$\frac{45-54+24}{4}=C\frac{5}{4}$$
$$\Rightarrow 15=5C\Rightarrow C=3$$
$$\frac{5x^2+9x+6}{(x^2-1)(2x+3)}=\frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)}=\frac{2}{x-1}+\frac{-1}{x+1}+\frac{3}{2x+3}$$
$$:\int\frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx=2\int\frac{1}{x-1}dx-1\int\frac{1}{x+1}dx+\frac{3}{2}\int\frac{2}{2x+3}dx$$
$$=2\ln|x-1|-ln|x+1|+\frac{3}{2}|2x+3|+c$$

Q12. $\int\frac{4+7x}{(1+x)^2(2+3x)}dx$

Solution:

$$\int \frac{\boldsymbol{4+7x}}{\boldsymbol{(1+x)^2(2+3x)}}\boldsymbol{dx}$$

Now

$$\frac{\boldsymbol{4+7x}}{\boldsymbol{(1+x)^2(2+3x)}}=\frac{\boldsymbol{A}}{\boldsymbol{1+x}}+\frac{\boldsymbol{B}}{\boldsymbol{(1+x)^2}}+\frac{\boldsymbol{C}}{\boldsymbol{2x+3}}$$

$$\Rightarrow 4+7x = A(1+x)(2x+3)+B(2x+3) + C(1+x)^2 \to (i)$$

Put $1+x=0 \Rightarrow x=-1\ in\ (i)$

$$\Rightarrow 4+7(-1)=A(0)+B(-2+3)+C(0)$$

$$\Rightarrow -3 = B \Rightarrow B=-3$$

$$Put\ 2+3x=0 \Rightarrow x=-\frac{2}{3}\ in(i)$$

$$4+7\left(-\frac{2}{3}\right)=A(0)+B(0)+C\left(1-\frac{2}{3}\right)^2$$

$$4-\frac{14}{3}=C\left(\frac{3-2}{3}\right)^2$$

$$\frac{12-14}{3}=C\left(\frac{1}{9}\right)$$

$$\Rightarrow -\frac{2}{3}=\frac{1}{9}C \ \Rightarrow -\frac{2}{3}\times\frac{9}{1}=C \Rightarrow C=-6$$

From (i)

$$4+7x = A(2+3x+2x+3x^2)+2B+3Bx + C(1+2x+x^2)$$

$$\Rightarrow 4+7x = 2A+5Ax+3x^2A+2B+3Bx+C + 2Cx+cx^2$$

Equating coefficient of x^2

$$0=3A+C \Rightarrow 3A=-C \Rightarrow 3A=-(-6)$$

$$\Rightarrow 3A=6 \Rightarrow A=\frac{6}{3}=2 \Rightarrow A=2$$

So,

$$\frac{\boldsymbol{4+7x}}{\boldsymbol{(1+x)^2(2+3x)}}=\frac{\boldsymbol{2}}{\boldsymbol{1+x}}+\frac{\boldsymbol{3}}{\boldsymbol{(1+x)^2}}-\frac{\boldsymbol{6}}{\boldsymbol{2x+3}}$$

$$\int\frac{4+7x}{(1+x)^2(2+3x)}dx = 2\int\frac{1}{1+x}dx+3\int(1+x^2)^{-2}dx + \frac{6}{3}\int\frac{3}{2+3x}dx$$

$$=2\ln|1+x|+\frac{3(1+x)^{-1}}{-1}-2\ln|2+3x|+c$$

$$\ln|1+x|^2-\frac{3}{1+x}-\ln|2+3x|^2+c$$

Q.13 $\int \frac{2x^2}{(x-1)^2(x+1)}dx$

Solution:

Now

$$\frac{2x^2}{(x-1)^2(x+1)} = \frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)}$$

$$\Rightarrow 2x^2 = A(x-1)(x+1)+B(x+1)+C(x-1)^2$$

Put$x-1=0 \Rightarrow x=1\ in(i)$

$$\Rightarrow 2(1)^2 = A(0)+B(1+1)+C(0)$$

$$\Rightarrow 2=2B \Rightarrow B=1$$

Put $x+1=0 \Rightarrow x=-1\ in\ (i)$

$$\Rightarrow 2(-1)^2=A(0)+B(o)+C(-1-1)^2$$

$$2=4C \Rightarrow C=\frac{1}{2}$$

From (i)

$$2x^2=A(x^2-1)+Bx+B+C(x^2+1-2x)$$

$$\Rightarrow 2x^2 = Ax^2-A+Bx+B+Cx^2+C-2Cx$$

Equating coefficients of x^2, *we have*

$$\Rightarrow 2=A+C \Rightarrow 2=A+\tfrac{1}{2} \Rightarrow A=2-\tfrac{1}{2}$$

$$\Rightarrow A=\frac{3}{2}$$

So,

$$\frac{2x^2}{(x-1)^2(x+1)} = \frac{3/2}{(x-1)}+\frac{1}{(x-1)^2}+\frac{1/2}{(x+1)}$$

$$\int\frac{2x^2}{(x-1)^2(x+1)}dx = \frac{3}{2}\int\frac{1}{x-1}dx+\int(x-1)^{-2}dx + \frac{1}{2}\int\frac{1}{x+1}dx$$

$$=\frac{3}{2}\ln|x-1|+\frac{(x-1)^{-1}}{-1}+\frac{1}{2}\ln|x+1|+c$$

$$=\frac{3}{2}\ln|x-1|-\frac{1}{x-1}+\frac{1}{2}\ln|x+1|+c$$

Q.14 $\int\frac{1}{(x-1)(x+1)^2}dx$

Solution: $\int\frac{1}{(x-1)(x+1)^2}dx$

Now

$$\frac{\boldsymbol{1}}{\boldsymbol{(x-1)(x+1)^2}}=\frac{\boldsymbol{A}}{\boldsymbol{x-1}}+\frac{\boldsymbol{B}}{\boldsymbol{x+1}}+\frac{\boldsymbol{C}}{\boldsymbol{(x+1)^2}}$$

$$\Rightarrow 1 = A(x+1)^2+B(x-1)(x+1)+C(x-1) \to (i)$$

Put $x-1=0 \Rightarrow x=1\ \ in\ (i)$

$$1=A(1+1)^2 \Rightarrow 1=4A \Rightarrow A=\frac{1}{4}$$

Put $x+1=0 \Rightarrow x=-1\ in\ (i)$

$$1=C(-1-1)$$

$$\Rightarrow 1=-2C \Rightarrow C=-\frac{1}{2}$$

From (i)

$$\Rightarrow 1 = A(x^2+2x+1)+B(x^2-1)+Cx-C$$

$$\Rightarrow 1=Ax^2+2Ax+A+Bx^2-B+Cx-C$$

Equating coefficient of x^2, *we have*

$$0=A+B \Rightarrow 0=\frac{1}{4}+B \Rightarrow B=-\frac{1}{4}$$

$$\frac{1}{(x-1)(x+1)^2}=\frac{1/4}{x-1}+\frac{-1/4}{x+1}+\frac{-1/2}{(x+1)^2}$$

$\int \frac{1}{(x-1)(x+1)^2}dx$

$= \frac{1}{4}\int \frac{1}{x-1}dx - \frac{1}{4}\int \frac{1}{x+1}dx - \frac{1}{2}\int (x+1)^{-2}dx$

$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\frac{(x+1)^{-1}}{-1} + C$

$= \{\frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1|\} + \frac{1}{2(x+1)} + C$

Q.15 $\int \frac{x+4}{x^3-3x^2+4}dx$

Solution: $\int \frac{x+4}{x^3-3x^2+4}dx$

Now

$\because x^3 - 3x^2 + 4 = x^3 + x^2 - 4x^2 + 4$

$= x^2(x+1) - 1(x^2-1)$

$= x^2(x+1) - 4(x-1)(x+1)$

$= (x+1)(x^2-4x+4)$

$\Rightarrow x^3 - 3x^2 + 4 = (x+1)(x-2)^2$

Now

$$\frac{x+4}{x^3-3x^2+4} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$\Rightarrow x + 4 = A(x-2)^2 + B(x+1)(x-2) + C(x+1) \rightarrow (i)$

Put $x + 1 = 0 \Rightarrow x = -1\ in (i)$

$\Rightarrow -1 + 4 = A(-1-2)^2 + B(0) + C(0)$

$\Rightarrow 3 = 9A \Rightarrow A = \frac{1}{3}$

Put $x - 2 = 0 \Rightarrow x = 2\ in\ (i)$

$\Rightarrow 2 + 4 = A(0) + B(0) + C(2+1)$

$\Rightarrow 6 = 3C \Rightarrow C = 2$

From (i)

$x + 4 = A(x^2-4x+4) + B(x^2-2x+x-2) + Cx + C$

$\Rightarrow x + 4 = Ax^2 - 4Ax + 4A + Bx^2 - Bx - 2B + Cx + C$

Equating coefficients of x^2

$\Rightarrow 0 = A + B \Rightarrow 0 = \frac{1}{3} + B \Rightarrow B = -\frac{1}{3}$

$$\frac{x+4}{x^3-3x^2+4} = \frac{1/3}{x+1} + \frac{-1/3}{x-2} + \frac{2}{(x-2)^2}$$

$\int \frac{x+4}{x^3-3x^2+4}dx$

$= \frac{1}{3}\int \frac{1}{x+1}dx - \frac{1}{3}\int \frac{1}{x-2} + 2\int (x-2)^{-2}dx$

$= \frac{1}{3}ln|x+1| - \frac{1}{3}\ln|x-2| + 2\frac{(x-2)^{-1}}{-1} + c$

$= \frac{1}{3}ln|x+1| - \frac{1}{3}\ln|x-2| - \frac{2}{x-2} + c$

$= \frac{1}{3}\{\ln|x+1| - \ln|x-2|\} - \frac{2}{x-2} + c$

Q16. $\int \frac{x^3-6x^2+25}{(x+1)^2(x-2)^2}dx$

Solution:

$$\frac{x^3-6x^2+25}{(x+1)^2(x-2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x-2)} + \frac{D}{(x-2)^2}$$

$\Rightarrow x^3 - 6x^2 + 25 = A(x+1)(x-2)^2 + B(x-2)^2 + C(x+1)^2(x-2) + D(x+1)^2 \rightarrow (i)$

Put $x + 1 = 0 \Rightarrow x = -1\ in\ (i)$

$\Rightarrow (-1)^3 - 6(-1)^2 + 25 = A(0) + B(-1-2)^2 + C(0) + D(0)$

$-1 - 6 + 25 = 9B$

$9B = 18 \Rightarrow B = 2$

Put $x - 2 = 0 \Rightarrow x = 2\ in\ (i)$

$\Rightarrow (2)^3 - 6(2)^2 + 25 = D(2+1)^2$

$\Rightarrow 8 - 24 + 25 = 9D$

$9 = 9D \Rightarrow D = 1$

From (i)

$x^3 - 6x^2 + 25 = A(x+1)(x^2-4x+4) + B(x^2-4x+4)$

$+C(x^2+1+2x)(x-2) + D(x^2+1+2x)$

$= A(x^3-4x62+4x+x62-4x+4) + Bx^2 - 4Bx + 4B$

$+C(x^3-2x^2+x-2+2x^2-4x) + Dx^2 + D + 2Dx$

$= Ax^3 - 3Ax^2 + 4A + Bx^2 - 4Bx + 4B + Cx^3 - 3Cx + Dx^2 + D + 2Dx$

Equating coefficients of x^3 and x^2

$FOr\ x^3$

$1 = A + C \rightarrow (ii)$

For x^2 $\quad -6 = -3A + B + D$

$-6 = -3A + 2 + 1$

$-6 - 3 = -3A \Rightarrow -9 = -3A \Rightarrow A = 3\ put\ in\ (ii)$

$1 = 3 + C \Rightarrow C = 1 - 3 = -2 \Rightarrow C = -2$

$$\frac{x^3-6x^2+25}{(x+1)^2(x-2)^2} = \frac{3}{x+1} + \frac{2}{(x+1)^2} - \frac{2}{(x-2)} + \frac{1}{(x-2)^2}$$

$\int \frac{x^3-6x^2+25}{(x+1)^2(x-2)^2}dx$

$= 3\int \frac{1}{x+1}dx + 2\int (x+1)^{-2}dx - 2\int \frac{1}{x-2}dx + \int (x-2)^{-2}dx$

$= 3ln|x+1| + 2\frac{(x+1)^{-1}}{-1} - 2ln|x-2| + \frac{(x-2)^{-1}}{-1} + C$

$= 3ln|x+1| - \frac{2}{x+1} - 2ln|x-2| - \frac{1}{x-2} + C$

Q.17 $\int \frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3}dx$

Solution:

$$\frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3} = \frac{A}{x-3}+\frac{B}{x+2}+\frac{C}{(x+2)^2}+\frac{D}{(x+2)^3}$$

$\Rightarrow x^3+22x^2+14x-17 = A(x+2)^3+B(x-3)(x+2)^2+C(x-3)(x+2)+D(x-3) \rightarrow (i)$

$$Put\ x-3=0 \Rightarrow x=3\ in\ (i)$$

$$\Rightarrow (3)^3+22(3)^2+14(3)-17 = A(3+2)^3$$

$\Rightarrow 27+198+42-17 = 125A$

$$\Rightarrow 250 = 125A \Rightarrow A = 2$$

$$Put\ x+2=0 \rightarrow x=-2\ in(i)$$

$$\Rightarrow (-2)^3+22(-2)^2+14(-2)-17 = D(-2-3)$$

$$-8+88-28-17 = -5D \Rightarrow 35 = -5D$$

$$D = -7$$

From (i)

$x^3+22x^2+14x-17$

$= A[x^3+6x^2+12x+8]+B(2-3)(x^2+4x+4)$
$+C(x^2+2x-3x-6)+Dx-3D$

$= Ax^3+6Ax^2+12Ax+8A$
$+B(x^3+4x^2+4x-3x^2-12x-12)+Cx^2-Cx-6c+Dx-3D$

Equating coefficients of x^2 and x^3

$$For\ x^3; 1 = A+B \Rightarrow 1 = 2+B \Rightarrow B = -1$$

$$For\ x^2;\quad 22 = 6A+B+C \Rightarrow 22 = 6(2)-1+C$$

$$\Rightarrow C = 22-12+1 = 11 \Rightarrow C = 11$$

So

$$\frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3} = \frac{2}{x-3}+\frac{1}{x+2}+\frac{11}{(x+2)^2}-\frac{7}{(x+2)^3}$$

$$\int\frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3}dx$$

$$= 2\int\frac{1}{x-3}-\int\frac{1}{x+3}dx+\int(x+2)^{-2}dx-7\int(x+2)^{-3}dx$$

$$= 2ln|x-3|-ln|x+2|+11\frac{(x+2)^{-1}}{-1}-7\frac{(x+2)^{-2}}{-2}+c$$

$$= 2ln|x-3|-ln|x+2|-\frac{11}{x+2}+\frac{7}{2}\frac{1}{(x+2)^2}+c$$

Q.18 $\int\frac{x-2}{(x+1)(x^2+1)}dx$

Solution:

$$\frac{x-2}{(x+1)(x^2+1)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$

$$\Rightarrow x-2 = A(x^2+1)+(Bx+C)(x+1) \rightarrow (i)$$

$$Put\ x+1=0 \Rightarrow x=-1\ in(i)$$

$$\Rightarrow -1-2 = A((-1)^2+1)$$

$$-3 = 2A \Rightarrow A = -\frac{3}{2}$$

From (i)

$$\Rightarrow x-2 = Ax^2+A+Bx^2+Bx+Cx+C$$

Equating coefficients of x^2 and x

$$For\ x^2;\quad 0 = A+B \Rightarrow 0 = -\frac{3}{2}+B \Rightarrow B = \frac{3}{2}$$

$$For\ x; 1 = B+C \Rightarrow 1 = \frac{3}{2}+C$$

$$\Rightarrow C = 1-\frac{3}{2} = -\frac{1}{2} \Rightarrow C = -\frac{1}{2}$$

So

$$\frac{x-2}{(x+1)(x^2+1)} = \frac{-3/2}{x+1}+\frac{\frac{3}{2}x-\frac{1}{2}}{x^2+1}$$

$$\int\frac{x-2}{(x+1)(x^2+1)}dx = -\frac{3}{2}\int\frac{1}{x+1}dx+\frac{1}{2}\int\frac{3x-1}{x^2+1}dx$$

$$= -\frac{3}{2}ln|x+1| +\frac{1}{2}\int\frac{3x}{x^2+1}dx-\frac{1}{2}\int\frac{1}{x^2+1}dx$$

$$= -\frac{3}{2}\ln|x+1|+\frac{3}{2}.\frac{1}{2}\ln|x^2+1|-\frac{1}{2}\tan^{-1}x$$

$$= -\frac{3}{2}\ln|x+!|+\frac{3}{4}\ln|x^2+1|-\frac{1}{2}\tan^{-1}x+c$$

Q.19 $\int\frac{x}{(x-1)(x^2+1)}dx$

Solution: $\int\frac{x}{(x-1)(x^2+1)}dx$

Now

$$\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$

$$\Rightarrow x = A(x^2+1)+(Bx+C)(x-1) \rightarrow (i)$$

$Put x-1=0 \Rightarrow x=1\ in(i)$

$$\Rightarrow 1 = A((1)^1+1)$$

$$\Rightarrow A = \frac{1}{2}$$

From (i)

$$\Rightarrow x = Ax^2+A+Bx^2-Bx+Cx-C$$

Equating coefficients of x^2 and x we have

$$For\ x^2\ ;\quad \Rightarrow 0 = A+B \Rightarrow 0 = \frac{1}{2}+B \Rightarrow B = -\frac{1}{2}$$

$$For\ x;\quad 1 = -B+C \Rightarrow 1 = -\left(-\frac{1}{2}\right)+C$$

$$\Rightarrow 1 = \frac{1}{2}+C \Rightarrow 1-\frac{1}{2} = C \Rightarrow C = \frac{1}{2}$$

So,

$$\frac{x}{(x-1)(x^2+1)} = \frac{1/2}{x-1}+\frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}$$

$$\int\frac{x}{(x-1)(x^2+1)}dx = \frac{1}{2}\int\frac{1}{x-1}dx-\frac{1}{2}\int\frac{x-1}{x^2+!}dx$$

$$= \frac{1}{2}ln|x-1|-\frac{1}{4}\int\frac{2x-2}{x^2+1}dx$$

$$\frac{1}{2}ln|x-1|-\frac{1}{4}\int\frac{2x}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x^2+1}dx$$
$$=\frac{1}{2}ln|x-1|-\frac{1}{4}ln|x^2+1|+\frac{1}{2}tan^{-1}x+c$$

Q.20 $\int\frac{9x-7}{(x+3)(x^2+1)}dx$

Solution: $\int\frac{9x-7}{(x+3)(x^2+1)}dx$

Now

$$\frac{9x-7}{(x+3)(x^2+1)}=\frac{A}{x+3}+\frac{Bx+C}{x^2+1}$$
$$\Rightarrow 9x-7=A(x^2+1)+(Bx+C)(x+3)\rightarrow(i)$$

Put $x+3=0\Rightarrow x=-3\ in\ (i)$
$$\Rightarrow 9(-3)-7=A((-3)^2+1)$$
$$-27-7=10A\Rightarrow -34=10A\Rightarrow A=-\frac{34}{10}$$
$$\Rightarrow A=-\frac{17}{5}$$

From (i)
$$9x-7=Ax^2+A+Bx^2+3Bx+Cx+3C$$

Equating coefficients of x^3 and x
$$For\ x^2\ ,\Rightarrow 0=A+B\Rightarrow 0=-\frac{17}{5}+B\Rightarrow B=\frac{17}{5}$$
$$and for\ x;3B+C=9\Rightarrow 3\left(\frac{17}{5}\right)+C=9\Rightarrow\frac{51}{5}+C=9$$
$$C=9-\frac{51}{5}=\frac{45-51}{5}\Rightarrow C=-\frac{6}{5}$$

So
$$\frac{9x-7}{(x+3)(x^2+1)}=\frac{-17/5}{x+3}+\frac{\frac{17}{5}x-\frac{6}{5}}{x^2+1}$$
$$\int\frac{9x-7}{(x+3)(x^2+1)}dx$$
$$=-\frac{17}{5}\int\frac{1}{x+3}dx$$
$$+\frac{17}{5}\int\frac{x}{x^2+1}dx-\frac{6}{5}\int\frac{1}{x^2+1}dx$$
$$=-\frac{17}{5}\ln|x+3|+\frac{17}{10}\ln|x^2+1|-\frac{6}{5}\tan^{-1}x+c$$

Q21. $\int\frac{1+4x}{(x-3)(x^2+4)}dx$

Solution: $\int\frac{1+4x}{(x-3)(x^2+4)}dx$

Now
$$\frac{1+4x}{(x-3)(x^2+4)}=\frac{A}{x-3}+\frac{Bx+C}{x^2+4}$$
$$\Rightarrow 1+4x=A(x^2+4)+(Bx+C)(x-3)\rightarrow(i)$$
$$Put\ x-3=0\Rightarrow x=3\ in\ (i)$$
$$1+4(3)=A((3)^2+4)+B(3)+C(0)$$
$$\Rightarrow 13=A(9+4)\Rightarrow 13=13A\Rightarrow A=1$$

From (i)
$$1+4x=Ax^2+4a+Bx^2-3Bx+Cx-3C$$

Equating Coefficients of x^2 and x
$$\Rightarrow 0=A+B\quad for\ x^2$$
$$0=1+B\Rightarrow B=-1$$
$$\Rightarrow 4=-3B+C\Rightarrow 4-3=C\Rightarrow C=1$$

So
$$\frac{1+4x}{(x-3)(x^2+4)}=\frac{1}{x-3}+\frac{(-)x+1}{x^2+4}$$

$$\int\frac{1+4x}{(x-3)(x^2+4)}dx$$
$$=\int\frac{1}{x-3}dx-\int\frac{x}{x^2+4}dx$$
$$+\int\frac{1}{x^2+4}dx$$
$$=\ln|x-3|-\frac{1}{2}\int\frac{2x}{x^2+4}dx+\int\frac{1}{x^2+4}dx$$
$$=\ln|x-3|-\frac{1}{2}|x^2+4|+\frac{1}{2}\tan^{-1}\frac{x}{2}+c$$

Q.22
$$\int\frac{12}{x^3+8}dx$$

Solution:

$\int\frac{12}{x^3+8}dx\quad\because a^3-b^3=(a+b)(a^2-ab+b^2)$

Now
$$\frac{12}{x^3+8}=\frac{A}{x+2}+\frac{Bx+C}{x^2-2x+4}$$
$$\Rightarrow 12=A(x^2-2x+4)+(Bx+C)(x+2)\rightarrow(i)$$

Put $x+2=0\Rightarrow x=-2\ in\ (i)$

$\Rightarrow 12=A(4+4+4)\Rightarrow 12=12A\Rightarrow A=1$

From (i)

$12=Ax^2-2Ax+4A+Bx^2+2Bx+Cx+2C$

Equating coefficients of x^2 and x we have

$for\ x^2;\ 0=A+B\Rightarrow 0=1+B\Rightarrow B=-1$

$for\ x;\ 0=-2(1)+2(-1)+C\Rightarrow 0=-2-2+C$

$\Rightarrow C=4$

So
$$\frac{12}{x^3+8}=\frac{1}{x+2}+\frac{-x+4}{x^2-2x+4}$$
$$\int\frac{12}{x^3+8}dx=\int\frac{1}{x+2}dx-\int\frac{x-4}{x^2-2x+4}dx$$
$$=\int\frac{1}{x+2}dx-\frac{1}{2}\int\frac{2x-8}{x^2-2x+4}dx$$
$$=ln|x+2|-\frac{1}{2}\int\frac{2x-2-6}{x^2-2x+4}dx$$
$$=ln\ |x+2|-\frac{1}{2}\int\frac{2x-2}{x^2-2x+4}dx$$
$$+\frac{6}{2}\int\frac{1}{x^2-2x+4}dx$$
$$=\ln|x+2|-\frac{1}{2}\ln|x^2-2x+4|$$
$$+3\int\frac{1}{x^2-2x+1+3}dx$$
$$=\ln|x+2|-\frac{1}{2}\ln|x^2-2x+4|$$
$$+3\int\frac{1}{(x-1)^2+\sqrt{3}}dx$$
$$=\ln|x+2|-\frac{1}{2}\ln|x^2-2x+4|+\frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right)+c$$
$$=\ln|x+2|-\frac{1}{2}\ln|x^2-2x+4|+\sqrt{3}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right)+c$$

Q23. $\int\frac{9x^2+6}{x^3-8}dx$

Solution:

$$\int \frac{9x^2+6}{x^3-8}dx$$

Now

$$\frac{9x^2+6}{x^3-8} = \frac{9x^2+6}{(x-2)(x^2+2x+4)}$$
$$= \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$$
$$\Rightarrow 9x+6 = A(x^3+2x+4)+(Bx+C)(x-2) \rightarrow (i)$$

Put $= x-2=0 \Rightarrow x=2$ in (i)

$$9(2)+6 = A[(2)^2+2(x)+4]+B(2)+C(0)$$

$\Rightarrow 24 = 12A \Rightarrow A = 2$

From (i)

$$9x+6 = Ax^2+2Ax+4A+Bx^2-2Bx+Cx-2C$$

Equating coefficient of x^2 and x

For $x^2; 0 = A+B \Rightarrow 0 = 2+B \Rightarrow B=-2$

$FOr\ x; 9 = 2A-2B+C \Rightarrow 9 = 2(2)-2(-2)+C$

$$\Rightarrow 9 = 4+4+C \Rightarrow 9-8 = C \Rightarrow C = 1$$

So

$$\frac{9x^2+6}{(x-2)(x^2+2x+4)} = \frac{2}{x-2}+\frac{-2x+1}{x^2+2x+4}$$
$$\int \frac{9x^2+6}{x^3-8}dx = 2\int\frac{1}{x-2}dx - \int\frac{2x-1}{x^2+2x+4}dx$$
$$= -2\int\frac{1}{x-2}dx - \int\frac{2x+2-2-1}{x^2+2x+4}dx$$
$$= -2\int\frac{1}{x-2}dx - \int\frac{2x+2}{x^2+2x+4}dx + 3\int\frac{1}{x^2+2x+4}dx$$
$$= 2\ln|x-2| - \ln|x^2+2x+4| + 3\int\frac{1}{(x+1)^2+(\sqrt{3})^2}dx$$
$$2\ln|x-2| - \ln|x^2+2x+4| + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+C$$
$$2\ln|x-2| - \ln|x^2+2x+4| + \sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+c$$

Q.24 $\int \frac{2x^2+5x+3}{(x-1)^2(x^2+4)}dx$

Solution: $\int \frac{2x^2+5x+3}{(x-1)^2(x^2+4)}dx$

Now

$$\frac{2x^2+5x+3}{(x-1)^2(x^2+4)} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+4}$$

$\Rightarrow 2x^2+5x+3 = A(x-1)(x^2+4)+B(x^2+4)+(Cx+D)(x-1)^2 \rightarrow (i)$

Put $x-1=0 \Rightarrow x = 1$ in (i)

$\Rightarrow 2(1)^2+5(1)+3 = B(1+4)$

$$2+5+3 = 5B \Rightarrow 10 = 5B \Rightarrow B = 2$$

From (i)

$$2x^2+5x+3 = A(x^3+4x-x^2-4)+Bx^2+4B+(Cx+D)(x^2+1+2x)$$
$$Ax^3+4Ax-Ax^2-4A+Bx^2+4B+Cx^3+Cx-2Cx^2+Dx^2+D-2Dx$$

Equating coefficient of x^3, x^2 *and* x *we get*

For $x^3 \Rightarrow 0 = A+C \Rightarrow C = -A \rightarrow (ii)$

For $x^2; 2 = -A+B-2C+D$

Put $B = 2$ and $C = -A$

$$2 = -A+2-2(-A)+D$$
$$\Rightarrow 2-2 = -A+2A+D \Rightarrow 0 = A+D$$
$$\Rightarrow D = -A \rightarrow (iii)$$

$For\ x; 5 = 4A+C-2D$ put $C = -A$ and $D = -A$

$$\Rightarrow 5 = 4A-A-2(-A)$$
$$5 = 3A+2A \Rightarrow 5 = 5A \Rightarrow A = 1$$

So (ii)$\Rightarrow C = -1$ and $(iii) \Rightarrow D = -1$

Thus

$$\frac{2x^2+5x+3}{(x-1)^2(x^2+4)} = \frac{1}{x-1}+\frac{2}{(x-1)^2}+\frac{(-1)x\pm 1}{x^2+4}$$

⇨ $\int \frac{2x^2+5x+3}{(x-1)^2(x^2+4)}dx = \int\frac{1}{x-1}dx + 2\int(x-1)^{-2} - \int\frac{x+1}{x^2+4}dx$

$$= \int\frac{1}{x-1}dx + 2\int(x-1)^{-2}dx - \int\frac{x}{x^2+4}dx - \int\frac{1}{x^2+4}dx$$
$$= \ln|x-1| + \frac{2(x-1)^{-1}}{-1} - \frac{1}{2}\int\frac{2x}{x^2+4}dx - \int\frac{1}{(x)^2+(2)^2}dx$$
$$= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\frac{x}{2}+c$$

Q25.

$$\int\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}dx$$

Solution:

$$\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)} = \frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{Cx+D}{x^2+x+1}$$

$$\Rightarrow 2x^2-x-7 = A(x+2)(x^2+x+1)+B(x^2+x+1)+(Cx+D)(x+2)^2$$

$$Put\ x+2=0 \Rightarrow x=-2$$

$$\Rightarrow 2(-2)^2-2(-2)-7 = B((-2)^2+(-2)+1)$$
$$\Rightarrow 8+2-7 = B(4-2+1)$$
$$\Rightarrow 3 = 3B \Rightarrow B = 1$$

From (i)

$$2x^2-x-7 = A(x^3+x^2+x+2x^2+2)+Bx^2+Bx+B+C(Cx+D)(x^2+4x+4)$$
$$= Ax^3+3Ax^2+3A+Bx^2+Bx+B+Cx^3+4Cx^2+4Cx+Dx^2+4Dx+4D$$

Equating coefficients of x^3, x^2 and x

$$for\ x^3;\quad 2+3A+B+4C+D$$

Put $B=1,\ C=-A \rightarrow (ii)$

For $x^2;\quad 2 = 3A+B+4C+D \Rightarrow 2-1 = -A+D$

$$\Rightarrow D = A+1 \rightarrow (iii)$$
$$For\ x;\quad -1 = 3A+B+4C+4D$$

Put $B=1, C=-A, D=A+1$

$$\Rightarrow -1 = 3A+1-4A+4A+4$$
$$-1-1-4 = 3A \Rightarrow -6 = 3A \Rightarrow A = -2$$

So $(ii) \Rightarrow C = 2$ and $(iii) \Rightarrow B = -1$

Thus,

$$\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)} = \frac{-2}{x+2}+\frac{1}{(x+2)^2}+\frac{2x-1}{x^2+x+1}$$

$$\int \frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}dx$$

$$= -2\int\frac{1}{x+2}dx + \int (x+2)^{-2}dx + \int \frac{2x+1-2}{x^2+x+1}dx$$

$$= -2\ln|x+2| + \frac{(x+2)^{-1}}{-1} + \int\frac{2x+1}{x^2+x+1}dx - 2\int\frac{1}{x^2+x+1}dx$$

$$= -2ln|x+2| - \frac{1}{x+2} + ln|x^2+x+1| - 2\int\frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}$$

$$= -2ln|x+2| - \frac{1}{x+2} + ln|x^2+x+1| - 2\int\frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}dx$$

$$-2ln|x+2| - \frac{1}{x+2} + ln|x^2+x+1| - 2\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -2ln|x+2| - \frac{1}{x+2}ln|x^2+x+1| - \frac{2}{\frac{\sqrt{3}}{2}}tan^{-1}\frac{\left(x+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + c$$

$$= -2ln|x+2| - \frac{1}{x+2} + ln|x^3+x+1| - \frac{4}{\sqrt{3}}tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

Q.26 $\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}dx$

Solution: $\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}dx$

$$\because \frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{Ax+B}{4x^2+1}+\frac{Cx+D}{(x^2-x+1)}$$

$\Rightarrow 3x+1 = (Ax+B)(x^2-x+1)+(Cx+D)(4x^2+1)$

$3x+1 = Ax^3-Ax^2+Ax+Bx^2-Bx+B+4Cx^3+Cx+4Dx^2+D$

Equating coefficients of x^3, x^2, x and constants terms.

For x^3; $0 = A+4C \to (i)$

For x^2; $0 = -A+B+4D \to (ii)$

for x; $3 = A+B+C \to (iii)$

For constant term; $1 = B+D \to (iv)$

From (i) $A = -4C$ and $(iv) \Rightarrow B = 1-D$

Put in (ii) and (iii)

$\Rightarrow 0 = -(-4C)+(1-D)+4D$ and $3 = -4C-(1-D)+C$

$0 = 4C+1+4D$ $\quad 3 = -4C-1+D+C$

$0 = 4C+3D+1 \to (v)$ $\quad 0 = -3C+D-4$

$\Rightarrow D = 3C+4$ put in (iv)

$\Rightarrow 0 = 4C+3(3C+4)+1$

$0 = 4C+3(3C+4)+1$

$0 = 4C+9C+12+1 \Rightarrow 0 = 13C+13$

$\Rightarrow -13C = 12 \Rightarrow C = -1$

As $A = -4C \Rightarrow A = -4(-1) \Rightarrow A = 4 \because C = -1$

As $D = 3C+4 \Rightarrow D = 3(-1)+4 = -3+4 \Rightarrow D = 1$

As $B = 1-D = 1-1 = 0 \Rightarrow B = 0$

Thus

$$\frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{4x+0}{4x^2+1}+\frac{(-1)x+1}{(x^2-x+1)}$$

$$\frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{1}{2}\frac{8x}{4x^2+1}+\frac{(-1)(x-1)}{x^2-x+1}$$

$$\int\frac{3x+1}{(4x^2+1)(x^2-x+1)}dx = \frac{1}{2}\int\frac{8x}{4x^3+1} - \frac{1}{2}\int\frac{2x-2}{x^2-x+1}dx$$

$$= \frac{1}{2}ln|4x^2+1| - \frac{1}{2}\int\frac{2x-1-1}{x^2-x+1}dx$$

$$= \frac{1}{2}\ln|x^2+1| - \frac{1}{2}\int\frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int\frac{1}{x^2-x+1}dx$$

$$= \frac{1}{2}ln|4x^2+1| - \frac{1}{2}ln|x^2-x+1| = \frac{1}{2}\int\frac{1}{x^2-x+\frac{1}{4}+\frac{3}{4}}dx$$

$$= \frac{1}{2}ln|4x^2+1| - \frac{1}{2}ln|x^2-x+1| + \frac{1}{2}\int\frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^3}$$

$$= \frac{1}{2}ln|4x^2+1| - \frac{1}{2}ln|x^2-x+1| + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}tan^{-1}\frac{\left(x-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + c$$

$$\frac{1}{2}ln|4x^2+1| - \frac{1}{2}ln|x^2-x+1| + \frac{1}{\sqrt{3}}tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c$$

Q27. $\int\frac{4x+1}{(x^2+4)(x^2+4x+5)}dx$

Solution:

$$\int\frac{4x+1}{(x^2+4)(x^2+4x+5)}dx$$

$$\because \frac{4x+1}{(x^2+4)(x^2+4x+5)} = \frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+4x+5}$$

$\Rightarrow 4x+1 = (Ax+B)(x^2+4x+5)+(Cx+D)(x^2+4)$

$\Rightarrow 4x+1 = Ax^3+4Ax^2+5Ax+Bx^2+4Bx+5B+Cx^3+4cx+Dx^2+4D$

Equating coefficients of x^3, x^2 , x and constant term.

$$Put\ x^3; 0 = A + C \rightarrow (i)$$
$$for\ x^2; 0 = 4A + B + D \rightarrow (ii)$$
$$for\ x; 4 = 5A + 4B + 4C \rightarrow (iiii)$$

For constant term $1 = 5B + 4D \rightarrow (iv)$

From $(i) \Rightarrow A = -C$ and $(iv) \Rightarrow 5B = 1 - 4D$

$$B = \frac{1-4D}{5} \text{ put in } (ii) \text{ and } (iii)$$

$$so\ (ii) \Rightarrow 0 = 4(-C) + \frac{1-4D}{5} + D \text{ and } (iii) \Rightarrow 4$$
$$= 5(-C) + 4\left(\frac{-4D}{5}\right) + 4C$$
$$\Rightarrow 0 = -4C + \frac{1-4D}{5} + D \quad \Rightarrow 20$$
$$= -25C + 4 - 16D + 20C$$
$$0 = -20C + 1 - 4D + 5D \quad \Rightarrow 16D$$
$$= -5C + 4 - 20$$
$$\Rightarrow 0 = -20C + D + 1 \qquad \Rightarrow D = \frac{-5C - 16}{16}$$
$$\rightarrow (vi)$$

$\Rightarrow D = 20C - 1 \rightarrow (v)$

$By\ (v) and\ (vi) \Rightarrow 20C - 1 = \frac{-5C-16}{16}$

$$\Rightarrow 320C - 16 = -5C - 16 \Rightarrow 320C + 5C = 0$$
$$\Rightarrow 320C = 0 \Rightarrow C = 0$$

As $a = -C \Rightarrow A = 0$

$$As\ D = 20C - 1 \Rightarrow D = 20(0) - 1 \Rightarrow D = -1$$
$$As\ B = \frac{1-4D}{5} \Rightarrow B = \frac{1-4(-1)}{5} = \frac{5}{3} = 1$$
$$B = 1$$

So

$$\frac{4x+1}{(x^2+4)(x^2+4x+5)} = \frac{0x+1}{x^2+4} + \frac{0x+(-1)}{x^2+4x+5}$$

$$\int \frac{4x+1}{(x^2+4)(x^2+4x+5)}$$
$$= \int \frac{1}{x^2+4} dx$$
$$- \int \frac{1}{x^2+4x+4+1} dx$$
$$= \frac{1}{2} tan^{-1} \frac{x}{2} - \int \frac{1}{(x-2)^2 + (1)^2} dx$$
$$= \frac{1}{2} \tan^{-1} \frac{x}{2} - \tan^{-1}(x-2) + c$$

Q28. $\int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)} dx$

$$\because \frac{6a^2}{(x^2+a^2)(x^2+4a^2)} = \frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{x^2+4a^2}$$
$$\Rightarrow 6a^2 = (Ax+B)(x^2+4a^2) + (Cx+D)(x^2 + a^2)$$
$$\Rightarrow 6a^2 = Ax^3 + 4a^2Ax + Bx^2 + 4Ba^2 + Cx^3 + Ca^2x + Dx^2 + Da^2$$

Equating coefficients of $x^3 + x^2, x$ and constants term.

$$Put\ x^3; 0 = A + C \rightarrow (i)$$
$$for\ x^2; 0 = B + D \rightarrow (ii)$$
$$for\ x; 0 = 4a^2A + a^2C \Rightarrow 0 = (4A + C)a^2$$
$$\Rightarrow 4A + C \rightarrow (iiii)$$

For constant term $1 = 5B + 4D \rightarrow (iv)$

From $(i) \Rightarrow A = -C$ and $(iv) \Rightarrow B = -D$

$$Put\ in\ (iii)\ and\ (iv) so$$
$$(iii)\ 4(-C) + C = 0 \Rightarrow -4C + c = 0 \Rightarrow -3C = 0$$
$$\Rightarrow C = 0$$
$$(iv) 4(-D) + D = 6 \Rightarrow -4D + d \Rightarrow -3D = 6$$
$$D = -2$$

As $A = -C \Rightarrow A = 0 \because C = 0$

$As\ B = -D \Rightarrow B = -(-2) \Rightarrow B = 2 \because D = -2$

So

$$\frac{6a^2}{(x^2+a^2)(x^2+4a^2)} = \frac{0x+2}{x^2+a^2} + \frac{0x+(-2)}{x^2+4a^2}$$
$$\int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)} dx = 2\int \frac{1}{x^2+a^2} dx - 2\int \frac{1}{x^2+(2a)^2} dx$$
$$= \frac{2}{a} tan^{-1} \frac{x}{a} - \frac{1}{a} tan^{-1} \frac{x}{2a} + c$$
$$= \frac{2}{a} tan^{-1} \frac{x}{a} - \frac{1}{a} tan^{-1} \frac{x}{2a} + c$$

Q29. $\int \frac{2x^2-2}{(x^4+x^2+1)(x^2-x+1)} dx$

Solution:

$$\int \frac{2x^2-2}{(x^2+x^2+1)(x^2-x+1)}$$
$$= \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1}$$
$$2x^2 - 2 = (Ax+B)(x^2-x+1) + (Cx+D)(x^2+x+1)$$
$$= Ax^3 - Ax^2 + Ax + Bx^2 - Bx + B + Cx^3 + Cx^2 + Cx + Dx^2 + Dx + D$$

Equating coefficients of x^3, x^2, x and constant term.

$$for\ x^3;\ 0 = A + C \rightarrow (i)$$
$$for\ x^2; 0 = A - B + C + D \rightarrow (ii)$$
$$for\ x; 2 = -A + B + C + D \rightarrow (iii)$$
$$for\ constant\ term;\ -2 = -A + C - 2$$
$$\Rightarrow 2 + 2 = -A + C \Rightarrow -A + C = 4 \rightarrow (v)$$

Put $A + C = 0$ $in(ii) \Rightarrow 0 = -B + D \rightarrow (vi)$

$$Now\ by\ (i) + (v) \Rightarrow 2C = 4 \Rightarrow C = 2$$
$$as\ A + C = 0 \Rightarrow A + 2 = 0 \Rightarrow A = -2$$

Now by (iv)+$(vi) \Rightarrow 2D = -D \Rightarrow D = -1$

$As\ B + D = -2 \Rightarrow B - 1 = -2 \Rightarrow B = -1$

So;

$$\frac{2x^2-2}{(x^2+x^2+1)(x^2-x+1)}$$
$$= \frac{-2x-1}{x^2+x+1} + \frac{2x-1}{x^2-x+1}$$
$$\int \frac{2x^2-2}{(x^2+x+1)(x^2-x+1)}$$
$$= -\int \frac{2x+1}{x^2+x+1} dx$$
$$+ \int \frac{2x-1}{x^2-x+1} dx$$
$$= -ln|x^2+x+1| + ln|x^2-x+1| + c$$

$$= ln\left|\frac{x^2-x+1}{x^2+x+1}\right| + c$$

Q 30. $\int \frac{3x-8}{(x^2-x+2)(x62+x+2)} dx$

Solution: $\int \frac{3x-8}{(x^2-x+2)(x62+x+2)} dx$

$$\because \frac{3x-8}{(x^2-x+2)(x^2+x+2)} = \frac{Ax+B}{x^2-x+2} + \frac{Cx+D}{x^2+x+2}$$

$$3x-8 = (Ax+B)(x^2+x+2)+(Cx+D)(x^2-x+2)$$
$$= Ax^3+Ax^2+2Ax+Bx^2+Bx+2B+Cx^3-Cx^2+2Cx+Dx^2-Dx+2D$$

Equating coefficients of x^3, x^2, x *and constant term.*

$for\ x^3;\ 0 = A+C \to (i)$

$for\ x^2; 0 = A+B-C+D \to (ii)$

$for\ x; 3 = 2A+B+2C-D \to (iii)$

$for\ constant\ term;\ -8 = 2B+2D \Rightarrow B+D = -4 \to (iv)$

From $(i) \Rightarrow A = -C$ and from$(iv) \Rightarrow B = -4-D$

Put *in (ii)and (iii)so*

$(ii)\Rightarrow 0 = -C+(-4-D)-C+D$

$0 = -C-4-B-C+D$

$0 = -2C-4$

$\Rightarrow 2C = -4 \Rightarrow C = -2\quad as\ A = -C \Rightarrow A = 2$

$(ii) \Rightarrow 3 = 2(-C)-4-D+2C-D$

$3 = -2C-4+2C-2D$

$\Rightarrow 3+4 = -2D \Rightarrow D = -\frac{7}{2}$

As $B = -4-D = -4\left(-\frac{7}{2}\right) = -4+\frac{7}{2} = \frac{-8+7}{2} = -1/2 \Rightarrow B = -\frac{1}{2}$

So

$$\frac{3x-8}{(x^2-x+2)(x^2+x+2)} = \frac{2x-1/2}{x^2-x+2} + \frac{-2x+(-7/2)}{x^2+x+2}$$

$$\int \frac{3x-8}{(x^2-x+2)(x^2-x+2)} = -\int \frac{2x+1-1-1/2}{x^2-x+2}dx + \int \frac{2x+1-1+7/2}{x^2+x+2}dx$$

$$= \int \frac{2x-1+\frac{1}{2}}{x^2-x+2}dx + \int \frac{2x+1+\frac{5}{2}}{x^2+x+2}dx$$

$$= \int \frac{2x-1}{x^2-x+2}dx + \frac{1}{2}\int \frac{1}{x^2-x+2}dx - \int \frac{2x+1}{x^2+x+2}dx - \int \frac{5/2}{x^2+x+2}dx$$

$$= ln|x^2-x+2| + \frac{1}{2}\int \frac{dx}{x^2-x+\frac{1}{4}-\frac{1}{4}+2} - ln|x^2+x+2| - \frac{5}{2}\int \frac{1}{x^2+x+\frac{1}{4}-\frac{1}{4}+2}dx$$

$$= ln|x^2-x+2| + \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}} - ln|x^2+x+2| - \frac{5}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}dx$$

$$= \boldsymbol{ln|x^2-x+1| + \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}}dx - ln|x^2+x+2| - \frac{5}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}dx}$$

$$= \boldsymbol{ln|x^2-x+1| + \frac{1}{2}.\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}dx - ln|x^2+x+1| - \frac{5}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}}$$

$$= \boldsymbol{ln|x^2-x+1| + \frac{1}{2}.\frac{1}{\frac{\sqrt{7}}{2}}tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right) - ln|x^2+x+1| - \frac{5}{2}.\frac{1}{\frac{\sqrt{7}}{2}}tan^{-1}\left(\frac{x+1/2}{\frac{\sqrt{7}}{2}}\right)}$$

$$\boldsymbol{ln|x^2-x+1| + \frac{1}{\sqrt{7}}tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right) - ln|x^2+x+1| - \frac{5}{\sqrt{7}}tan^{-1}\left(\frac{2x+1}{\sqrt{7}}\right) + c}$$

Q31. $\int \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}dx$

Solution: $\int \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}dx$

$$\because \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2+2x+3}$$

$$\Rightarrow 3x^3+4x^2+9x+5 = (Ax+B)(x^2+2x+3)+(Cx+D)(x^2+x+1)$$
$$ax^3+2Ax^2+3Ax+Bx^2+2Bx+3B+Cx^3+Cx^2+Cx+Dx^2+Dx+D$$

Equation coefficients of x^3, x^2, x *and constant term.*

$\boldsymbol{for\ x^3; 3 = A+C \to (i)}$

For $x^2; 4 = 2A+B+C+D \to (ii)$

For $x; 9 = 3A+2B+C+D \to (iii)$

For constant term; $5 = 3B+2B+C+D \to (iv)$

From $(i) \Rightarrow A = 3-C$ *and from* $(iv) \Rightarrow D = 5-3B$

Put in (ii) and (iii)

$(ii) \Rightarrow 4 = 2(3-C)+B+C+5-3B$

$4 = 6-2C+B+C+5-3B$

$4-6-5 = -C-2B \Rightarrow -7 = -(C+2B)$

$\Rightarrow C+2B = 7 \to (v)$

$(iii) \Rightarrow 9 = 3(3-C)+2B+C+5-3B$

$\Rightarrow 9 = 9-3C+2B+C+5-3B$

$\Rightarrow 9-9-5 = -2C-B \Rightarrow B = -2C+5\ put\ in(v)$

$\Rightarrow C+2(-2C+5) = 7 \Rightarrow C-4C+10 = 7$

$\Rightarrow -3C+10+7 \Rightarrow -3C = 7-10$

$\Rightarrow -3C = -3 \Rightarrow C = 1$

As B= $5-2C = 5-2(1) = 3 \Rightarrow B = 3$

$As\ D = 5-3B = 5-3(3) = 5-9 = -4 \Rightarrow D = -4$

$As\ A = 3-C = 3-1 = 2 \Rightarrow A = 2$

So

$$\frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)} = \frac{2x+3}{x^2+x+1} + \frac{x-4}{x^2+2x+3}$$

$$\int \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}dx$$
$$= \int \frac{2x+1+2}{x^2+x+1}dx + \frac{1}{2}\int \frac{2x-8}{x^2+2x+3}dx$$
$$= \int \frac{2x+1}{x^2+x+1}dx + 2\int \frac{1}{x^2+x+1}dx + \frac{1}{2}\int \frac{2x+2-10}{x^2+2x+3}dx$$
$$= ln|x^2+x+1| + 2\int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}dx$$
$$+\frac{1}{2}\int \frac{2x+2}{x^2+2x+3}dx$$
$$-5\int \frac{1}{x^2+2x+3}dx$$
$$= ln|x^2+x+1| + 2\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$
$$+\frac{1}{2}ln|x^2+2x+3|$$
$$-5\int \frac{1}{x^2+2x+1+2}dx$$
$$= ln|x^2+x+1| + 2.\frac{1}{\frac{\sqrt{3}}{2}}tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + \frac{1}{2}ln|x^2+2x+3|$$
$$-5\int \frac{1}{(x+1)^2+(\sqrt{2})^2}dx$$
$$= ln|x^2+x+1| + \frac{4}{\sqrt{3}}tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{2}ln|x^2+2x+3|$$
$$-\frac{5}{\sqrt{2}}tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c$$
$$= ln|x^2+x+1| + ln|x^2+2x+3|^{\frac{1}{2}} + \frac{4}{\sqrt{3}}tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$
$$-\frac{5}{2}tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c$$
$$= ln|x^2+x+1|\sqrt{x^2+2x+3} + \frac{4}{\sqrt{3}}tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$
$$-\frac{5}{2}tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c$$

The Definite integrals:

If $\int f(x)dx = \emptyset(x)+c$, *then the integral of* $f(x)$ *from a to b is denoted by* $\int_a^b f(x)dx$

And read $a + cs$ definite integral of $f(x)$ here a is called lower limit and b is called upper limit.

*the interval $[a, b]$ is called range of integration.

We evaluate $\int_a^b f(x)dx$ as;

Consider $\int f(x)dx = \emptyset(x)+c$

$$\Rightarrow \int_a^b f(x)dx = |\emptyset(x)+c|_a^b$$
$$= [\emptyset(b)+c]-[\emptyset(a)+c]$$
$$= \emptyset(b)+c-\emptyset(a)-c$$
$$\Rightarrow \int_a^b f(x)dx = \emptyset(b)-\emptyset(a)$$

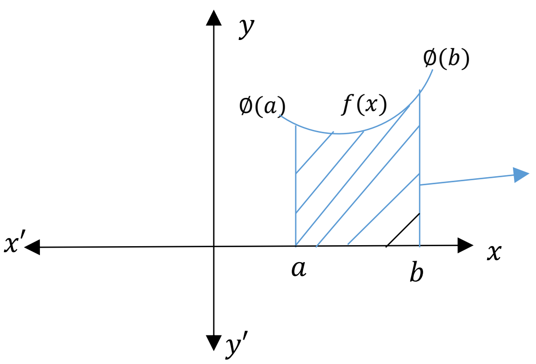
Note: if the lower limit is a constant and upper limit is a variable, then the integral is a function of the upper limit.

$$\int_a^x \boldsymbol{f(t)dt} = |\boldsymbol{\emptyset(t)}|_a^x = \boldsymbol{\emptyset(x)-\emptyset(a)}$$

The area under the curve

$$\int_a^b f(x)dx = \emptyset(b)-\emptyset(a)$$

Represented the "area of region" bounded under the curve of function $f(x)$ the $x-axis$ and between two ordinates $x = a, x = b$ *as shown in figure.*



Fundamental theorem of calculus:

If $f(x)$is continuous $\forall\, x \in [a, b]$ *and* $\emptyset'(x) = f(x)$

$$\int_a^b f(x)dx = \emptyset(b)-\emptyset(a)$$

Is called fundamental theorem of integral calculus.

Properties of Definite integral

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$
$$= \emptyset(b)-\emptyset(a)$$
$$= -[\emptyset(a)-\emptyset(b)]$$

$$= -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

(b) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad a < c < b$

Proof:

$$\int_a^c f(x)dx = \emptyset(c) - \emptyset(a)$$

$$\int_c^b f(x)dx = \emptyset(b) - \emptyset(c)$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx =$$

$$= \emptyset(c) - \emptyset(a) + \emptyset(b) - \emptyset(c)$$

$$= \emptyset(b) - \emptyset(a)$$

$$\int_a^b f(x)dx$$

$$\Rightarrow \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

(c) $\int_a^a f(x)dx = 0$

Proof:

$$\int_a^a f(x)dx = \emptyset(a) - \emptyset(a)$$

$$= 0$$

$$\Rightarrow \int_a^a f(x)dx = 0$$

Also member $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

and $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Exercise 3.6

Evaluate the following indefinate integrls:

Q.1: $\int_1^2 (x^2 + 1)\, dx$

SOLUTION:

$$\int_1^2 (x^2 + 1)\, dx$$

$$= \left|\frac{x^3}{3} + x\right|_1^2$$

$$= \left(\frac{2^3}{3} + 2\right) - \left(\frac{1^3}{3} + 1\right)$$

$$= \left(\frac{8}{3} + 2\right) - \left(\frac{1}{3} + 1\right)$$

$$= \left(\frac{8+6}{3}\right) - \left(\frac{1+3}{3}\right)$$

$$= \frac{14}{3} - \frac{4}{3}$$

$$= \frac{14-4}{3} = \frac{10}{3}$$

Q.2: $\int_{-1}^{1} \left(x^{\frac{1}{3}} + 1\right) dx$

SOLUTION:

$$= \int_{-1}^{1} \left(x^{\frac{1}{3}}.1 + 1\right) dx$$

$$= \left|\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + x\right|_{-1}^{1}$$

$$= \left|\frac{x^{\frac{4}{3}}}{\frac{4}{3}} + x\right|_{-1}^{1}$$

$$= \left|\frac{3}{4}x^{\frac{4}{3}} + x\right|_{-1}^{1}$$

$$= \left(\frac{3}{4}(1)^{\frac{4}{3}} + 1\right) - \left(\frac{3}{4}(-1)^{\frac{4}{3}} + (-1)\right)$$

$$= \left(\frac{3}{4}.1 + 1\right) - \left(\frac{3}{4}.1 - 1\right)$$

$$= \left(\frac{3+4}{4}\right) - \left(\frac{3-4}{4}\right)$$

$$= \frac{7}{4} - \frac{-1}{4} = \frac{7}{4} + \frac{1}{4}$$

$$= \frac{7+1}{4} = \frac{8}{4} = 2$$

Q.3: $\int_{-2}^{0} \frac{1}{(2x-1)^2}\, dx$

SOLUTION:

$$= \int_{-2}^{0} (2x-1)^{-2}\, dx$$

$$= \frac{1}{2}\int_{-2}^{0} (2x-1)^{-2}.2\, dx$$

$$= \frac{1}{2}\left|\frac{(2x-1)^{-2+1}}{-2+1}\right|_{-2}^{0}$$

$$= \frac{1}{2}\left|\frac{(2x-1)^{-1}}{-1}\right|_{-2}^{0}$$

$$= -\frac{1}{2}\left|\frac{1}{2x-1}\right|_{-2}^{0}$$

$$= -\frac{1}{2}\left[\left(\frac{1}{2(0)-1}\right) - \left(\frac{1}{2(-2)-1}\right)\right]$$

$$= -\frac{1}{2}\left[\left(\frac{1}{-1}\right) - \left(\frac{1}{-5}\right)\right]$$

$$= -\frac{1}{2}\left[-1 + \frac{1}{5}\right] = -\frac{1}{2}\left[\frac{-5+1}{5}\right]$$

$$= -\frac{1}{2}\left[\frac{-4}{5}\right] = \frac{2}{5}$$

Q.4: $\int_{-6}^{2} \sqrt{3-x}\, dx$

SOLUTION:

$= \int_{-6}^{2} (3-x)^{\frac{1}{2}}\, dx$

$= (-1)\int_{-6}^{2} (3-x)^{\frac{1}{2}}(-1)dx$

$= -\left|\frac{(3-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{-6}^{2}$

$= -\left|\frac{(3-x)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{-6}^{2}$

$= -\frac{2}{3}\left|(3-x)^{\frac{3}{2}}\right|_{-6}^{2}$

$= -\frac{2}{3}\left[\left((3-2)^{\frac{3}{2}}\right) - \left((3-(-6))^{\frac{3}{2}}\right)\right]$

$= -\frac{2}{3}\left[\left((1)^{\frac{3}{2}}\right) - \left((9)^{\frac{3}{2}}\right)\right]$

$= -\frac{2}{3}\left[1 - \left((3^2)^{\frac{3}{2}}\right)\right]$

$= -\frac{2}{3}[1-27] = \frac{52}{3}$

$Q.5$: $\int_{1}^{\sqrt{5}} \sqrt{(2t-1)^3}\, dt$

***SOLUTION*:**

$= \int_{1}^{\sqrt{5}} (2t-1)^{\frac{3}{2}}\, dt$

$= \frac{1}{2}\int_{1}^{\sqrt{5}} (2t-1)^{\frac{3}{2}}.2\, dt$

$= \frac{1}{2}\left|\frac{(2t-1)^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right|_{1}^{\sqrt{5}} = \frac{1}{2}\left|\frac{(2t-1)^{\frac{5}{2}}}{\frac{5}{2}}\right|_{1}^{\sqrt{5}}$

$= \frac{1}{2}\cdot\frac{2}{5}\left[\left((2\sqrt{5}-1)^{\frac{5}{2}}\right) - \left((2(1)-1)^{\frac{5}{2}}\right)\right]$

$= \frac{1}{5}\left[(2\sqrt{5}-1)^{\frac{5}{2}} - 1\right]$

$Q.6$: $\int_{2}^{\sqrt{5}} x\sqrt{x^2-1}\, dx$

***SOLUTION*:**

$= \int_{2}^{\sqrt{5}} (x^2-1)^{\frac{1}{2}}\, x\, dx$

$= \frac{1}{2}\int_{2}^{\sqrt{5}} (x^2-1)^{\frac{1}{2}}\, .2x\, dx$

$= \frac{1}{2}\left|\frac{(x^2-1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{2}^{\sqrt{5}} = \frac{1}{2}\left|\frac{(x^2-1)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{2}^{\sqrt{5}}$

$= \frac{1}{2}\cdot\frac{2}{3}\left[\left(\left((\sqrt{5})^2-1\right)^{\frac{3}{2}}\right) - \left(\left((2)^2-1\right)^{\frac{3}{2}}\right)\right]$

$= \frac{1}{3}\left[\left((4)^{\frac{3}{2}}\right) - \left(3^{\frac{3}{2}}\right)\right] = \frac{1}{3}\left[\left((2^2)^{\frac{3}{2}}\right) - \left(3^{\frac{3}{2}}\right)\right]$

$= \frac{1}{3}[8 - 3\sqrt{3}] = \frac{8}{3} - \sqrt{3}$

$Q.7$: $\int_{1}^{2} \frac{x}{x^2+2}\, dx$

***SOLUTION*:**

$= \frac{1}{2}\int_{1}^{2} \frac{2x}{x^2+2}\, dx$

$= \frac{1}{2}|\ln(x^2+2)|_{1}^{2}$

$= \frac{1}{2}[\ln(2^2+2) - \ln(1^2+2)]$

$= \frac{1}{2}[\ln(6) - \ln(3)]$

$= \frac{1}{2}\left[\ln\left(\frac{6}{3}\right)\right] = \frac{1}{2}\ln 2$

$= \ln 2^{\frac{1}{2}} = \ln\sqrt{2}$

Properties of natural log

$\ln(AB) = \ln A + \ln B$

$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$

$\ln A^B = B\ln A$

$Q.8$: $\int_{2}^{3} \left(x - \frac{1}{x}\right)^2\, dx$

***SOLUTION*:**

$= \int_{2}^{3} \left(x^2 + \frac{1}{x^2} - 2x.\frac{1}{x}\right)\, dx$

$= \int_{2}^{3} (x^2 + x^{-2} - 2)\, dx$

$= \left|\frac{x^{2+1}}{2+1} + \frac{x^{-2+1}}{-2+1} - 2x\right|_{2}^{3}$

$= \left|\frac{x^3}{3} + \frac{x^{-1}}{-1} - 2x\right|_{2}^{3} = \left|\frac{x^3}{3} - \frac{1}{x} - 2x\right|_{2}^{3}$

$= \left(\frac{3^3}{3} - \frac{1}{3} - 2(3)\right) - \left(\frac{2^3}{3} - \frac{1}{2} - 2(2)\right)$

$= \left(\frac{27}{3} - \frac{1}{3} - 6\right) - \left(\frac{8}{3} - \frac{1}{2} - 4\right)$

$= \left(\frac{27-1-18}{3}\right) - \left(\frac{16-3-24}{6}\right)$

$= \left(\frac{8}{3}\right) - \left(\frac{-11}{6}\right) = \frac{16+11}{6} = \frac{27}{6} = \frac{9}{2}$

$Q.9$: $\int_{-1}^{1} \left(x + \frac{1}{2}\right)\sqrt{x^2+x+1}\, dx$

***SOLUTION*:**

$= \int_{-1}^{1} (x^2+x+1)^{\frac{1}{2}}\left(\frac{2x+1}{2}\right)\, dx$

$= \frac{1}{2}\int_{-1}^{1} (x^2+x+1)^{\frac{1}{2}}(2x+1)\, dx$

$= \frac{1}{2}\left[\frac{(x^2+x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_{-1}^{1} = \frac{1}{2}\left[\frac{(x^2+x+1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-1}^{1}$

$= \frac{1}{2}.\frac{2}{3}\left[\left((1^2+1+1)^{\frac{3}{2}}\right) - \left(((-1)^2+(-1)+1)^{\frac{3}{2}}\right)\right]$

$= \frac{1}{3}\left[\left((3)^{\frac{3}{2}}\right) - \left((1-1+1)^{\frac{3}{2}}\right)\right]$

$= \frac{1}{3}\left[\left((3)^{\frac{3}{2}}\right) - \left((1)^{\frac{3}{2}}\right)\right]$

$= \frac{1}{3}[3\sqrt{3}-1] = \sqrt{3} - \frac{1}{3}$ *ANS.*

$3^{\frac{3}{2}} = \left(3^{\frac{1}{2}}\right)^3 = (\sqrt{3})^3 = (\sqrt{3})^2(\sqrt{3})^1 = 3\sqrt{3}$

Q. 10: $\int_0^3 \frac{dx}{x^2+9}$

SOLUTION:

$= \int_0^3 \frac{1}{3^2+x^2}\,dx = \left|\frac{1}{3}\tan^{-1}\frac{x}{3}\right|_0^3$

$= \frac{1}{3}\left[\left(\tan^{-1}\frac{3}{3}\right) - \left(\tan^{-1}\frac{0}{3}\right)\right]$

$= \frac{1}{3}[(\tan^{-1}1) - (\tan^{-1}0)]$

$= \frac{1}{3}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{12}$

Q. 11: $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos t\,dt$

SOLUTION:

$= |\sin t|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \left(\sin\frac{\pi}{3}\right) - \left(\sin\frac{\pi}{6}\right)$

$= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$

Q. 12: $\int_1^2 \left(x+\frac{1}{x}\right)^{\frac{1}{2}}\left(1-\frac{1}{x^2}\right)dx$

SOLUTION:

Here $f(x) = x + \frac{1}{x}$

$f'(x) = 1 - \frac{1}{x^2}$

$\left|\frac{\left(x+\frac{1}{x}\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_1^2 = \left|\frac{\left(x+\frac{1}{x}\right)^{\frac{3}{2}}}{\frac{3}{2}}\right|_1^2$

$= \frac{2}{3}\left[\left(\left(2+\frac{1}{2}\right)^{\frac{3}{2}}\right) - \left(\left(1+\frac{1}{1}\right)^{\frac{3}{2}}\right)\right]$

$= \frac{2}{3}\left[\left(\left(\frac{5}{2}\right)^{\frac{3}{2}}\right) - \left((2)^{\frac{3}{2}}\right)\right]$

$= \frac{2}{3}\left[\frac{5}{2}\sqrt{\frac{5}{2}} - 2\sqrt{2}\right]$

$= \frac{2}{3}.\frac{5}{2}\frac{\sqrt{5}}{\sqrt{2}} - \frac{2}{3}.2\sqrt{2}$

$= \frac{5\sqrt{5}}{3\sqrt{2}} - \frac{4\sqrt{2}}{3} = \frac{5\sqrt{5}-4(2)}{3\sqrt{2}}$

$= \frac{5\sqrt{5}-8}{3\sqrt{2}}$

Q. 13: $\int_1^2 \ln x\,dx$

SOLUTION:

Consider

$\int \ln x\,dx = \int \ln x .1\,dx$

Here $U = \ln x$, $V = 1$

Using $\int U.V = U.\int V\,dx - \int[U'.\int V\,dx]\,dx$

$= \ln x . \int 1\,dx - \int[(\ln x)'.\int 1\,dx]\,dx$

$= \ln x .x - \int\left[\frac{1}{x}.x\right]dx$

$= \ln x .x - \int 1\,dx$

$= x\ln x - x + c$

Taking limits

$\int_1^2 \ln x\,dx = |x\ln x - x|_1^2$

$= (2\ln 2 - 2) - (1\ln 1 - 1)$

$= (2\ln 2 - 2) - (1(0) - 1)$

$= (2\ln 2 - 2)$

Q. 14: $\int_0^2 \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)dx$

SOLUTION:

$\int_0^2 \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)dx = \left|\frac{e^{\frac{x}{2}}}{\frac{1}{2}} - \frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right|_0^2 = \left|2e^{\frac{x}{2}} + 2e^{-\frac{x}{2}}\right|_0^2$

$= 2\left[\left(e^{\frac{2}{2}} + e^{-\frac{2}{2}}\right) - \left(e^{\frac{0}{2}} + e^{-\frac{0}{2}}\right)\right]$

$= 2[(e^1 + e^{-1}) - (e^0 + e^0)]$

$= 2\left[e + \frac{1}{e} - 1 - 1\right] = 2\left[e + \frac{1}{e} - 2\right]$

$= 2\left[\frac{e^2+1-2e}{e}\right] = \frac{2}{e}(e^2 + 1^2 - 2e)$

Q. 15: $\int_0^{\frac{\pi}{4}} \frac{\cos\theta+\sin\theta}{\cos 2\theta+1}\,d\theta$

SOLUTION:

$= \int_0^{\frac{\pi}{4}} \frac{\cos\theta+\sin\theta}{1+\cos 2\theta}\,dx = \int_0^{\frac{\pi}{4}} \frac{\cos\theta+\sin\theta}{2\cos^2\theta}\,d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}}\left[\frac{\cos\theta}{\cos^2\theta}+\frac{\sin\theta}{\cos^2\theta}\right]\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}}\left[\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta.\cos\theta}\right]\,d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}}[\sec\theta+\sec\theta\tan\theta]\,d\theta$

$= \frac{1}{2}|\ln|\sec\theta+\tan\theta|+\sec\theta|_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\left(\ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right|+\sec\frac{\pi}{4}\right)-(\ln|\sec 0+\tan 0|+\sec 0)\right]$

$= \frac{1}{2}[(\ln|\sqrt{2}+1|+\sqrt{2})-(\ln|1+0|+1)]$

$= \frac{1}{2}[(\ln|\sqrt{2}+1|+\sqrt{2})-(0+1)]$

$= \frac{1}{2}[\ln|\sqrt{2}+1|+\sqrt{2}-1]$

Q. 16: $\int_0^{\frac{\pi}{6}}\cos^3\theta\ d\theta$

SOLUTION:

$= \int_0^{\frac{\pi}{6}}\cos\theta\ \cos^2\theta\ d\theta = \int_0^{\frac{\pi}{6}}\cos\theta\ (1-\sin^2\theta)\,d\theta$

$= \int_0^{\frac{\pi}{6}}(\cos\theta-\sin^2\theta\cos\theta)\ d\theta = \left|\sin\theta-\frac{\sin^3\theta}{3}\right|_0^{\frac{\pi}{6}}$

$= \left(\sin\frac{\pi}{6}-\frac{\sin^3\frac{\pi}{6}}{3}\right)-\left(\sin 0-\frac{\sin^2 0}{3}\right)$

$= \left(\frac{1}{2}-\frac{\left(\frac{1}{2}\right)^3}{3}\right)-\left(0-\frac{0}{3}\right) = \left(\frac{1}{2}-\frac{1}{24}\right)-(0) = \frac{12-1}{24} = \frac{11}{24}$

Q. 17: $\int_0^{\frac{\pi}{6}}\cos^2\theta\ \cot^2\theta\,d\theta$

SOLUTION:

$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\cos^2\theta\ (\operatorname{cosec}^2\theta-1)\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}}(\cos^2\theta\operatorname{cosec}^2\theta-\cos^2\theta)\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\frac{\cos^2\theta}{\sin^2\theta}-\cos^2\theta\right)d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}}(\cot^2\theta-\cos^2\theta)\,d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\operatorname{cosec}^2\theta-1-\frac{1+\cos 2\theta}{2}\right)d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\frac{2\operatorname{cosec}^2\theta-2-1-\cos 2\theta}{2}\right)d\theta$

$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}(2\operatorname{cosec}^2\theta-3-\cos 2\theta)\,d\theta$

$= \frac{1}{2}\left[-2\cot\theta-3\theta-\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\left(-2\cot\frac{\pi}{4}-3\frac{\pi}{4}-\frac{\sin 2\left(\frac{\pi}{4}\right)}{2}\right)-2\cot\frac{\pi}{6}-3\frac{\pi}{6}-\frac{\sin 2\left(\frac{\pi}{6}\right)}{2}\right)$

$= \frac{1}{2}\left[\left(-2(1)-\frac{3\pi}{4}-\frac{1}{2}\sin\frac{\pi}{2}\right)-\left(-2\sqrt{3}-\frac{\pi}{2}-\frac{1}{2}\sin\frac{\pi}{3}\right)\right]$

$= \frac{1}{2}\left[\left(-2-\frac{3\pi}{4}-\frac{1}{2}.1\right)-2\sqrt{3}-\frac{\pi}{2}-\frac{1}{2}\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2}\left[-2-\frac{3\pi}{4}-\frac{1}{2}+2\sqrt{3}+\frac{\pi}{2}+\frac{\sqrt{3}}{4}\right]$

$= \frac{1}{2}\left[-2-\frac{1}{2}+2\sqrt{3}+\frac{\sqrt{3}}{4}-\frac{3\pi}{4}+\frac{\pi}{2}\right]$

$= \frac{1}{2}\left[-2-\frac{1}{2}+2\sqrt{3}+\frac{\sqrt{3}}{4}-\frac{3\pi}{4}+\frac{\pi}{2}\right]$

$= \frac{1}{2}\left[\frac{-8-2+8\sqrt{3}+\sqrt{3}-3\pi+2\pi}{4}\right]$

$= \frac{1}{2}\left[\frac{-10+9\sqrt{3}-\pi}{4}\right]$

$= \frac{-10+9\sqrt{3}-\pi}{8}$

Q. 18: $\int_0^{\frac{\pi}{4}}\cos^4 t\ dt$

SOLUTION:

$\int_0^{\frac{\pi}{4}}\cos^4 t\ dt = \int_0^{\frac{\pi}{4}}(\cos^2 t)^2\,dt = \int_0^{\frac{\pi}{4}}\left(\frac{1+\cos 2t}{2}\right)^2\ dt$

$= \int_0^{\frac{\pi}{4}}\frac{1+\cos^2 2t+2\cos 2t}{4}\ dt$

$= \frac{1}{4}\int_0^{\frac{\pi}{4}}\left(1+\frac{1+\cos 4t}{2}+2\cos 2t\right)\ dt$

$= \frac{1}{4}\int_0^{\frac{\pi}{4}}\left(\frac{2+1+\cos 4t+4\cos 2t}{2}\right)\ dt = \frac{1}{8}\int_0^{\frac{\pi}{4}}(3+\cos 4t+4\cos 2t)\ dt$

$= \frac{1}{8}\left[3t+\frac{\sin 4t}{4}+4\frac{\sin 2t}{2}\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{8}\left[\left(3.\frac{\pi}{4}+\frac{\sin 4\left(\frac{\pi}{4}\right)}{4}+4\frac{\sin 2\left(\frac{\pi}{4}\right)}{2}\right)-\left(3(0)+\frac{\sin 4(0)}{4}+4\frac{\sin 2(0)}{2}\right)\right]$

$= \frac{1}{8}\left[\left(\frac{3\pi}{4}+\frac{\sin\pi}{4}+2\sin\left(\frac{\pi}{2}\right)\right)-\left(0+\frac{\sin 0}{4}+4\frac{\sin 0}{2}\right)\right]$

$= \frac{1}{8}\left[\left(\frac{3\pi}{4}+\frac{0}{4}+2.1\right)-\left(0+\frac{0}{4}+4.\frac{0}{2}\right)\right] = \frac{1}{8}\left[\left(\frac{3\pi}{4}+0+2\right)-0\right]$

$= \frac{1}{8}\left[\left(\frac{3\pi}{4}+2\right)\right] = \frac{1}{8}\left[\left(\frac{3\pi+8}{4}\right)\right] = \frac{3\pi+8}{32}$

Q. 19: $\int_0^{\frac{\pi}{3}}\cos^2\theta\ \sin\theta\,d\theta$

SOLUTION:

$\int_0^{\frac{\pi}{3}} \cos^2\theta\ \sin\theta\, d\theta = -\int_0^{\frac{\pi}{3}} \cos^2\theta\,(-\sin\theta)\, d\theta =$

$-\left|\frac{\cos^3\theta}{3}\right|_0^{\frac{\pi}{3}} = -\frac{1}{3}\left[\left(\cos^3\frac{\pi}{3}\right) - (\cos^3 0)\right] =$

$-\frac{1}{3}\left[\cos^3\frac{\pi}{3} - \cos^3 0\right]$

$= -\frac{1}{3}\left[\left(\frac{1}{2}\right)^3 - (1)^3\right] = -\frac{1}{3}\left[\frac{1}{8} - 1\right] = -\frac{1}{3}\left[\frac{1-8}{8}\right] =$

$-\frac{1}{3}\left[\frac{-7}{8}\right] = \frac{7}{24}$

Q.20: $\int_0^{\frac{\pi}{4}} (1+\cos^2\theta)\tan^2\theta\ d\theta$

SOLUTION:

$= \int_0^{\frac{\pi}{4}} (\tan^2\theta + \tan^2\theta\cos^2\theta)\, d\theta$

$= \int_0^{\frac{\pi}{4}} (\tan^2\theta + \tan^2\theta\cos^2\theta)\, d\theta =$

$\int_0^{\frac{\pi}{4}} \left(\tan^2\theta + \frac{\sin^2\theta}{\cos^2\theta}\cos^2\theta\right) d\theta$

$= \int_0^{\frac{\pi}{4}} (\tan^2\theta + \sin^2\theta)\, dt = \int_0^{\frac{\pi}{4}} \left[\sec^2\theta - 1 + \frac{1-\cos 2\theta}{2}\right] d\theta$

$= \int_0^{\frac{\pi}{4}} \left[\sec^2\theta - 1 + \frac{1}{2} - \frac{\cos 2\theta}{2}\right] d\theta = \int_0^{\frac{\pi}{4}} \left[\sec^2\theta - \frac{1}{2} - \frac{1}{2}\cos 2\theta\right] d\theta$

$= \left[\tan\theta - \frac{1}{2}\theta - \frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = \left[\tan\theta - \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}}$

$= \left[\left(\tan\frac{\pi}{4} - \frac{1}{2}\frac{\pi}{4} - \frac{1}{4}\sin 2\left(\frac{\pi}{4}\right)\right) - \left(\tan 0 - \frac{1}{2}0 - \frac{1}{4}\sin 2(0)\right)\right]$

$= \left[\left(1 - \frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2}\right) - \left(0 - 0 - \frac{1}{4}\sin(0)\right)\right]$

$= \left[\left(1 - \frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2}\right) - (0)\right] = \left[\left(1 - \frac{\pi}{8} - \frac{1}{4}(1)\right) - (0)\right]$

Q.21: $\int_0^{\frac{\pi}{4}} \frac{\sec\theta}{\sin\theta+\cos\theta}\ d\theta$

SOLUTION:

Divide uo and down by $\cos\theta$

$= \int_0^{\frac{\pi}{4}} \frac{\sec\theta}{\sin\theta+\cos\theta}\, d\theta = \int_0^{\frac{\pi}{4}} \frac{\frac{\sec\theta}{\cos\theta}}{\frac{\sin\theta+\cos\theta}{\cos\theta}}\, d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{\sec\theta\ \sec\theta}{\tan\theta+1}\, d\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\tan\theta+1}\, d\theta$

Here $f(x) = \tan\theta + 1 \quad \Rightarrow f'(x) = \sec^2\theta$

$= [\ln|\tan\theta + 1|]_0^{\frac{\pi}{4}}$

$= \left[\left(\ln\left|\tan\frac{\pi}{4} + 1\right|\right) - (\ln|\tan 0 + 1|)\right]$

$= [(\ln|1+1|) - (\ln|0+1|)]$

$= \ln|2| - \ln|1| = \ln 2 - 0 = \ln 2$

Q.22: $\int_{-1}^{5} |x-3|\, dx$

SOLUTION:

$\int_{-1}^{5} |x-3|\, dx$

$= \int_{-1}^{3} |x-3|\, dx + \int_{3}^{5} |x-3|\, dx$

$= \int_{-1}^{3} -(x-3)\, dx + \int_{3}^{5} (x-3)\, dx$

$= -\int_{-1}^{3} (x-3).1\, dx + \int_{3}^{5} (x-3).1\, dx$

$= -\left|\frac{(x-3)^2}{2}\right|_{-1}^{3} + \left|\frac{(x-3)^2}{2}\right|_{3}^{5}$

$= -\frac{1}{2}[((3-3)^2) - ((-1-3)^2)] + \frac{1}{2}[((5-3)^2) - ((3-3)^2)]$

$= -\frac{1}{2}[0-16] + \frac{1}{2}[4-0]$

$= -\frac{1}{2}[0-16] + \frac{1}{2}[4-0] = 8 + 2 = 10$

Q.23: $\int_{\frac{1}{8}}^{1} \frac{\left(x^{\frac{1}{3}}+2\right)^2}{x^{\frac{2}{3}}}\ dx$

$\int_{\frac{1}{8}}^{1} \frac{\left(x^{\frac{1}{3}}+2\right)^2}{x^{\frac{2}{3}}}\ dx \ = 3\int_{\frac{1}{8}}^{1} \left(x^{\frac{1}{3}}+2\right)^2 . \frac{1}{3}x^{-\frac{2}{3}}\, dx =$

$3\left|\frac{\left(x^{\frac{1}{3}}+2\right)^{2+1}}{2+1}\right|_{\frac{1}{8}}^{1} = \frac{3}{3}\left|\left(x^{\frac{1}{3}}+2\right)^3\right|_{\frac{1}{8}}^{1} = \left(\left((1)^{\frac{1}{3}}+2\right)^3\right) - \left(\left(\left(\frac{1}{8}\right)^{\frac{1}{3}}+2\right)^3\right)$

$= ((1+2)^3) - \left(\left((2^{-3})^{\frac{1}{3}}+2\right)^3\right) = (3)^3 - \left(\frac{1}{2}+2\right)^3 = 27 - \left(\frac{1+4}{2}\right)^3 = 27 - \frac{125}{8} = \frac{216-125}{8} = \frac{91}{8}$

$\int_1^3 \frac{x^2-2}{x+1}\, dx \quad$ *(Improper fraction)*

$$\begin{array}{r} x-1 \\ x+1\,\overline{)\,x^2-2} \\ \pm x^2 \pm x \\ \hline -x-2 \\ \mp x \mp 1 \\ \hline -1 \end{array}$$

$= \int_1^3 \left(Q + \frac{R}{D}\right)\, dx$

$= \int_1^3 \left(x - 1 - \frac{1}{1+x}\right)\, dx$

$= \left|\frac{x^2}{2} - x - \ln|x+1|\right|_1^3$

$= \left(\frac{3^2}{2} - 3 - \ln|3+1|\right) - \left(\frac{1}{2} - 1 - \ln|1+1|\right)$

$= \left(\frac{9}{2} - 3 - \ln 4\right) - \left(\frac{1}{2} - 1 - \ln 2\right)$

$= \frac{9}{2} - 3 - \ln 4 - \frac{1}{2} + 1 + \ln 2$

$= \frac{9}{2} - 2 - \frac{1}{2} - \ln 4 + \ln 2$

$= \frac{9-4-1}{2} - \ln 2^2 + \ln 2$

$= 2 - 2\ln 2 + \ln 2 = 2 - \ln 2$

***Q*.25**: $\int_2^3 \frac{3x^2-2x+1}{(x-1)(x^2+1)}\, dx$

SOLUTION:

$\int_2^3 \frac{3x^2-2x+1}{(x-1)(x^2+1)}\, dx = \int_2^3 \frac{3x^2-2x+1}{x^3-x^2+x-1}\, dx = |\ln|x^3 - x^2 + x - 1||_2^3$

$= (\ln|3^3 - 3^2 + 3 - 1|) - (\ln|2^3 - 2^2 + 2 - 1|)$

$= (\ln|27 - 9 + 3 - 1|) - (\ln|8 - 4 + 2 - 1|)$

$= \ln 20 - \ln 5 = \ln\frac{20}{5} = \ln 4$

***Q*.26**: $\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x}\, dx$

SOLUTION:

$= \int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x}\right) dx$

$= \int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos x \cos x} - \frac{1}{\cos^2 x}\right) dx$

$= \int_0^{\frac{\pi}{4}} (\sec x \tan x - \sec^2 x)\, dx$

$= |\sec x - \tan x|_0^{\frac{\pi}{4}}$

$= \left(\sec\frac{\pi}{4} - \tan\frac{\pi}{4}\right) - (\sec 0 - \tan 0)$

$= (\sqrt{2} - 1) - (1 + 0)$

$= \sqrt{2} - 1 - 1 = \sqrt{2} - 2$

***Q*.27**: $\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x}\, dx$

SOLUTION:

$= \int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\, dx = \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{1-\sin^2 x}\, dx$

$= \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{\cos^2 x}\, dx = -\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x}\, dx$

$= -\int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x}\right) dx$

$= -\int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos x \cos x} - \frac{1}{\cos^2 x}\right) dx$

$= -\int_0^{\frac{\pi}{4}} (\sec x \tan x - \sec^2 x)\, dx$

$= -|\sec x - \tan x|_0^{\frac{\pi}{4}}$

$= -\left[\left(\sec\frac{\pi}{4} - \tan\frac{\pi}{4}\right) - (\sec 0 - \tan 0)\right]$

$= -[(\sqrt{2} - 1) - (1 + 0)]$

$= -\sqrt{2} + 1 + 1 = 2 - \sqrt{2}$

***Q*.28**: $\int_0^1 \frac{3x}{\sqrt{4-3x}}\, dx$

SOLUTION:

$\int_0^1 \frac{3x}{\sqrt{4-3x}}\, dx = -\int_0^1 \frac{-3x}{\sqrt{4-3x}}\, dx = -\int_0^1 \frac{4-3x-4}{\sqrt{4-3x}}\, dx = -\int_0^1 \frac{4-3x}{\sqrt{4-3x}}\, dx - \int_0^1 \frac{-4}{\sqrt{4-3x}}\, dx = -\int_0^1 \sqrt{4-3x}\, dx + 4\int_0^1 \frac{1}{\sqrt{4-3x}}\, dx$

$= -\frac{1}{-3}\int_0^1 (4-3x)^{\frac{1}{2}}(-3)dx + \frac{4}{-3}\int_0^1 (4-3x)^{-\frac{1}{2}}(-3)dx$

$= \frac{1}{3}\left|\frac{(4-3x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_0^1 - \frac{4}{3}\left|\frac{(4-3x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right|_0^1 = \frac{1}{3}\left|\frac{(4-3x)^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^1 - \frac{4}{3}\left|\frac{(4-3x)^{\frac{1}{2}}}{\frac{1}{2}}\right|_0^1 = \frac{1}{3}.\frac{2}{3}\left|(4-3x)^{\frac{3}{2}}\right|_0^1 - \frac{4}{3}.\frac{2}{1}\left|(4-3x)^{\frac{1}{2}}\right|_0^1$

$= \frac{2}{9}\left|(4-3x)^{\frac{3}{2}}\right|_0^1 - \frac{8}{3}\left|(4-3x)^{\frac{1}{2}}\right|_0^1 = \frac{2}{9}\left[\left((4-3(1))^{\frac{3}{2}}\right) - \left((4-3(0))^{\frac{3}{2}}\right)\right] - \frac{8}{3}\left[\left((4-3(1))^{\frac{1}{2}}\right) - \left((4-3(0))^{\frac{1}{2}}\right)\right]$

$= \frac{2}{9}\left[(1)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\right] - \frac{8}{3}\left[(1)^{\frac{1}{2}} - (4)^{\frac{1}{2}}\right] = \frac{2}{9}\left[1 - (2^2)^{\frac{3}{2}}\right] - \frac{8}{3}\left[(1)^{\frac{1}{2}} - (2^2)^{\frac{1}{2}}\right] = \frac{2}{9}[1 - 2^3] - \frac{8}{3}[1 - 2^1]$

$= \frac{2}{9}[1-8] - \frac{8}{3}[1-2] = \frac{2}{9}[-7] - \frac{8}{3}[-1] = \frac{-14}{9} + \frac{8}{3} = \frac{-14+24}{9} = \frac{10}{9}$

***Q*.29**: $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x(2+\sin x)}\, dx$

SOLUTION:

$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{2}{\sin x(2+\sin x)}.\cos x\ dx = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{(2+\sin x)-\sin x}{\sin x(2+\sin x)}.\cos x\ dx$

$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[\frac{(2+\sin x)}{\sin x(2+\sin x)} - \frac{\sin x}{\sin x(2+\sin x)}\right]\cos x\ dx$

$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[\frac{1}{\sin x} - \frac{1}{2+\sin x}\right]\cos x\ dx = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[\frac{\cos x}{\sin x} - \frac{\cos x}{2+\sin x}\right] dx$

$= \frac{1}{2}\left|ln|\sin x| - ln|2+\sin x|\right|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\left(ln\left|\sin\frac{\pi}{2}\right| - ln\left|2+\sin\frac{\pi}{2}\right|\right) - \left(ln\left|\sin\frac{\pi}{6}\right| - ln\left|2+\sin\frac{\pi}{6}\right|\right)\right]$

$= \frac{1}{2}\left[(\ln(1) - ln|2+1|) - \left(ln\frac{1}{2} - ln\left|2+\frac{1}{2}\right|\right)\right]$

$= \frac{1}{2}\left[0 - \ln 3 - ln\frac{1}{2} + ln\frac{5}{2}\right] = \frac{1}{2}[-\ln 3 - (ln1 - ln2) + (\ln 5 - ln2)]$

$= \frac{1}{2}[-\ln 3 - ln1 + ln2 + \ln 5 - ln2] = \frac{1}{2}[-\ln 3 - 0 + \ln 5]$

$= \frac{1}{2}[\ln 5 - \ln 3] = \frac{1}{2}\left[\ln\frac{5}{3}\right] = \frac{1}{2}\ln\frac{5}{3}$

$\boldsymbol{Q.30}$: $\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)(2+\cos x)}\, dx$

SOLUTION:

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)(2+\cos x)}\, dx = \int_0^{\frac{\pi}{2}} \frac{1}{(1+\cos x)(2+\cos x)} \sin x\, dx$

$= \int_0^{\frac{\pi}{2}} \frac{(2+\cos x)-(1+\cos x)}{(1+\cos x)(2+\cos x)} \sin x\, dx$

$= \int_0^{\frac{\pi}{2}} \left[\frac{(2+\cos x)}{(1+\cos x)(2+\cos x)} - \frac{(1+\cos x)}{(1+\cos x)(2+\cos x)}\right] \sin x\, dx$

$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{(1+\cos x)} - \frac{1}{(2+\cos x)}\right] \sin x\, dx$

$= \int_0^{\frac{\pi}{2}} \left[\frac{\sin x}{(1+\cos x)} - \frac{\sin x}{(2+\cos x)}\right] dx$

$= \frac{-1}{-1}\int_0^{\frac{\pi}{2}} \left[\frac{\sin x}{(1+\cos x)} - \frac{\sin x}{(2+\cos x)}\right] dx$

$= \frac{1}{-1}\int_0^{\frac{\pi}{2}} \left[\frac{-\sin x}{(1+\cos x)} - \frac{-\sin x}{(2+\cos x)}\right] dx$

$= -[\ln|1+\cos x| - \ln|2+\cos x|]_0^{\frac{\pi}{2}}$

$= -\left[\left(\ln\left|1+\cos\frac{\pi}{2}\right| - \ln\left|2+\cos\frac{\pi}{2}\right|\right) - (\ln|1+\cos 0| - \ln|2+\cos 0|)\right]$

$= -[(\ln|1+0| - \ln|2+0|) - (\ln|1+1| - \ln|2+1|)]$

$= -[\ln(1) - \ln(2) - \ln(2) + \ln(3)]$

$= -[0 - 2\ln(2) + \ln 3] = 2\ln(2) - \ln(3)$

$= \ln(2)^2 - \ln(3) = \ln 4 - \ln(3) = \ln\frac{4}{3}$

$\because \begin{pmatrix} a\ln b = \ln b^a \\ \because \ln a - \ln b \\ = \ln\frac{a}{b} \end{pmatrix}$

Application of definate integral: Area under the curve:

Case1. *if* $f(x) \geq 0\ \forall x \in [a,b]$ *then curve lies above* $x - axis$.

$$A = \int_a^b f(x)dx \ \ where\ a < b$$

A is area of region above x − axis. under the curve of function $y = f(x)$ *from a to b*

Case II *if* $f(x) \leq 0\ \forall x \in [a,b]$ *then curve lies below*

$$x - axis. so\ A = -\int_a^b f(x)dx \ \ where\ a < b$$

A is area of region below x − axis, under The curve of function $y = f(x)$ *from a to b*

Exercise 3.7

Q.1: Find the area between the $\boldsymbol{x}$ − ***axis and the curve*** $\boldsymbol{y = x^2 + 1}$ ***from*** $\boldsymbol{x = 1}$ ***to*** $\boldsymbol{x = 2}$

SOLUTION:

Given $y = x^2 + 1$ As $y = x^2 + 1 > 0$ *in* $[1,2]$, *therefore curve is above x − axis*

$$Required\ Area = \int_1^2 y\ dx = \int_1^2 (x^2+1)\ dx$$

$$= \left|\frac{x^3}{3} + x\right|_1^2 = \left(\frac{8}{3}+2\right) - \left(\frac{1}{3}+1\right)$$

$$= \left(\frac{2^3}{3}+2\right) - \left(\frac{1^3}{3}+1\right)$$

$$= \left(\frac{8+6}{3}\right) - \left(\frac{1+3}{3}\right) = \frac{14}{3} - \frac{4}{3}$$

$$= \frac{14-4}{3} = \frac{10}{3} \quad square\ unit$$

Q.2: Find the area above the $\boldsymbol{x}$ − ***axis and under the curve*** $\boldsymbol{y = 5 - x^2}$ ***from*** $\boldsymbol{x = -1}$ ***to*** $\boldsymbol{x = 2}$

SOLUTION:

Given $y = 5 - x^2$ *As* $y = 5 - x^2 > 0$ *in* $[-1,2]$

, *therefore curve is above x − axisRequired Area*

$$= \int_{-1}^2 y\ dx = \int_{-1}^2 (5 - x^2)\ dx = \left|5x - \frac{x^3}{3}\right|_{-1}^2$$

$$= \left(5.2 - \frac{2^3}{3}\right) - \left(5(-1) - \frac{(-1)^3}{3}\right)$$

$$= \left(10 - \frac{8}{3}\right) - \left(-5 - \frac{-1}{3}\right)$$

$$=\left(\frac{30-8}{3}\right)-\left(\frac{-15+1}{3}\right)$$

$$=\frac{22}{3}-\frac{-14}{3}=\frac{22+14}{3}=\frac{36}{3}=12 \quad sq.unit$$

Q.3: Find the area below the curve $y=3\sqrt{x}$ and above the $x-$axis between $x=1$ to $x=4$

SOLUTION:

Given $y=3\sqrt{x}$ As $y=3\sqrt{x}>0$ in $[1,4]$, therefore curve is above $x-axis$

$$Required\ Area=\int_1^4 y\ dx=3\int_{-1}^{2}\sqrt{x}\ dx$$

$$=3\int_1^4 (x)^{\frac{1}{2}}.1\ dx=3\left|\frac{(x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_1^4$$

$$=3\left|\frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right|_1^4=3.\frac{2}{3}\left|(x)^{\frac{3}{2}}\right|_1^4$$

$$=2\left((4)^{\frac{3}{2}}-(1)^{\frac{3}{2}}\right)=2\left((2^2)^{\frac{3}{2}}-1\right)$$

$$=2(2^3-1)=2(8-1)=2(7)$$

$$=14 \quad sq.unit$$

Q.4: Find the area bounded by cos function from $x=-\frac{\pi}{2}$ to $x=\frac{\pi}{2}$

SOLUTION:

Given $y=\cos x$ As $y=\cos x\geq 0$ in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, therefore curve is above $x-axis$

$$Required\ Area=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y\ dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\ dx$$

$$=|\sin x|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\sin\left(\frac{\pi}{2}\right)-\sin\left(-\frac{\pi}{2}\right)$$

$$=1-(-1)=1+1=2 \quad sq.unit$$

Q.5: Find the area betweeen the $x-$axis and the curve $y=4x-x^2$.

SOLUTION:

Put $y=0$ $\quad 4x-x^2=0$

$x(4-x)=0$

$x=0$ or $4-x=0$

$x=4$

Given $y=4x-x^2$

As $y=4x-x^2\geq 0$ in $[0,4]$, therefore curve is above $x-axis$

$$Required\ Area=\int_0^4 y\ dx=\int_0^4(4x-x^2)\ dx$$

$$=\left|4\frac{x^2}{2}-\frac{x^3}{3}\right|_0^4$$

$$=\left(2.4^2-\frac{4^3}{3}\right)-\left(0^2-\frac{0^3}{3}\right)$$

$$=\left(32-\frac{64}{3}\right)-(0-0)=\frac{96-64}{3}$$

$$=\frac{32}{3}\ sq.unit$$

Q.6: Determine the area bounded by the parabola $y=x^2+2x-3$ and the $x-$axis.

SOLUTION:

Given $y=x^2+2x-3$

As $y=\ y=x^2+2x-3\leq 0$ in $[-3,1]$, therefore curve is below $x-axis$

$$Required\ Area=-\int_{-3}^{1} y\ dx$$

$$=-\int_{-3}^{1}(x^2+2x-3)\ dx$$

$$=-\left|\frac{x^3}{3}+2\frac{x^2}{2}-3x\right|_{-3}^{1}$$

$$=-\left|\frac{x^3}{3}+x^2-3x\right|_{-3}^{1}$$

$$=-\left[\left(\frac{1^3}{3}+1^2-3.1\right)-\left(\frac{(-3)^3}{3}+(-3)^2-3(-3)\right)\right]$$

$$=-\left[\left(\frac{1}{3}+1-3\right)-\left(\frac{-27}{3}+9+9\right)\right]$$

$$=-\left[\frac{1}{3}+1-3+\frac{27}{3}-9-9\right]$$

$$=-\left[\frac{1}{3}+\frac{27}{3}-20\right]$$

$$=-\left[\frac{1+27-60}{3}\right]$$

$$=-\left[\frac{-32}{3}\right]=\frac{32}{3}\ sq.unit$$

Put $y=o$ $\quad x^2+2x-3=0$

$x^2+3x-x-3=0$

$x(x+3)-1(x+3)=0$

$(x+3)-(x-1)=0$

$x+3=0$ or $x-1=0$

$x=-3$, $x=0$

Q.7: Find the area bounded by the curve $y=x^3+1$, the $x-$axis and line $x=2$

SOLUTION:

Given $y=x^3+1$ $\quad$ Put $y=0$ $\quad x^3+1=0$

$(x+1)(x^2-x+1)=0$

$x+1=0$ or $x^2-x+1=0$ (Solve itself)

$x=-1$

After solving this equation gives imaginary roots so neglect.

As $y=x^3+1\geq 0$ in $[-1,2]$, therefore curve is above $x-axis$

$$Required\ Area = \int_{-1}^{2} y\ dx = \int_{-1}^{2} (x^3+1)\ dx$$
$$= \left|\frac{x^4}{4}+x\right|_{-1}^{2}$$
$$= \left(\frac{2^4}{4}+2\right) - \left(\frac{(-1)^4}{4}+(-1)\right)$$
$$= (4+2) - \left(\frac{1}{4}-1\right)$$
$$= (6) - \left(\frac{1-4}{4}\right) = 6 - \frac{-3}{4} = 6 + \frac{3}{4}$$
$$= \frac{24+3}{4} = \frac{27}{4}\quad square\ unit.$$

Q.8: Find the area bounded by the curve $y = x^3 - 4x$, and the $x-axis$.

SOLUTION:

Given $y = x^3 - 4x$

Put $y=0$ $x^3-4x=0$
$x(x^2-4)=0$
$x(x-2)(x+2)=0$
$x=0$ or $x-2=0$, or $x+2=0$
$x=0$ or $x=2$, or $x=-2$

As $y = x^3 - 4x \geq 0$ in $[-2,0]$, therefore the curve is above $x-axis$

As $y = x^3 - 4x \leq 0$ in $[0,2]$, therefore the curve is below $x-axis$

$$Required\ Area = \int_{-2}^{0} y\ dx - \int_{0}^{2} y\ dx$$
$$= \int_{-2}^{0} (x^3-4x)\ dx - \int_{0}^{2} (x^3-4x)\ dx$$
$$= \left|\frac{x^4}{4} - 4\frac{x^2}{2}\right|_{-2}^{0} - \left|\frac{x^4}{4} - 4\frac{x^2}{2}\right|_{0}^{2}$$
$$= \left[\left(\frac{0^4}{4} - 2(0)^2\right) - \left(\frac{(-2)^4}{4} - 2(-2)^2\right)\right] - \left[\left(\frac{(2)^4}{4} - 2(2)^2\right) - \left(\frac{(0)^4}{4} - 2(0)^2\right)\right]$$
$$= [(0-0)-(4-8)] - [(4-8)-(0-0)]$$
$$= 0+4+4+0$$
$$= 8\quad square\ unit.$$

Q.9: Find the area between the curve $y = x(x-1)(x+1)$, and the $x-axis$.

SOLUTION:

Given $y = x(x-1)(x+1) = x^3 - x$

As $y = x^3 - x \geq 0$ in $[-1,0]$, therefore the curve is above $x-axis$

As $y = x^3 - x \leq 0$ in $[0,1]$, therefore the curve is below $x-axis$

$$Required\ Area = \int_{-1}^{0} y\ dx - \int_{0}^{1} y\ dx$$
$$= \int_{-1}^{0} (x^3-x)\ dx - \int_{0}^{1} (x^3-x)\ dx$$
$$= \left|\frac{x^4}{4} - \frac{x^2}{2}\right|_{-1}^{0} - \left|\frac{x^4}{4} - \frac{x^2}{2}\right|_{0}^{1}$$
$$= \left[\left(\frac{0^4}{4} - \frac{(0)^2}{2}\right) - \left(\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right)\right] - \left[\left(\frac{(1)^4}{4} - \frac{(1)^2}{2}\right) - \left(\frac{(0)^4}{4} - \frac{(0)^2}{2}\right)\right]$$
$$= \left[(0-0) - \left(\frac{1}{4}-\frac{1}{2}\right)\right] - \left[\left(\frac{1}{4}-\frac{1}{2}\right) - (0-0)\right]$$
$$= 0 - \frac{1}{4} + \frac{1}{2} - \frac{1}{4} + \frac{1}{2} + 0$$
$$= \frac{-1+2-1+2}{4} = \frac{1}{2}$$

Q.10: Find the area above the $x-axis$, bounded by the curve $y^2 = 3 - x$ from $x = -1$ to $x = 2$

SOLUTION:

Given $y^2 = 3 - x \Rightarrow y = \sqrt{3-x}$ As $y = \sqrt{3-x} \geq 0$ in $[-1,2]$, therefore curve is above $x-axis$

$$Required\ Area = \int_{-1}^{2} y\ dx = \int_{-1}^{2} \sqrt{3-x}\ dx$$
$$= -\int_{-1}^{2} (3-x)^{\frac{1}{2}}(-1)\ dx$$
$$= -\left|\frac{(3-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{-1}^{2}$$
$$= -\frac{2}{3}\left|(3-x)^{\frac{3}{2}}\right|_{-1}^{2}$$
$$= -\frac{2}{3}\left[\left((3-2)^{\frac{3}{2}}\right) - \left((3-(-1))^{\frac{3}{2}}\right)\right]$$
$$= -\frac{2}{3}\left[1 - \left((4)^{\frac{3}{2}}\right)\right] = -\frac{2}{3}[1-8]$$
$$= \frac{14}{3}\quad sq.unit$$

Q.11: Find the area betweeen the $x-axis$ and the curve $y = \cos\frac{1}{2}x$ from $x = -\pi$ to π.

SOLUTION:

Given $y=\cos\frac{1}{2}x$ *As* $y=\cos\frac{1}{2}x \geq 0$ *in* $[-\pi,\pi]$, *therefore curve is above* $x-axis$

Required Area $=\int_{-\pi}^{\pi} y\ dx=\int_{-\pi}^{\pi}\cos\frac{1}{2}x\ dx$

$=\left|\frac{\sin\frac{1}{2}x}{\frac{1}{2}}\right|_{-\pi}^{\pi}=2\left[\left(\sin\frac{1}{2}(\pi)\right)-\left(\sin\frac{1}{2}(-\pi)\right)\right]$

$=2[1-(-1)]=4$

Q.12: Find the area betweeen the $x-$ ***axis and the curve*** $y=\sin 2x$ ***from*** $x=0$ ***to*** $\frac{\pi}{3}$.

SOLUTION:

Given $y=\sin 2x$ *As* $y=\sin 2x\geq 0$ *in* $\left[0,\frac{\pi}{3}\right]$, *therefore curve is above* $x-axis$

Required Area $=\int_0^{\frac{\pi}{3}} y\ dx=\int_0^{\frac{\pi}{3}}\sin 2x\ dx$

$=\left|\frac{-\cos 2x}{2}\right|_0^{\frac{\pi}{3}}=-\frac{1}{2}\left[\left(\cos\frac{2\pi}{3}\right)-(\cos 2(0))\right]$

$=-\frac{1}{2}\left[-\frac{1}{2}-1\right]=\frac{3}{4}$ *sq.unit*

Q.13: Find the area betweeen the $x-$ ***axis and the curve*** $y=\sqrt{2ax-x^2}$ ***when*** $a>0$.

SOLUTION:

Given $y=\sin 2x$ *As* $y=\sin 2x\geq 0$ *in* $\left[0,\frac{\pi}{3}\right]$, *therefore curve is above* $x-axis$

Required Area $=\int_0^{2a} y\ dx=\int_0^{2a}\sqrt{2ax-x^2}\ dx=\int_0^{2a}\sqrt{a^2-a^2+2ax-x^2}\ dx$

$=\int_0^{2a}\sqrt{a^2-(a^2-2ax+x^2)}\ dx$

$=\int_0^{2a}\sqrt{a^2-(x-a)^2}\ dx$

Using formula $\int\sqrt{a^2-x^2}dx$

$=\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}+c$

$=\left[\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)+\left(\frac{x-a}{2}\right)\sqrt{a^2-(x-a)^2}\right]_0^{2a}$

$=\left(\frac{a^2}{2}\sin^{-1}\left(\frac{2a-a}{a}\right)+\left(\frac{2a-a}{2}\right)\sqrt{a^2-(2a-a)^2}\right)-\left(\frac{a^2}{2}\sin^{-1}\left(\frac{0-a}{a}\right)+\left(\frac{0-a}{2}\right)\sqrt{a^2-(0-a)^2}\right)$

$=\left(\frac{a^2}{2}\sin^{-1}(1)+\left(\frac{a}{2}\right)\sqrt{a^2-a^2}\right)-\left(\frac{a^2}{2}\sin^{-1}(-1)+\left(\frac{-a}{2}\right)\sqrt{a^2-a^2}\right)=\left(\frac{a^2}{2}.\frac{\pi}{2}+0\right)-\left(\frac{a^2}{2}\left(-\frac{\pi}{2}\right)-0\right)=\frac{a^2\pi}{4}+\frac{a^2\pi}{4}$

$=\frac{a^2\pi+a^2\pi}{4}=\frac{2a^2\pi}{4}=\frac{a^2\pi}{2}$ *square unit.*

Differential equation:
An equation containing atleast one derivative of a dependent variable with respect to an independent variable is called differential equation.e.g.

$y\frac{dy}{dx}+2x=0$ *and* $x\frac{d^2y}{dx^2}+\frac{dy}{dx}-2x=0$

Order of Differential equation:
The order of the differential equation is the order of the highest derivative in the equation.

$y\frac{dy}{dx}+2x=0$ (1*st order differential equation*)

$x\frac{d^2y}{dx^2}+\frac{dy}{dx}-2x=0$ (2*nd order differential equation*)

Degree of Differential equation:
The degree of a differential equation is the greatest power of the highest order derivative in the equation.

$x\frac{d^4y}{dx^4}+\frac{dy}{dx}-x\left(\frac{d^2y}{dx^2}\right)^4+\frac{dy}{dx}+2x=0$

(1*st degree differential equation*)

$x\left(\frac{d^4y}{dx^4}\right)^3+\frac{dy}{dx}-x\left(\frac{d^2y}{dx^2}\right)^4+\frac{dy}{dx}+2x=0$

(3*rd degree differential equation*)

General solution:
The solution of differential equation which contains arbitrary constants is called general solution.

Particular solution:
The solution obtained from general solution by applying the initial conditions is called particular solution.

Initial value conditions:
The arbitrary constants involving in the solution of a differential equation can be determine by the someven conditions. such conditions are called Initial value c

TIT BIT: *General soltion of the differential equation of order n contains n arbitrary constants which can be determined by n initial values condtions.*

Exercise 3.8

Q.1: Check that each of the following equations written against the differential equation is its solution.

i) $x\frac{dy}{dx}=1+y$

Prove that $y=cx-1$

SOLUTION:

$x\frac{dy}{dx}=1+y$

Separating the variables:

$x\,dy=(1+y)dx$

$\frac{1}{1+y}dy=\frac{1}{x}\ dx$

Integrating both sides

$\int\frac{1}{1+y}dy=\int\frac{1}{x}\ dx$

$ln|1+y|=ln|x|+ln|c|$

$ln|1+y|=ln|cx|$

$1+y=cx$

$y = cx - 1$

ii) $\boldsymbol{x^2(2y+1)\frac{dy}{dx} - 1 = 0}$

prove that $\boldsymbol{y^2 + y = c - \frac{1}{x}}$

SOLUTION:

$x^2(2y+1)\frac{dy}{dx} - 1 = 0$

Separating the variables:

$x^2(2y+1)\frac{dy}{dx} = 1$

$(2y+1)dy = \frac{1}{x^2}\, dx$

Integrating both sides

$\int (2y+1)dy = \int x^{-2}\, dx$

$2\frac{y^2}{2} + y = \frac{x^{-2+1}}{-2+1}$

$y^2 + y = \frac{x^{-1}}{-1} + c$

$y^2 + y = -\frac{1}{x} + c$

iii) $\boldsymbol{y\frac{dy}{dx} - 1 = 0}$

prove that $\boldsymbol{y^2 + y = c - \frac{1}{x}}$

SOLUTION:

$x^2(2y+1)\frac{dy}{dx} - 1 = 0$

Separating the variables:

$x^2(2y+1)\frac{dy}{dx} = 1$

$(2y+1)dy = \frac{1}{x^2}\, dx$

Integrating both sides

$\int (2y+1)dy = \int x^{-2}\, dx$

$2\frac{y^2}{2} + y = \frac{x^{-2+1}}{-2+1}$

$y^2 + y = \frac{x^{-1}}{-1} + c$

$y^2 + y = -\frac{1}{x} + c$

iv) $\boldsymbol{\frac{1}{x}\frac{dy}{dx} - 2y = 0}$

Prove that $\boldsymbol{y = ce^{x^2}}$

SOLUTION:

$\frac{1}{x}\frac{dy}{dx} = 2y$

Separating the variables:

$\frac{1}{y}\, dy = 2x\, dx$

Integrating both sides

$\int \frac{1}{y} dy = \int 2x\, dx$

$ln|y| = 2\frac{x^2}{2} + c_1$

$ln|y| = x^2 + c_1$

$e^{ln|y|} = e^{x^2 + c_1}$

$y = e^{x^2} + e^{c_1}$

$y = e^{x^2} + c$

v) $\boldsymbol{\frac{dy}{dx} = \frac{y^2+1}{e^{-x}}}$

Prove $\boldsymbol{y = \tan(e^x + c)}$

SOLUTION:

$\frac{dy}{dx} = \frac{y^2+1}{e^{-x}}$

Separating the variables:

$\frac{1}{y^2+1}\, dy = \frac{1}{e^{-x}} dx$

$\frac{1}{1+y^2}\, dy = e^x dx$

Integrating both sides

$\int \frac{1}{1+y^2} dy = \int e^x\, dx$

$\tan^{-1} y = e^x + c$

$y = \tan(e^x + c)$

Q.2: $\boldsymbol{\frac{dy}{dx} = -y}$

SOLUTION:

$\frac{dy}{dx} = -y$

Separating the variables:

$\frac{1}{y} dy = -\, dx$

Integrating both sides

$\int \frac{1}{y} dy = -\int 1\, dx$

$\ln y = -x + c_1$

$e^{\ln y} = e^{-x+c_1}$

$y = e^{-x}\, e^{c_1}$

$y = c\, e^{-x}$

Q.3: $\boldsymbol{y\, dx + x\, dy = 0}$

SOLUTION:

$y\, dx + x\, dy = 0$

Separating the variables:

$x\, dy = -y\, dx$

$\frac{1}{y}\, dy = -\frac{1}{x}\, dx$

Integrating both sides

$\int \frac{1}{y} dy = -\int \frac{1}{x}\, dx$

$\ln y = -\ln x + \ln c$

$\ln y + \ln x = +\ln c$

$\ln(xy) = +\ln c$

$xy = c$

Q.4: $\boldsymbol{\frac{dy}{dx} = \frac{1-x}{y}}$

SOLUTION:

$\frac{dy}{dx} = \frac{1-x}{y}$

Separating the variables:

$y\, dy = (1-x)\, dx$

Integrating both sides

$\int y\, dy = \int (1-x)\, dx$

$\frac{y^2}{2} = x - \frac{x^2}{2} + c_1$

$y^2 = 2x - x^2 + 2c_1$

$y^2 = x(2-x) + c$

Q.5: $\boldsymbol{\frac{dy}{dx} = \frac{y}{x^2}}$

SOLUTION:

$\frac{dy}{dx} = \frac{y}{x^2}$

Separating the variables:

$\frac{1}{y}\, dy = \frac{1}{x^2}\, dx$

Integrating both sides

$\int \frac{1}{y} dy = \int x^{-2}\, dx$

$\ln y = \frac{x^{-2+1}}{-2+1} + c_1$

$\ln y = \frac{x^{-1}}{-1} + c_1$

$\ln y = -\frac{1}{x} + c_1$

$e^{\ln y} = e^{-\frac{1}{x}+c_1}$

$y = e^{-\frac{1}{x}}\ e^{c_1}$

$y = c\, e^{-\frac{1}{x}}$

Q.6: $\sin y \csc x \frac{dy}{dx} = 1$

SOLUTION:

$\sin y \csc x \frac{dy}{dx} = 1$

Separating the variables:

$\sin y \frac{1}{\sin x}\ dy = dx$

$\sin y\ dy = \sin x\, dx$

Integrating both sides

$\int \sin y\ dy = \int \sin x\ dx$

$-\cos y = -\cos x + c_1$

$-\cos y = -(\cos x - c_1)$

$\cos y = \cos x - c_1$

$\cos y = \cos x + c$

Q.7: $x\, dy + y(x-1)dx = 0$

SOLUTION:

$x\, dy + y(x-1)dx = 0$

Separating the variables:

$x\, dy = -y(x-1)dx$

$\frac{1}{y}\ dy = -\left(\frac{x-1}{x}\right)\ dx$

$\frac{1}{y}\ dy = -\left(1-\frac{1}{x}\right)\ dx$

Integrating both sides

$\int \frac{1}{y} dy = \int -1 + \frac{1}{x}\ dx$

$\ln y = -x + \ln x + \ln c$

$\ln y = \ln(xc) - x$

$\ln y - \ln(xc) = -x$

$\ln\left(\frac{y}{xc}\right) = -x$

$e^{\ln\left(\frac{y}{xc}\right)} = e^{-x}$

$\frac{y}{xc} = e^{-x}$

$y = cx\, e^{-x}$

Q.8: $\frac{x^2+1}{y+1} = \frac{x}{y} \cdot \frac{dy}{dx}$

SOLUTION:

$\frac{x^2+1}{y+1} = \frac{x}{y} \cdot \frac{dy}{dx}$

Separating the variables:

$(x^2+1)y\, dx = x(y+1)dy$

$x(y+1)\, dy = (x^2+1)y\, dx$

$\frac{y+1}{y}\ dy = \frac{x^2+1}{x}\ dx$

$1 + \frac{1}{y}\ dy = x + \frac{1}{x}\ dx$

Integrating both sides

$\int \left(1+\frac{1}{y}\right) dy = \int \left(x + \frac{1}{x}\right)\ dx$

$y + \ln y = \frac{x^2}{2} + \ln(x) + \ln c$

$y + \ln y = \frac{x^2}{2} + \ln(xc)$

$\ln y - \ln(xc) = \frac{x^2}{2} - y$

$\ln\left(\frac{y}{xc}\right) = \frac{x^2}{2} - y$

$e^{\ln\left(\frac{y}{xc}\right)} = e^{\frac{x^2}{2}-y}$

$\frac{y}{xc} = e^{\frac{x^2}{2}}\ e^{-y}$

$\frac{y}{e^{-y}} = xc\ e^{\frac{x^2}{2}}$

$y\, e^{y} = cx\ e^{\frac{x^2}{2}}$

Q.9: $\frac{1}{x}\frac{dy}{dx} = \frac{1}{2}\ (1+y^2)$

SOLUTION:

$\frac{1}{x}\frac{dy}{dx} = \frac{1}{2}\ (1+y^2)$

Separating the variables:

$\frac{1}{1+y^2} dy = \frac{1}{2}\ (x)dx$

Integrating both sides

$\int \frac{1}{1+y^2}\ dy = \frac{1}{2}\int\ x\, dx$

$\tan^{-1} y = \frac{1}{2}\,\frac{x^2}{2} + c$

$y = \tan\left(\frac{x^2}{4} + c\right)$

Q.10: $2x^2y\frac{dy}{dx} = x^2 - 1$

SOLUTION:

$2x^2y\frac{dy}{dx} = x^2 - 1$

Separating the variables:

$y\, dy = \frac{x^2-1}{2x^2}dx$

$y\, dy = \frac{1}{2}\left(\frac{x^2-1}{x^2}\right)dx$

Integrating both sides

$\int y\, dy = \frac{1}{2}\int \left(1-\frac{1}{x^2}\right)\ dx$

$\int y\, dy = \frac{1}{2}\int\ (1-x^{-2})\ dx$

$\frac{y^2}{2} = \frac{1}{2}\left(x - \frac{x^{-1}}{-1}\right) + c$

$y^2 = x + \frac{1}{x} + c$

Q.11: $\frac{dy}{dx} + \frac{2xy}{2y+1} = x$

SOLUTION:

$\frac{dy}{dx} + \frac{2xy}{2y+1} = x$

Separating the variables:

$\frac{dy}{dx} = x - \frac{2xy}{2y+1}$

$\frac{dy}{dx} = \frac{x(2y+1)-2xy}{2y+1}$

$\frac{dy}{dx} = \frac{2xy+x-2xy}{2y+1}$

$\frac{dy}{dx} = \frac{x}{2y+1}$

$(2y+1)dy = xdx$

Integrating both sides

$\int (2y+1)\ dy = \int\ x\, dx$

$2\frac{y^2}{2} + y = \frac{x^2}{2} + c$

$y(y+1) = \frac{x^2}{2} + c$

Q.12: $(x^2 - yx^2)\frac{dy}{dx} + y^2 + xy^2$

SOLUTION:

$(x^2 - yx^2)\frac{dy}{dx} + y^2 + xy^2$

Separating the variables:

$x^2(1-y)\frac{dy}{dx} = -y^2(1+x)$

$x^2(1-y)dy = -y^2(1+x)dx$

$\frac{1-y}{y^2}\ dy = -\frac{1+x}{x^2}dx$

$\frac{1}{y^2} - \frac{y}{y^2}\ dy = -\left(\frac{1}{x^2} + \frac{x}{x^2}\right)dx$

$\frac{1}{y^2} - \frac{1}{y}\ dy = -\left(\frac{1}{x^2} + \frac{1}{x}\right)dx$

$\left(y^{-2} - \frac{1}{y}\right)\ dy = -\left(x^{-2} + \frac{1}{x}\right)dx$

Integrating both sides

$\int \left(y^{-2} - \frac{1}{y}\right)\ dy = -\int \left(x^{-2} + \frac{1}{x}\right)\ dx$

$\frac{y^{-2+1}}{-2+1} - \ln y = -\left(\frac{x^{-2+1}}{-2+1} + \ln x\right) + c_1$

$\frac{y^{-1}}{-1} - \ln y = -\left(\frac{x^{-1}}{-1} + \ln x\right) + c_1$

$-\frac{1}{y} - \ln y = -\left(-\frac{1}{x} + \ln x\right) + c_1$

$\ln y + \frac{1}{y} = \left(-\frac{1}{x} + \ln x\right) - c_1$

$\ln y + \frac{1}{y} = \ln x - \frac{1}{x} + c$

$Q.13$: $\sec^2 x\ \tan y\ dx + \sec^2 y\ \tan x\ dy = 0$

SOLUTION:

$\sec^2 x\ \tan y\ dx + \sec^2 y\ \tan x\ dy = 0$

Separating the variables:

$\sec^2 y\ \tan x\ dy = \sec^2 x\ \tan y\ dx$

$\frac{\sec^2 y}{\tan y}\ dy = \frac{\sec^2 x}{\tan x}\ dx$

Integrating both sides

$\int \frac{\sec^2 y}{\tan y}\ dy = -\int \frac{\sec^2 x}{\tan x}\ dx$

$\ln(\tan y) = -\ln(\tan x) + \ln c$

$\ln(\tan y) + \ln(\tan x) = \ln c$

$\ln(\tan y \tan x) = \ln c$

$\tan y \tan x = c$

$Q.14$: $\left(y - x\frac{dy}{dx}\right) = 2\left(y^2 + \frac{dy}{dx}\right)$

SOLUTION:

$\left(y - x\frac{dy}{dx}\right) = 2\left(y^2 + \frac{dy}{dx}\right)$

Separating the variables:

$y - x\frac{dy}{dx} = 2y^2 + 2\frac{dy}{dx}$

$-x\frac{dy}{dx} - 2\frac{dy}{dx} = 2y^2 - y$

Multiplying both sides by -1

$x\frac{dy}{dx} + 2\frac{dy}{dx} = y - 2y^2$

$(x+2)\frac{dy}{dx} = y - 2y^2$

$(x+2)dy = y(1-2y)\ dx$

$\frac{1}{y(1-2y)}dy = \frac{1}{x+2}\ dx$

Integrating both sides

$\int \frac{(1-2y)+2y}{y(1-2y)}\ dy = \int \frac{1}{x+2}\ dx$

$\int \left[\frac{(1-2y)}{y(1-2y)} + \frac{2y}{y(1-2y)}\right]\ dy = \int \frac{1}{x+2}\ dx$

$\int \left[\frac{1}{y} + \frac{2}{(1-2y)}\right]\ dy = \int \frac{1}{x+2}\ dx$

$\int \frac{1}{y}\ dy - \int \frac{2}{2y-1}\ dy = \int \frac{1}{x+2}\ dx$

$\ln(y) + \ln(2y-1) = \ln(x+2) + \ln(c)$

$\ln\left(\frac{y}{2y-1}\right) = \ln\ c(x+2)$

$\frac{y}{2y-1} = c(x+2)$

$Q.15$: $1 + \cos x \tan y \frac{dy}{dx} = 0$

SOLUTION:

$1 + \cos x \tan y \frac{dy}{dx} = 0$

Separating the variables:

$\cos x \tan y \frac{dy}{dx} = -1$

$\tan y\, dy = -\frac{1}{\cos x}\ dx$

Integrating both sides

$\int \frac{-\sin y}{\cos y}\ dy = \int \sec x\ dx$

$\ln(\cos y) = \ln(\sec x + \tan x) + \ln c$

$\ln(\cos y) = \ln[c(\sec x + \tan x)]$

$\cos y = c(\sec x + \tan x)$

$Q.16$: $y - x\frac{dy}{dx} = 3\left(1 + x\frac{dy}{dx}\right)$

SOLUTION:

$y - x\frac{dy}{dx} = 3\left(1 + x\frac{dy}{dx}\right)$

Separating the variables:

$y - x\frac{dy}{dx} = 3 + 3x\frac{dy}{dx}$

$y - 3 = 3x\frac{dy}{dx} + x\frac{dy}{dx}$

$y - 3 = 4x\frac{dy}{dx}$

$4x\frac{dy}{dx} = y - 3$

$\frac{1}{y-3}\ dy = \frac{1}{4x}dx$

Integrating both sides

$\int \frac{1}{y-3}\ dy = \frac{1}{4}\int \frac{1}{x}\ dx$

$\ln(y-3) = \frac{1}{4}\ln x + \ln c$

$\ln(y-3) = \ln x^{\frac{1}{4}} + \ln c$

$\ln(y-3) = \ln\left(cx^{\frac{1}{4}}\right)$

$y - 3 = cx^{\frac{1}{4}}$

$y = 3 + cx^{\frac{1}{4}}$

$Q.17$: $\sec x + \tan y \frac{dy}{dx} = 0$

SOLUTION:

$\sec x + \tan y \frac{dy}{dx} = 0$

Separating the variables:

$\tan y \frac{dy}{dx} = -\sec x$

$\tan y\ dy = -\sec x\ dx$

Integrating both sides

$\int \tan y\ dy = -\int \sec x\ dx$

$\int \frac{-\sin y}{\cos y}\ dy = \int \sec x\ dx$

$\ln(\cos y) = \ln(\sec x + \tan x) + \ln c$

$\ln(\cos y) = \ln[c(\sec x + \tan x)]$

$\cos y = c(\sec x + \tan x)$

$Q.18$: $(e^x + e^{-x})\frac{dy}{dx} = e^x - e^{-x}$

SOLUTION:

$(e^x + e^{-x})\frac{dy}{dx} = e^x - e^{-x}$ *Separating the variables:*

$dy = \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$

Integrating both sides

$\int 1\ dy = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$

$y = \ln(e^x + e^{-x}) + c$

Q. 19: Find the general solution of the following equation $\frac{dy}{dx} - x = xy^2$. Also find the Particular solution if $y = 1$ when $x = 0$

SOLUTION:

$\frac{dy}{dx} - x = xy^2$

Separating the variables:

$\frac{dy}{dx} = xy^2 + x$

$\frac{dy}{dx} = x(y^2 + 1)$

$\frac{1}{y^2+1} dy = x\, dx$

Integrating both sides

$\int \frac{1}{1+y^2}\, dy = \int x\, dx$

$\text{Tan}^{-1} y = \frac{x^2}{2} + c$

(*General solution*) (1)

At $x = 0$, $y = 1$

$\tan^{-1}(1) = \frac{(0)^2}{2} + c$

$\frac{\pi}{4} = c$ (*Put in* 1)

$\tan^{-1} y = \frac{x^2}{2} + \frac{\pi}{4}$

(*Particular solution*)

Q. 20: Solve the differential equation $\frac{dx}{dt} = 2x$ given that $x = 4$ when $t = 0$

SOLUTION:

$\frac{dx}{dt} = 2x$

Separating the variables:

$dx = 2x\, dt$

$\frac{1}{x}\, dx = 2\, dt$

Integrating both sides

$\int \frac{1}{x}\, dy = 2\int 1\, dx$

$\ln x = 2t + c_1$

$e^{\ln x} = e^{2t + c_1}$

$x = e^{2t}\ e^{c_1}$

$x = ce^{2t}$ *Where* $e^{c_1} = c$

(*General solution*) (1)

At $x = 4$, $t = 0$

$4 = ce^{2(0)}$

$4 = ce^0$

$4 = c$ *Put in* (1) $\because e^0 = 1$

$x = 4e^{2t}$

(*Particular solution*)

Q. 21: Solve the differential equation $\frac{ds}{dt} + 2st = 0$. Also find the Particular solution if $s = 4e$ when $t = 0$

SOLUTION:

$\frac{ds}{dt} + 2st = 0$

Separating the variables:

$\frac{ds}{dt} = -2st$

$\frac{1}{s} ds = -2t\, dt$

Integrating both sides

$\int \frac{1}{s} ds = -\int 2t\, dx$

$\ln s = -2\frac{t^2}{2} + c_1$

$\ln s = -t^2 + c_1$

$s = e^{-t^2 + c_1}$

$s = e^{-t^2}\ e^{c_1}$

$s = ce^{-t^2}$ *Where* $e^{c_1} = c$

(*General solution*) (1)

At $s = 4e$, $t = 0$

$4e = ce^{-(0)^2}$

$4e = ce^0$

$4e = c$ *Put in* (1) $\because e^0 = 1$

$s = 4e \,.\, e^{-t^2}$

$s = 4\, e^{1-t^2}$

(*Particular solution*)

Q22. In a culture, bacteria increase number of bacteria present. If bacteria are 200 initially and are doubled in 2 hours, find the number of bacteria present four hours later.

Solution:

Let P be numbers of bacteria then

$$\frac{dP}{dt} \propto P$$

⇨ $\frac{dP}{dt} = kP$

⇨ $\frac{1}{P}\, dP = kdt$

take integral

⇨ $\int \frac{1}{P} dP = k\int dt$

⇨ $lnP = kt + lnc$

⇨ $lnp - lnc = kt$

⇨ $\ln\frac{p}{c} = kt$

⇨ $\frac{P}{c} = e^{kt}$

⇨ $p = ce^{kt} \to (i)$

put p=200 , t=0 (condition1)

$200 = ce^{k(0)} = ce^0$

⇨ $c = 200$ $\because e^0 = 1$

So $(i) p = 200e^{kt} \to (ii)$

Put $p = 400$ *when* $t = 2$ (*conditionII*)

$so(i) \Rightarrow 400 = 200e^{kt}$

⇨ $2 = e^{kt} \Rightarrow ln2 = lne^{2k}$

⇨ $2k = ln2$

⇨ $k = \frac{1}{2} ln2$

So $(ii) \Rightarrow p = 200e^{\frac{1}{2}ln2}$

⇨ $p = 200e^{\frac{ln2}{2}(4)}$ $for\ t = 4$

⇨ $p = 200^{2ln2} = 200e^{ln2^2} = 200e^{ln4}$

⇨ $p = 200(4) \Rightarrow p = 800$

Which is required number of bacteria present four latter.

Q.23 a ball is thrown vertically upward with a velocity of 2450cm/sec neglecting air resistance, find

i. **Velocity of ball at any time t**
ii. **Distance traveled in any time t**
iii. **Maximum height attained by the ball**

Solution:

Let $v\ is\ velocity\ and\ g\ is\ acceleration, so$

$$i)\ \frac{dv}{dt} = -g \quad for\ upward$$

⇨ $dv = -gdt$

⇨ $\int dv = -g \int dt$

⇨ $v = -gt + c_1$

$Put\ v = 2450\ , t = 0\ so$

$$2450 = -g(0) + c_1 \Rightarrow c_1 = 2450$$

$$v = -gt + 2450 \quad \because g = 9.8m/sec$$

Thus $v = -980t + 2450 \quad \Rightarrow g = 980cm/sec$

ii) $let\ h\ be\ height\ so$

$$v = \frac{dh}{dt}$$

⇨ $\frac{dh}{dt} = v$

⇨ $\frac{dh}{dt} = -980 + 2450$

⇨ $dh = -980tdt + 2450dt$

⇨ $\int dh = -980 \int tdt + 2450 \int dt$

⇨ $h = -980\frac{t^2}{2} + 2450t + c_2$

$$put\ h = 0\ , t = 0$$

$$0 = -490(0)^2 + 2450(0) + c_2$$

$$\Rightarrow c_2 = 0$$

$$so\ \ h = -490t^2 + 2450$$

$$(iii)\ \ For\ max. hight, v = 0$$

So $0 = -980t^2 + 2450\ from\ (i)$

⇨ $980t = \frac{2450}{980}$

⇨ $t = \frac{5}{2}$

So $h = 2450\left(\frac{5}{2}\right) - 490\left(\frac{5}{2}\right)^2$

$$= 6125 - 30625$$

$$\Rightarrow h = 3062.5$$

$$So\ max. hight = 3062.5cm$$

$$\max hight = 30.6m \ \ (\div\ by\ 100)$$