Average rate of change

Let

f be a real valued function the (difference quotient) $\frac{f(x_1)-f(x)}{x_1-x}$ is called average rate of cahnge.

Derivative:

let $f(x)$ be a function, then its derivative is denoted by $f'(x)$ or $\frac{df}{dx}$ and defined as;

$$f'(x)=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

- The process of finding f' is called differentiation.

Notation for Derivative:

For the function $y=f(x)$

⇨ $\boldsymbol{y+\delta y=f(x+\delta x)\Rightarrow \delta y=f(x+\delta x)-f(x)}$

Where δy is the increment of y (change in the valve of y corresponding to δx (increment of x) dividing (i) by δx on both sides, we get

⇨ $\frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}$

Taking limit on both sides as $\delta x\to 0$ we get

⇨ $\lim_{\delta x\to o}\frac{\delta y}{\delta x}=\lim_{\delta x\to o}\frac{f(x+\delta x)-f(x)}{\delta x}\to(ii)$

$\lim_{\delta x\to o}\frac{\delta y}{\delta x}$ is denoted by $\frac{dy}{dx}$, so (ii) is written as $\frac{dy}{dx}=f'(x)$

Note: the symbol $\frac{dy}{dx}$ is used for the derivative of y with replace to x and here it is not a quotient of dy and dx is also denoted by y'

*Different mathematician used symbols given below.

Leibniz $\frac{dy}{dx}$ or $\frac{df}{dx}$	**Newton** $f'(x)$
Lagrange $f'(x)$	**Cauchy** $D(fx)$

Finding $f'(x)$ from definition of derivative:

Given a function

$f, f(x)$ if it exist can be found by the following four Steps.

Step1. Find $f(x+\delta x)$

Step2. Simplify $f(x+\delta x-f(x)$

Step3. Divide $f(x+\delta x)-f(x)$ by δx to get $\frac{f(x+\delta x)-f(x)}{\delta x}$

And simplify it.

Step4. Find $\lim_{\delta x\to o}\frac{f(x+\delta x)-f(x)}{\delta x}$ the method of finding derivatives by this process is called differentiation by definition or by from first principle.

Derivation of x^n when $n\in Z$

a) Let $y=x^n$ when n is $+ve$ integer.

⇨ $y=x^n\to(i)$, $y+\delta y=(x+\delta x)^n\to(ii)$

by $(ii)-(i)\Rightarrow y+\delta y-\boldsymbol{y}=\boldsymbol{(x+\delta x)^n-x^n}$

⇨ $\boldsymbol{\delta y=(x+\delta x)^n-x^n}$

Using binomial theorem, we have

$$=\delta x\left[nx^{n-1}+\frac{(n-1)}{2!}x^{n-2}(\delta x)+\cdots+(\delta x)^n-x^n\right]$$

Dividing both side by δx

$$\Rightarrow \boldsymbol{\delta y=\left[x^n+nx^{n-1}.\delta x+\frac{n(n-1)(n-2)}{2!}x^{n-2}.(\delta x)^2+\cdots+(\delta x)^{n-1}\right]}$$

⇨ $x^n+nx^{n-1}.\delta x+\frac{n(n-1)(n-2)}{2!}x^{n-2}.(\delta x)^2+\cdots(\delta x)^{n-1}$

$$\frac{\delta y}{\delta x}=\frac{\delta x}{\delta x}\left[nx^{(n-1)}+\frac{(n-1)}{2!}x^{n-2}(\delta x)+\cdots+(\delta x)^{n-1}\right]$$

applying both sides by $\delta x\to 0$

$$\lim_{\delta x\to o}\frac{\delta y}{\delta x}=\lim_{\delta x\to o}\frac{\delta x}{\delta x}\left[nx^{(n-1)}+\frac{(n-1)}{2!}x^{n-2}(\delta x)+\cdots+(\delta x)^{n-1}\right]$$

$$\frac{dy}{dx}=nx^{n-1}+\frac{n-1}{2!}x^{n-1}(0)+\cdots+(0)^{n-1}$$

⇨ $\frac{dy}{dx}=nx^{(n-1)}$

b)

Let $y=x^n$ where n is $-ve$ integer.

let $n=-m$ (m is $+ve$ integer). then

$$y=x^{-m}\Rightarrow y=\frac{1}{x^m}\to(i)$$

⇨ $y+\delta y=\frac{1}{(x+\delta x)^m}\to(ii)$

$$by\ (ii)-(i)\Rightarrow y+\delta y-y=\frac{1}{(x+\delta x)^m}-\frac{1}{x^m}$$

$$\Rightarrow \delta y=\frac{x^m-(x+\delta x)^m}{x^m(x+\delta x)^m}\ \text{using binomial theorem}$$

$$=\frac{x^m-(x^m+mx^{m-1}.\delta x+\frac{m(m-1)}{2!}x^{m-2}.(\delta x)^2+\cdots+(\delta x)^2}{x^m(x+\delta x)^m}$$

$$=\frac{x^m-x^m-mx^{m-1}.\delta x-\frac{mx^{m-1}}{2!}x^{m-2}.(\delta x)^2-\cdots-(\delta x)^2}{x^m(x+\delta x)^m}$$

$$\delta y=\frac{-\delta x}{x^m(x+\delta x)^m}\left(mx^{m-1}+\frac{m(m-1)}{2!}x^{m-2}.(\delta x)+\cdots+(\delta x)^{m-1}\right)$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x}=\frac{-\delta x}{x^m(x+\delta x)^m}\left(\frac{1}{\delta x}\right)\left(mx^{m-1}+\frac{m(m-1)}{2!}x^{m-2}.(\delta x)+\cdots+(\delta x)^{m-1}\right)$$

Applying limit as $\delta x\to 0$

$$\lim_{\delta x\to o}\frac{\delta y}{\delta x}=\lim_{\delta x\to o}\left[\frac{-1}{x^m(x+\delta x)^m}\left(mx^{m-1}+\frac{m(m-1)}{2!}x^{m-2}.(\delta x)+\cdots+(\delta x)^{m-1}\right)\right]$$

$$\frac{dy}{dx}=-\frac{1}{x^m.x^m}\left(mx^{m-1}+\frac{m(m-1)}{2!}x^{m-2}.(0)+\cdots+(0)^{m-1}\right)$$

$$\frac{dy}{dx}=\frac{-mx^{m-1}}{x^{2m}}=-mx^{m-1-2m}=-mx^{-m-1}$$

$$\Rightarrow\frac{dy}{dx}=-mx^{m-1}\ \text{or}\ \frac{dy}{dx}=nx^{n-1}$$

So for we have proved that $\frac{dy}{dx}=nx^{n-1}$, $n\in Z$

Note that $\frac{d}{dx}(x^n)=nx^{n-1}$ i s called power rule and holds if $n\in Q-Z$

Exercise 2.1

Question # 1.

Find the definition, the derivatives w.r.t ' ' x of the following functions defined as:

(i). $2x^2+1$

Solution.

Let $y=2x^2+1 \rightarrow (i)$

$y+\delta y=2(x+\delta x)^2+1 \rightarrow (ii)$

$by\ (ii)-(i)$

$\delta y=2(x+\delta x)^2+1-y$

$\delta y=2(x+\delta x)^2+1-2x^2-1$

$\delta y=2(x^2+\delta x^2+2x\delta x)-2x^2$

$\delta y=2x^2+2\delta x^2+4x\delta x-2x^2$

$\delta y=2\delta x^2+4x\delta x$

$\delta y=\delta x(2\delta x+4x)$

Dividing both sides by δx

$$\frac{\delta y}{\delta x}=(2\delta x+4x)$$

Taking limit when $\delta x\rightarrow 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}(2\delta x+4x)$$

$$\frac{dy}{dx}=2(0)+4x$$

$$\boldsymbol{\frac{dy}{dx}=4x}$$

Which is the required.

(ii). $2-\sqrt{x}$

Solution.

Let $y=2-\sqrt{x}\rightarrow (i)$

$y+\delta y=2-\sqrt{x+\delta x}\rightarrow (ii)$

$eq(ii)-eq(i)$

$\delta y=2-\sqrt{x+\delta x}-y$

$\delta y=2-\sqrt{x+\delta x}-2+\sqrt{x}$

$\delta y=\sqrt{x}-\sqrt{x+\delta x}$

$\delta y=x^{\frac{1}{2}}-(x+\delta x)^{\frac{1}{2}}$

$$\delta y=x^{\frac{1}{2}}-x^{\frac{1}{2}}\left(1+\frac{\delta x}{x}\right)^{\frac{1}{2}}$$

$$\delta y=x^{\frac{1}{2}}-x^{\frac{1}{2}}\left(1+\frac{1}{2}.\frac{\delta x}{x}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\right)$$

$$\delta y=x^{\frac{1}{2}}-x^{\frac{1}{2}}-x^{\frac{1}{2}}\left(\frac{1}{2}.\frac{\delta x}{x}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\right)$$

$$\delta y=-x^{\frac{1}{2}}\delta x\left(\frac{1}{2x}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{\delta x}{x^2}+\cdots\right)$$

Dividing both sides byδx , we have

$$\frac{\delta y}{\delta x}=-x^{\frac{1}{2}}\left(\frac{1}{2x}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{\delta x}{x^2}+\cdots\right)$$

Taking limit when $\delta x\rightarrow 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}-x^{\frac{1}{2}}\left(\frac{1}{2x}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{\delta x}{x^2}+\cdots\right)$$

$$\frac{dy}{dx}=\lim_{\delta x\to 0}-x^{\frac{1}{2}}\left(\frac{1}{2x}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{(0)}{x^2}+\cdots\right)$$

$$\frac{dy}{dx}=-x^{\frac{1}{2}}\left(\frac{1}{2x}\right)$$

$$\frac{dy}{dx}=-\frac{1}{2}x^{\frac{1}{2}-1}$$

$$\frac{dy}{dx}=-\frac{1}{2}x^{-\frac{1}{2}}$$

$$\boldsymbol{\frac{dy}{dx}=-\frac{1}{2x^{\frac{1}{2}}}}$$

Which is the required.

(iii). $\frac{1}{\sqrt{x}}$

Solution.

Let $y=\frac{1}{\sqrt{x}}=x^{-\frac{1}{2}}\rightarrow (i)$

$y+\delta y=(x+\delta x)^{-\frac{1}{2}}\rightarrow (ii)$

$eq(ii)-eq(i)$

$\delta y=(x+\delta x)^{-\frac{1}{2}}-y$

$\delta y=(x+\delta x)^{-\frac{1}{2}}-x^{-\frac{1}{2}}$

$\delta y=(x+\delta x)^{-\frac{1}{2}}-x^{-\frac{1}{2}}$

$$\delta y=x^{-\frac{1}{2}}\left(1+\frac{\delta x}{x}\right)^{-\frac{1}{2}}-x^{-\frac{1}{2}}$$

$$\delta y=x^{-\frac{1}{2}}\left(1-\frac{1}{2}.\frac{\delta x}{x}+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\right)-x^{-\frac{1}{2}}$$

$$\delta y=x^{-\frac{1}{2}}+x^{-\frac{1}{2}}\left(-\frac{1}{2}.\frac{\delta x}{x}+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\right)-x^{-\frac{1}{2}}$$

$$\delta y=x^{-\frac{1}{2}}\delta x\left(-\frac{1}{2x}+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}\frac{\delta x}{x^2}+\cdots\right)$$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x}=x^{-\frac{1}{2}}\left(-\frac{1}{2x}+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}\frac{\delta x}{x^2}+\cdots\right)$$

Taking limit when $\delta x\rightarrow 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}x^{-\frac{1}{2}}\left(-\frac{1}{2x}+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}\frac{\delta x}{x^2}+\cdots\right)$$

$$\frac{dy}{dx}=x^{-\frac{1}{2}}\left(-\frac{1}{2x}+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}\frac{(0)}{x^2}+\cdots\right)$$

$$\frac{dy}{dx} = x^{-\frac{1}{2}}\left(-\frac{1}{2x}\right)$$
$$\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{1}{2}-1}$$
$$\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$$
$$\boldsymbol{\frac{dy}{dx} = -\frac{1}{2x^{\frac{3}{2}}}}$$

Which is the required.

(iv) $\frac{1}{x^3}$

Solution.

Let $y = \frac{1}{x^3} = x^{-3} \rightarrow (i)$

$y + \delta y = (x+\delta x)^{-3} \rightarrow (ii)$

$eq(ii) - eq(i)$

$$\delta y = (x+\delta x)^{-3} - y$$
$$\delta y = (x+\delta x)^{-3} - x^{-3}$$
$$\delta y = x^{-3}(1+\frac{\delta x}{x})^{-3} - x^{-3}$$
$$\delta y = x^{-3}((1+\frac{\delta x}{x})^{-3} - 1)$$
$$\delta y = x^{-3}(1 - \frac{3\delta x}{x} + \frac{-3(-3-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots - 1)$$
$$\delta y = x^{-3}\left(-\frac{3\delta x}{x} + \frac{-3(-3-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots\right)$$
$$\delta y = x^{-3}\delta x\left(-\frac{3}{x} + \frac{-3(-3-1)}{2!}\frac{\delta x}{x^2} + \cdots\right)$$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = x^{-3}\left(-\frac{3}{x} + \frac{-3(-3-1)}{2!}\frac{\delta x}{x^2} + \cdots\right)$$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} x^{-3}\left(-\frac{3}{x} + \frac{-3(-3-1)}{2!}\frac{\delta x}{x^2} + \cdots\right)$$
$$\frac{dy}{dx} = x^{-3}\left(-\frac{3}{x} + \frac{-3(-3-1)}{2!}\frac{(0)}{x^2} + \cdots\right)$$
$$\frac{dy}{dx} = -\frac{3}{x^{1+3}}$$
$$\frac{dy}{dx} = -\frac{3}{x^4}$$

Which is the required.

(v). $\frac{1}{x-a}$

Solution.

Let $y = \frac{1}{x-a} = (x-a)^{-1} \rightarrow (i)$

$y + \delta y = (x+\delta x - a)^{-1} \rightarrow (ii)$

$eq(ii) - eq(i)$

$$\delta y = (x+\delta x-a)^{-3} - y$$
$$\delta y = (x-a+\delta x)^{-1} - (x-a)^{-1}$$
$$\delta y = (x-a)^{-1}(1+\frac{\delta x}{x-a})^{-1} - (x-a)^{-1}$$
$$\delta y = (x-a)^{-1}((1+\frac{\delta x}{x-a})^{-1} - 1)$$
$$\delta y = (x-a)^{-1}(1 - \frac{\delta x}{x-a} + \frac{-1(-1-1)}{2!}\left(\frac{\delta x}{x-a}\right)^2 + \cdots - 1)$$
$$\delta y = (x-a)^{-1}\left(-\frac{\delta x}{x-a} + \frac{-1(-1-1)}{2!}\left(\frac{\delta x}{x-a}\right)^2 + \cdots\right)$$
$$\delta y = (x-a)^{-1}\delta x\left(-\frac{1}{x-a} + \frac{-1(-1-1)}{2!}\frac{\delta x}{(x-a)^2} + \cdots\right)$$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = (x-a)^{-1}\left(-\frac{1}{x-a} + \frac{-1(-1-1)}{2!}\frac{\delta x}{(x-a)^2} + \cdots\right)$$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}(x-a)^{-1}\left(-\frac{1}{x-a} + \frac{-1(-1-1)}{2!}\frac{\delta x}{(x-a)^2} + \cdots\right)$$
$$\frac{dy}{dx} = (x-a)^{-1}\left(-\frac{1}{x-a} + \frac{-1(-1-1)}{2!}\frac{(0)}{(x-a)^2} + \cdots\right)$$
$$\frac{dy}{dx} = -\frac{1}{(x-a)^{1+1}}$$
$$\frac{dy}{dx} = -\frac{1}{(x-a)^2}$$

Which is the required.

(vi). $x(x-3)$

Solution.

Let $y = x(x-3) = x^2 - 3x$

$y + \delta y = (x+\delta x)^2 - 3(x+\delta x)$

$\delta y = (x+\delta x)^2 - 3x - 3\delta x - y$

$\delta y = x^2 + \delta x^2 + 2x\delta x - 3x - 3\delta x - x^2 + 3x$

$\delta y = \delta x^2 + 2x\delta x - 3\delta x$

$\delta y = \delta x(\delta x + 2x - 3)$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = (\delta x + 2x - 3)$$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}(\delta x + 2x - 3)$$
$$\frac{dy}{dx} = (0 + 2x - 3)$$
$$\boldsymbol{\frac{dy}{dx} = 2x - 3}$$

Which is the required.

(vii) $\frac{2}{x^4}$

Solution:

Let $y = \frac{2}{x^4} = 2x^{-4} \rightarrow (i)$

$y + \delta y = 2(x+\delta x)^{-4} \rightarrow (ii)$

$e(ii) - e(i)$

$\delta y = 2(x+\delta x)^{-4} - y$

$\delta y = 2(x+\delta x)^{-4} - 2x^{-4}$

$$\delta y = x^{-4}(1+\frac{\delta x}{x})^{-4} - 2x^{-4}$$

$$\delta y = 2x^{-4}((1+\frac{\delta x}{x})^{-4} - 1)$$

$$\delta y = 2x^{-4}(1 - \frac{4\delta x}{x} + \frac{-4(-4-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots - 1)$$

$$\delta y = 2\,x^{-4}\left(-\frac{4\delta x}{x} + \frac{-4(-4-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots\right)$$

$$\delta y = 2x^{-4}\delta x\left(-\frac{4}{x} + \frac{-4(-4-1)}{2!}\frac{\delta x}{x^2} + \cdots\right)$$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = 2x^{-4}\left(-\frac{4}{x} + \frac{-4(-4-1)}{2!}\frac{\delta x}{x^2} + \cdots\right)$$

Taking limit when $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} 2x^{-4}\left(-\frac{4}{x} + \frac{-4(-4-1)}{2!}\frac{\delta x}{x^2} + \cdots\right)$$

$$\frac{dy}{dx} = 2x^{-4}\left(-\frac{4}{x} + \frac{-4(-4-1)}{2!}\frac{(0)}{x^2} + \cdots\right)$$

$$\frac{dy}{dx} = -\frac{8}{x^{1+4}}$$

$$\frac{dy}{dx} = -\frac{8}{x^5}$$

Which is the required.

(viii). $(x+4)^{\frac{1}{3}}$

Solution.

Let $y = (x+4)^{\frac{1}{3}} \to (i)$

$y + \delta y = (x + \delta x + 4)^{\frac{1}{3}} \to (ii)$

$eq(ii) - eq(i)$

$$\delta y = (x + \delta x + 4)^{\frac{1}{3}} - y$$

$$\delta y = (x + 4 + \delta x)^{\frac{1}{3}} - (x+4)^{\frac{1}{3}}$$

$$\delta y = (x+4)^{\frac{1}{3}}\left(1 + \frac{\delta x}{x+4}\right)^{\frac{1}{3}} - (x+4)^{\frac{1}{3}}$$

$$\delta y = (x+4)^{\frac{1}{3}}\left(\left(1 + \frac{1}{3}\cdot\frac{\delta x}{x+4} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{\delta x}{x+4}\right)^2 + \cdots\right) - 1\right)$$

$$\delta y = (x+4)^{\frac{1}{3}}\left(1 + \frac{1}{3}\cdot\frac{\delta x}{x+4} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{\delta x}{x+4}\right)^2 + \cdots - 1\right)$$

$$\delta y = (x+4)^{\frac{1}{3}}\delta x\left(\frac{1}{3}\cdot\frac{1}{x+4} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\frac{\delta x}{(x+4)^2} + \cdots\right)$$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = (x+4)^{\frac{1}{3}}\left(\frac{1}{3}\cdot\frac{1}{x+4} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\frac{\delta x}{(x+4)^2} + \cdots\right)$$

Taking limit when $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}(x+4)^{\frac{1}{3}}\left(\frac{1}{3}\cdot\frac{1}{x+4} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\frac{\delta x}{(x+4)^2} + \cdots\right)$$

$$\frac{dy}{dx} = (x+4)^{\frac{1}{3}}\left(\frac{1}{3}\cdot\frac{1}{x+4} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\frac{(0)}{(x+4)^2} + \cdots\right)$$

$$\frac{dy}{dx} = (x+4)^{\frac{1}{3}}\cdot\frac{1}{3(x+4)}$$

$$\frac{dy}{dx} = \frac{1}{3(x+4)^{1-\frac{1}{3}}}$$

$$\boldsymbol{\frac{dy}{dx} = \frac{1}{3(x+4)^{\frac{2}{3}}}}$$

Which is the required.

(ix).$x^{\frac{3}{2}}$

Solution.

Let $y = x^{\frac{3}{2}} \to (i)$

$y + \delta y = (x + \delta x)^{\frac{3}{2}} \to (ii)$

$eq(ii) - eq(i)$

$$\delta y = (x + \delta x)^{\frac{3}{2}} - y$$

$$\delta y = (x + \delta x)^{\frac{3}{2}} - x^{\frac{3}{2}}$$

$$\delta y = x^{\frac{3}{2}}\left(1 + \frac{\delta x}{x}\right)^{\frac{3}{2}} - x^{\frac{3}{2}}$$

$$\delta y = x^{\frac{3}{2}}\left(1 + \frac{3}{2}\cdot\frac{\delta x}{x} + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots\right) - x^{\frac{3}{2}}$$

$$\delta y = x^{\frac{3}{2}} + x^{\frac{3}{2}}\left(\frac{3}{2}\cdot\frac{\delta x}{x} + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots\right) - x^{\frac{3}{2}}$$

$$\delta y = x^{\frac{3}{2}}\delta x\left(\frac{3}{2x} + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}\frac{\delta x}{x^2} + \cdots\right)$$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = x^{\frac{3}{2}}\left(\frac{3}{2x} + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}\frac{\delta x}{x^2} + \cdots\right)$$

Taking limit when $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} x^{\frac{3}{2}}\left(\frac{3}{2x} + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}\frac{\delta x}{x^2} + \cdots\right)$$

$$\frac{dy}{dx} = x^{\frac{3}{2}}\left(\frac{3}{2x} + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}\frac{(0)}{x^2} + \cdots\right)$$

$$\frac{dy}{dx} = x^{\frac{3}{2}}\left(\frac{3}{2x}\right)$$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{3}{2}-1}$$

$$\boldsymbol{\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}}$$

Which is the required.

(x).$x^{\frac{5}{2}}$

Solution.

Let $y = x^{\frac{5}{2}} \to (i)$

$y+\delta y=(x+\delta x)^{\frac{5}{2}} \rightarrow (ii)$
$\delta y=(x+\delta x)^{\frac{5}{2}}-y$
$\delta y=(x+\delta x)^{\frac{5}{2}}-x^{\frac{5}{2}}$
$\delta y=x^{\frac{5}{2}}\left(1+\frac{\delta x}{x}\right)^{\frac{5}{2}}-x^{\frac{5}{2}}$
$\delta y=x^{\frac{5}{2}}\left(1+\frac{5}{2}.\frac{\delta x}{x}+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\right)-x^{\frac{5}{2}}$
$\delta y=x^{\frac{5}{2}}+x^{\frac{5}{2}}\left(\frac{5}{2}.\frac{\delta x}{x}+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\right)-x^{\frac{5}{2}}$
$\delta y=x^{\frac{5}{2}}\delta x\left(\frac{5}{2x}+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}\frac{\delta x}{x^2}+\cdots\right)$

Dividing both sides by x , we have

$\frac{\delta y}{\delta x}=x^{\frac{5}{2}}\left(\frac{5}{2x}+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}\frac{\delta x}{x^2}+\cdots\right)$

Taking limit when $\delta x \rightarrow 0$

$\lim_{\delta x\rightarrow 0}\frac{\delta y}{\delta x}=\lim_{\delta x\rightarrow 0}x^{\frac{5}{2}}\left(\frac{5}{2x}+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}\frac{\delta x}{x^2}+\cdots\right)$
$\frac{dy}{dx}=x^{\frac{5}{2}}\left(\frac{5}{2x}+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}\frac{(0)}{x^2}+\cdots\right)$
$\frac{dy}{dx}=x^{\frac{5}{2}}\left(\frac{5}{2x}\right)$
$\frac{dy}{dx}=\frac{5}{2}x^{\frac{5}{2}-1}$
$\boldsymbol{\frac{dy}{dx}=\frac{5}{2}x^{\frac{3}{2}}}$

Which is the required.

(xi). $\boldsymbol{x^m}$, $\boldsymbol{m\in N}$

Solution.

Let $y=x^m \rightarrow (i)$
$y+\delta y=(x+\delta x)^m \rightarrow (ii)$
$eq(ii)-eq(i)$
$\delta y=(x+\delta x)^m-y$
$\delta y=(x+\delta x)^m-x^m$
$\delta y=x^m(1+\frac{\delta x}{x})^m-x^m$
$\delta y=x^m((1+\frac{\delta x}{x})^m-1)$
$\delta y=x^m(1+\frac{m\delta x}{x}+\frac{m(m-1)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots-1)$
$\delta y=x^m\left(\frac{m\delta x}{x}+\frac{m(m-1)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\right)$
$\delta y=x^m\delta x\left(\frac{m}{x}+\frac{m(m-1)}{2!}\frac{\delta x}{x^2}+\cdots\right)$

Dividing both sides by x , we have

$\frac{\delta y}{\delta x}=x^m\left(\frac{m}{x}+\frac{m(m-1)}{2!}\frac{\delta x}{x^2}+\cdots\right)$

Taking limit when $\delta x \rightarrow 0$

$\lim_{\delta x\rightarrow 0}\frac{\delta y}{\delta x}=\lim_{\delta x\rightarrow 0}x^m\left(\frac{m}{x}+\frac{m(m-1)}{2!}\frac{\delta x}{x^2}+\cdots\right)$
$\frac{dy}{dx}=x^m\left(\frac{m}{x}+\frac{m(m-1)}{2!}\frac{(0)}{x^2}+\cdots\right)$
$\boldsymbol{\frac{dy}{dx}=mx^{m-1}}$

Which is the required.

(xii). $\boldsymbol{\frac{1}{x^m}}$, $\boldsymbol{m\in N}$

Solution.

Let $y=x^{-m} \rightarrow (i)$
$y+\delta y=(x+\delta x)^{-m} \rightarrow (ii)$
$eq(ii)-eq(i)$
$\delta y=(x+\delta x)^{-m}-y$
$\delta y=(x+\delta x)^{-m}-x^{-m}$
$\delta y=x^{-m}(1+\frac{\delta x}{x})^{-m}-x^{-m}$
$\delta y=x^{-m}(\left(1+\frac{\delta x}{x}\right)^{-m}-1)$
$\delta y=x^{-m}(1+\frac{-m\delta x}{x}+\frac{-m(-m-1)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots-1)$
$\delta y=x^{-m}\left(\frac{-m\delta x}{x}+\frac{-m(-m-1)}{2!}\left(\frac{\delta x}{x}\right)^2+\cdots\right)$
$\delta y=x^{-m}\delta x\left(\frac{-m}{x}+\frac{-m(-m-1)}{2!}\frac{\delta x}{x^2}+\cdots\right)$

Dividing both sides by x , we have

$\frac{\delta y}{\delta x}=x^{-m}\left(\frac{-m}{x}+\frac{-m(-m-1)}{2!}\frac{\delta x}{x^2}+\cdots\right)$

Taking limit when $\delta x \rightarrow 0$

$\lim_{\delta x\rightarrow 0}\frac{\delta y}{\delta x}=\lim_{\delta x\rightarrow 0}x^{-m}\left(\frac{-m}{x}+\frac{-m(-m-1)}{2!}\frac{\delta x}{x^2}+\cdots\right)$
$\frac{dy}{dx}=x^{-m}\left(\frac{-m}{x}+\frac{-m(-m-1)}{2!}\frac{(0)}{x^2}+\cdots\right)$
$\frac{dy}{dx}=-\frac{m}{x^{1+m}}$
$\boldsymbol{\frac{dy}{dx}=-\frac{m}{x^{m+1}}}$

Which is the required.

(xiii). $\boldsymbol{x^{40}}$

Solution.

Let $y = x^{40} \rightarrow (i)$

$y + \delta y = (x + \delta x)^{40} \rightarrow (ii)$

$eq(ii) - eq(i)$

$\delta y = (x + \delta x)^{40} - y$

$\delta y = (x + \delta x)^{40} - x^{40}$

$\delta y = x^{40}(1 + \frac{\delta x}{x})^{40} - x^{40}$

$\delta y = x^{40}((1 + \frac{\delta x}{x})^{40} - 1)$

$\delta y = x^{40}(1 + \frac{40\delta x}{x} + \frac{40(40-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots - 1)$

$\delta y = x^{40}\left(\frac{40\delta x}{x} + \frac{40(40-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots\right)$

$\delta y = x^{40}\delta x\left(\frac{40}{x} + \frac{40(40-1)}{2!}\frac{\delta x}{x^2} + \cdots\right)$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = x^{40}\left(\frac{40}{x} + \frac{40(40-1)}{2!}\frac{\delta x}{x^2} + \cdots\right)$$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0} x^{40}\left(\frac{40}{x} + \frac{40(40-1)}{2!}\frac{\delta x}{x^2} + \cdots\right)$$

$$\frac{dy}{dx} = x^{40}\left(\frac{40}{x} + \frac{40(40-1)}{2!}\frac{(0)}{x^2} + \cdots\right)$$

$$\frac{dy}{dx} = 40x^{40-1}$$

$$\frac{dy}{dx} = 40x^{39}$$

Which is the required.

(xiv).x^{-100}

Solution.

Let $y = x^{-100} \rightarrow (i)$

$y + \delta y = (x + \delta x)^{-100} \rightarrow (ii)$

$\delta y = (x + \delta x)^{-100} - y$

$\delta y = (x + \delta x)^{-100} - x^{-100}$

$\delta y = x^{-100}(1 + \frac{\delta x}{x})^{-100} - x^{-100}$

$\delta y = x^{-100}(\left(1 + \frac{\delta x}{x}\right)^{-100} - 1)$

$\delta y = x^{-100}(1 + \frac{-100\delta x}{x} + \frac{-100(-100-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots - 1)$

$\delta y = x^{-100}\left(\frac{-100\delta x}{x} + \frac{-100(-100-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots\right)$

$\delta y = x^{-100}\delta x\left(\frac{-100}{x} + \frac{-100(-100-1)}{2!}\frac{\delta x}{x^2} + \cdots\right)$

Dividing both sides by x , we have

$\frac{\delta y}{\delta x} = x^{-100}\left(\frac{-100}{x} + \frac{-100(-100-1)}{2!}\frac{\delta x}{x^2} + \cdots\right)$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0} x^{-100}\left(\frac{-100}{x} + \frac{-100(-100-1)}{2!}\frac{\delta x}{x^2} + \cdots\right)$$

$\delta y = x^{-100}\delta x\left(\frac{-100}{x} + \frac{-100(-100-1)}{2!}\frac{\delta x}{x^2} + \cdots\right)$

Dividing both sides byδx , we have

$\frac{\delta y}{\delta x} = x^{-100}\left(\frac{-100}{x} + \frac{-100(-100-1)}{2!}\frac{\delta x}{x^2} + \cdots\right)$

Taking limit when $\delta x \rightarrow 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0} x^{-100}\left(\frac{-100}{x} + \frac{-100(-100-1)}{2!}\frac{\delta x}{x^2} + \cdots\right)$$

$\frac{dy}{dx} = x^{-100}\left(\frac{-100}{x} + \frac{-100(-100-1)}{2!}\frac{(0)}{x^2} + \cdots\right)$

$\frac{dy}{dx} = -\frac{100}{x^{1+100}}$

$\boldsymbol{\frac{dy}{dx} = -\frac{100}{x^{101}}}$

Which is the required.

Q2. Find $\frac{dy}{dx}$ $\boldsymbol{froom\ first\ principal\ if}$

(i) $\boldsymbol{\sqrt{x+2}}$

Solution.

Let $y = (x+2)^{\frac{1}{2}} \rightarrow (i)$

$y + \delta y = (x + \delta x + 2)^{\frac{1}{2}} \rightarrow (ii)$

$\delta y = (x + \delta x + 2)^{\frac{1}{2}} - y$

$\delta y = (x + 2 + \delta x)^{\frac{1}{2}} - (x+2)^{\frac{1}{2}}$

$\delta y = (x+2)^{\frac{1}{2}}\left(1 + \frac{\delta x}{x+2}\right)^{\frac{1}{2}} - (x+2)^{\frac{1}{2}}$

$\delta y = (x+2)^{\frac{1}{2}}(\left(1 + \frac{\delta x}{x+2}\right)^{\frac{1}{2}} - 1)$

$\delta y = (x+2)^{\frac{1}{2}}(1 + \frac{1}{2}\frac{\delta x}{(x+2)} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x+2}\right)^2 + \cdots - 1)$

$\delta y = (x+2)^{\frac{1}{2}}\delta x(\frac{1}{2}\frac{1}{(x+2)} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{\delta x}{(x+2)^2} + \cdots)$

Dividing both sides by x , we have

$\frac{\delta y}{\delta x} = (x+2)^{\frac{1}{2}}(\frac{1}{2}\frac{1}{(x+2)} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{\delta x}{(x+2)^2} + \cdots)$

Taking limit when $\delta x \rightarrow 0$

$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}(x+2)^{\frac{1}{2}}(\frac{1}{2}\frac{1}{(x+2)} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{\delta x}{(x+2)^2} + \cdots)$

$\frac{dy}{dx} = (x+2)^{\frac{1}{2}}(\frac{1}{2}\frac{1}{(x+2)} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{(0)}{(x+2)^2} + \cdots)$

$\frac{dy}{dx} = (x+2)^{\frac{1}{2}}\frac{1}{2}\frac{1}{(x+2)}$

$$\frac{dy}{dx} = \frac{1}{2(x+2)^{1-\frac{1}{2}}}$$

$$\boldsymbol{\frac{dy}{dx} = \frac{1}{2(x+2)^{\frac{1}{2}}}}$$

Which is the required.

(ii). $\frac{1}{\sqrt{x+a}}$

Solution.

Let $y = (x+a)^{-\frac{1}{2}} \to (i)$

$y + \delta y = (x + \delta x + a)^{-\frac{1}{2}} \to (ii)$

$\delta y = (x + \delta x + 2)^{\frac{1}{2}} - y$

$\delta y = (x + a + \delta x)^{-\frac{1}{2}} - (x+a)^{-\frac{1}{2}}$

$$\delta y = (x+a)^{-\frac{1}{2}}\left(1 + \frac{\delta x}{x+a}\right)^{-\frac{1}{2}} - (x+a)^{-\frac{1}{2}}$$

$$\delta y = (x+a)^{-\frac{1}{2}}(\left(1 + \frac{\delta x}{x+a}\right)^{-\frac{1}{2}} - 1)$$

$$\delta y = (x+a)^{-\frac{1}{2}}(1 - \frac{1}{2}\frac{\delta x}{(x+a)} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x+a}\right)^2 + \cdots - 1)$$

$$\delta y = (x+a)^{-\frac{1}{2}}\delta x(-\frac{1}{2}\frac{1}{(x+a)} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}\frac{\delta x}{(x+a)^2} + \cdots)$$

Dividing both sides by x , we have

$$\frac{\delta y}{\delta x} = (x+a)^{-\frac{1}{2}}(-\frac{1}{2}\frac{1}{(x+a)} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}\frac{\delta x}{(x+a)^2} + \cdots)$$

Taking limit when $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}(x+a)^{-\frac{1}{2}}(-\frac{1}{2}\frac{1}{(x+a)} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}\frac{\delta x}{(x+a)^2} + \cdots)$$

$$\frac{dy}{dx} = (x+a)^{-\frac{1}{2}}(-\frac{1}{2}\frac{1}{(x+a)} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}\frac{(0)}{(x+a)^2} + \cdots)$$

$$\frac{dy}{dx} = (x+a)^{-\frac{1}{2}}\left(-\frac{1}{2}\frac{1}{(x+a)}\right)$$

$$\frac{dy}{dx} = -\frac{1}{2(x+a)^{1+\frac{1}{2}}}$$

$$\boldsymbol{\frac{dy}{dx} = -\frac{1}{2(x+a)^{\frac{3}{2}}}}$$

Which is the required.

Differentiation of expression of the types.

$(ax+b)^n$ $and \frac{1}{(ax+b)^n}$ $, n = 1,2,3$

Exercise 2.2

Question # 1

Find from first principles, the derivatives of the following expansions w.r.t. their respective independent variables:

(i) $\boldsymbol{(ax+b)^3}$

Solution.

Let $\boldsymbol{y = (ax+b)^3 \to (i)}$

$y + \delta y = (a(x+\delta x) + b)^3 \to (ii)$

$eq(ii) - eq(i)$

$\delta y = (ax + a\delta x + b)^3 - y$

$\delta y = ((ax+b) + a\delta x)^3 - (ax+b)^3$

$$\delta y = (ax+b)^3\left(1 + \frac{a\delta x}{(ax+b)}\right)^3 - (ax+b)^3$$

$$\delta y = (ax+b)^3(\left(1 + \frac{a\delta x}{(ax+b)}\right)^3 - 1)$$

$$\delta y = (ax+b)^3(1 + \frac{3\,a\delta x}{(ax+b)} + \frac{3(3-1)}{2!}\left(\frac{a\delta x}{ax+b}\right)^2 + \cdots - 1)$$

$$\delta y = (ax+b)^3\delta x(\frac{3\,a}{(ax+b)} + \frac{3(3-1)}{2!}\frac{a^2\delta x}{(ax+b)^2} + \cdots)$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = (ax+b)^3(\frac{3\,a}{(ax+b)} + \frac{3(3-1)}{2!}\frac{a^2\delta x}{(ax+b)^2} + \cdots)$$

Taking limit when $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}(ax+b)^3(\frac{3\,a}{(ax+b)} + \frac{3(3-1)}{2!}\frac{a^2\delta x}{(ax+b)^2} + \cdots)$$

$$\frac{dy}{dx} = (ax+b)^3(\frac{3\,a}{(ax+b)} + \frac{3(3-1)}{2!}\frac{a^2(0)}{(ax+b)^2} + \cdots)$$

$$\frac{dy}{dx} = (ax+b)^3(\frac{3\,a}{(ax+b)})$$

$$\frac{dy}{dx} = 3a\,(ax+b)^{3-1}$$

$$\boldsymbol{\frac{dy}{dx} = 3a\,(ax+b)^2}$$

Which is the required.

(ii) $\boldsymbol{(2x+3)^5}$

Solution.

Let $y = (2x+3)^5 \to (i)$

$y + \delta y = (2(x+\delta x) + 3)^5 \to (ii)$

$eq(ii) - eq(i)$

$\delta y = (2x + 2\delta x + 3)^5 - y$

$\delta y = ((2x+3) + 2\delta x)^5 - (2x+3)^5$

$$\delta y = (2x+3)^5\left(1 + \frac{2\delta x}{(2x+3)}\right)^5 - (2x+3)^5$$

$$\delta y = (2x+3)^5(\left(1 + \frac{2\delta x}{(2x+3)}\right)^5 - 1)$$

$$\delta y = (2x+3)^5(1 + \frac{5\,(2)\delta x}{(2x+3)} + \frac{5(5-1)}{2!}\left(\frac{2\delta x}{2x+3}\right)^2 + \cdots - 1)$$

$\delta y=(2x+3)^5\delta x(\frac{5\ (2)}{(2x+3)}+\frac{5(5-1)}{2!}\frac{2^2\delta x}{(2x+3)^2}+\cdots)$

Dividing both sides by δx

$\frac{\delta y}{\delta x}=(2x+3)^5(\frac{5\ (2)}{(2x+3)}+\frac{5(5-1)}{2!}\frac{2^2\delta x}{(2x+3)^2}+\cdots)$

Taking limit when $\delta x\to 0$

$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}(2x+3)^5(\frac{5\ (2)}{(2x+3)}+\frac{5(5-1)}{2!}\frac{2^2\delta x}{(2x+3)^2}+\cdots)$

$\frac{dy}{dx}=(2x+3)^5(\frac{5\ (2)}{(2x+3)}+\frac{5(5-1)}{2!}\frac{2^2(0)}{(2x+3)^2}+\cdots)$

$\frac{dy}{dx}=(2x+3)^5(\frac{5\ (2)}{(2x+3)})$

$\frac{dy}{dx}=10(2x+3)^{5-1}$

$\boldsymbol{\frac{dy}{dx}=10(2x+3)^4}$

Which is the required.

(iii) $\boldsymbol{(3t+2)^{-2}}$

Solution.

Let $y=(3t+2)^{-2}\to(i)$

$y+\delta y=(3(t+\delta t)+2)^{-2}-(ii)$

$eq(ii)-eq(i)$

$\delta y=(3t+3\delta t+2)^{-2}-y$

$\delta y=((3t+2)+3\delta t)^{-2}-(3t+2)^{-2}$

$\delta y=(3t+2)^{-2}\left(1+\frac{3\delta t}{(3t+2)}\right)^{-2}-(3t+2)^{-2}$

$\delta y=(3t+2)^{-2}(\left(1+\frac{3\delta t}{(3t+2)}\right)^{-2}-1)$

$\delta y=(3t+2)^{-2}(1+\frac{-2\ (3)\delta t}{(3t+2)}+\frac{-2(-2-1)}{2!}\left(\frac{3\delta t}{3t+2}\right)^2+\cdots-1)$

$\delta y=(3t+2)^{-2}\delta t(-\frac{2(3)}{(3t+2)}+\frac{-2(-2-1)}{2!}\frac{3\delta t}{(3t+2)^2}+\cdots)$

Dividing both sides by δt

$\frac{\delta y}{\delta t}=(3t+2)^{-2}(-\frac{2(3)}{(3t+2)}+\frac{-2(-2-1)}{2!}\frac{3\delta t}{(3t+2)^2}+\cdots)$

Taking limit when $\delta t\to 0$

$\lim_{\delta t\to 0}\frac{\delta y}{\delta t}=\lim_{\delta t\to 0}(3t+2)^{-2}(-\frac{2(3)}{(3t+2)}+\frac{-2(-2-1)}{2!}\frac{3\delta t}{(3t+2)^2}+\cdots)$

$\frac{dy}{dt}=(3t+2)^{-2}(-\frac{2(3)}{(3t+2)}+\frac{-2(-2-1)}{2!}\frac{3(0)}{(3t+2)^2}+\cdots)$

$\frac{dy}{dt}=(3t+2)^{-2}(-\frac{2(3)}{(3t+2)})$

$\frac{dy}{dt}=-\frac{6}{(3t+2)^{1+2}}$

$\boldsymbol{\frac{dy}{dt}=-\frac{6}{(3t+2)^3}}$

Which is the required.

(iv) $\boldsymbol{(ax+b)^{-5}}$

Solution.

Let $y=(ax+b)^{-5}\to(i)$

$y+\delta y=(a(x+\delta x)+b)^{-5}\to(ii)$

$eq(ii)-eq(i)$

$\delta y=(ax+a\delta x+b)^{-5}-y$

$\delta y=((ax+b)+a\delta x)^{-5}-(ax+b)^{-5}$

$\delta y=(ax+b)^{-5}\left(1+\frac{a\delta x}{(ax+b)}\right)^{-5}-(ax+b)^{-5}$

$\delta y=(ax+b)^{-5}(\left(1+\frac{a\delta x}{(ax+b)}\right)^{-5}-1)$

$\delta y=(ax+b)^{-5}(1-\frac{5\ a\delta x}{(ax+b)}+\frac{-5(-5-1)}{2!}\left(\frac{a\delta x}{ax+b}\right)^2+\cdots-1)$

$\delta y=(ax+b)^{-5}\delta x(-\frac{5\ a}{(ax+b)}+\frac{-5(-5-1)}{2!}\frac{a^2\delta x}{(ax+b)^2}+\cdots)$

Dividing both sides by δx

$\frac{\delta y}{\delta x}=(ax+b)^{-5}(-\frac{5\ a}{(ax+b)}+\frac{-5(-5-1)}{2!}\frac{a^2\delta x}{(ax+b)^2}+\cdots)$

Taking limit when $\delta x\to 0$

$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}(ax+b)^{-5}(-\frac{5\ a}{(ax+b)}+\frac{-5(-5-1)}{2!}\frac{a^2\delta x}{(ax+b)^2}+\cdots)$

$\frac{dy}{dx}=(ax+b)^{-5}(-\frac{5\ a}{(ax+b)}+\frac{-5(-5-1)}{2!}\frac{a^2(0)}{(ax+b)^2}+\cdots)$

$\frac{dy}{dx}=(ax+b)^{-5}(-\frac{5\ a}{(ax+b)})$

$\frac{dy}{dx}=-\frac{5a}{(ax+b)^{1+5}}$

$\boldsymbol{\frac{dy}{dx}=-\frac{5a}{(ax+b)^6}}$

Which is the required.

(v) $\boldsymbol{\frac{1}{(az-b)^7}}$

Solution.

Let $y=(az-b)^{-7}\to(i)$

$y+\delta y=(a(z+\delta z)-b)^{-7}\to(ii)$

$eq(ii)-eq(i)$

$\delta y=(az+a\delta z-b)^{-7}-y$

$\delta y=((az-b)+a\delta z)^{-7}-(az-b)^{-7}$

$\delta y=(az-b)^{-7}\left(1+\frac{a\delta z}{(ax-b)}\right)^{-7}-(az-b)^{-7}$

$\delta y=(az-b)^{-7}(\left(1+\frac{a\delta z}{(az-b)}\right)^{-7}-1)$

$\delta y=(az-b)^{-7}(1-\frac{7\ a\delta z}{(az-b)}+\frac{-7(-7-1)}{2!}\left(\frac{a\delta z}{az-b}\right)^2+\cdots-1)$

$\delta y=(az-b)^{-7}\delta z(-\frac{7\ a}{(az-b)}+\frac{-7(-7-1)}{2!}\frac{a^2\delta z}{(az-b)^2}+\cdots)$

Dividing both sides by δz

$\frac{\delta y}{\delta z}=(az-b)^{-7}(-\frac{7\ a}{(az-b)}+\frac{-7(-7-1)}{2!}\frac{a^2\delta z}{(az-b)^2}+\cdots)$

Taking limit when $\delta z\to 0$

$\lim_{\delta z\to 0}\frac{\delta y}{\delta z}=\lim_{\delta z\to 0}(az-b)^{-7}(-\frac{7\ a}{(az-b)}+\frac{-7(-7-1)}{2!}\frac{a^2\delta z}{(az-b)^2}+\cdots)$

$\frac{dy}{dz}=(az-b)^{-7}(-\frac{7\ a}{(az-b)}+\frac{-7(-7-1)}{2!}\frac{a^2(0)}{(az-b)^2}+\cdots)$

$$\frac{dy}{dz} = (az-b)^{-7}\left(-\frac{7\,a}{(az-b)}\right)$$
$$\frac{dy}{dz} = -\frac{7a}{(az-b)^{1+7}}$$
$$\frac{dy}{dz} = -\frac{7a}{(az-b)^{8}}$$

Which is the required.

Theorems on Differentiation:

We have, so far proved the following two formulas;

1. $\frac{d}{dx}(c) = 0$

   $i.e\ \ derivative\ of\ constant\ function\ is\ zero.$

2. $\frac{d}{dx}(x^n) = nx^{n-1}$

   $power\ rule\ when\ constant\ function\ is\ zero.$

3. Derivative of $y = cf(x)\ i.e \frac{dy}{dx} = cf'(x)$

Proof:

$$y = cf(x) \to (i) \qquad y+\delta y = cf(x+\delta x) \to (ii)$$
$$by\ (ii)-(i) \Rightarrow \quad y+\delta y - y = cf(x+\delta x) - cf(x)$$
$$\Rightarrow \delta y = c(f(x+\delta x) - f)x)$$

Dividing by $\delta x\ and\ take\ limit\ \delta x \to 0$

$$\Rightarrow \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x\to 0}\frac{1}{\delta x}[c(f(x+\delta x) - f(x)]$$
$$\Rightarrow \frac{dy}{dx} = cf'(x)$$

Reciprocal Rule:

If $f(x)$ is differentiable $at\ x\ and\ g(x) \neq 0\ then\ \frac{1}{g(x)}$ $is\ differentiable\ at\ x\ and\ \frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{[g(x)]^2}$

Exercise 2.3`

Differentiate $w.r.t$ "x"

Question # 1. $x^4 + 2x^3 + x^2$

Solution.

Let $y = x^4 + 2x^3 + x^2$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 + 2x^3 + x^2)$$
$$\frac{dy}{dx} = \frac{d}{dx}x^4 + 2\frac{d}{dx}x^3 + \frac{d}{dx}x^2$$
$$\frac{dy}{dx} = 4x^{4-1} + 2.3x^{3-1} + 2x^{2-1}$$
$$\frac{dy}{dx} = 4x^3 + 6x^2 + 2x$$

Which is required.

Question # 2. $x^{-3} + 2x^{-\frac{3}{2}} + 3$

Solution.

Let $y = x^{-3} + 2x^{-\frac{3}{2}} + 3$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{-3} + 2x^{-\frac{3}{2}} + 3\right)$$
$$\frac{dy}{dx} = \frac{d}{dx}x^{-3} + 2\frac{d}{dx}x^{-\frac{3}{2}} + \frac{d}{dx}(3)$$
$$\frac{dy}{dx} = -3x^{-3-1} - 2.\frac{3}{2}x^{-\frac{3}{2}-1} + 0$$
$$\frac{dy}{dx} = -3x^{-4} - 3x^{-\frac{5}{2}}$$
$$\boldsymbol{\frac{dy}{dx} = -3\left(\frac{1}{x^4} + \frac{1}{x^{\frac{5}{2}}}\right)}$$

Which is required.

Question#3. $x^{-3} + 2x^{-\frac{3}{2}} + 3$

Solution.

Let $y = \frac{a+x}{a-x}$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{a+x}{a-x}\right)$$
$$\frac{dy}{dx} = \frac{(a-x)\frac{d}{dx}(a+x) - (a+x)\frac{d}{dx}(a-x)}{(a-x)^2}$$
$$\frac{dy}{dx} = \frac{(a-x)(1) - (a+x)(-1)}{(a-x)^2}$$
$$\frac{dy}{dx} = \frac{a-x+a+x}{(a-x)^2}$$
$$\boldsymbol{\frac{dy}{dx} = \frac{2a}{(a-x)^2}}$$

Which is required.

Question#4. $\frac{2x-3}{2x+1}$

Solution.

Let $y = \frac{2x-3}{2x+1}$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x-3}{2x+1}\right)$$
$$\frac{dy}{dx} = \frac{(2x+1)\frac{d}{dx}(2x-3) - (2x-3)\frac{d}{dx}(2x+1)}{(2x+1)^2}$$
$$\frac{dy}{dx} = \frac{(2x+1)(2) - (2x-3)(2)}{(2x+1)^2}$$
$$\frac{dy}{dx} = \frac{2(2x+1-2x+3)}{(2x+1)^2}$$
$$\frac{dy}{dx} = \frac{2(4)}{(2x+1)^2}$$
$$\boldsymbol{\frac{dy}{dx} = \frac{8}{(2x+1)^2}}$$

Which is required.

Question#5. $(x-5)(3-x)$

Solution.

Let $y = (x-5)(3-x)$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}((x-5)(3-x))$$
$$\frac{dy}{dx} = \frac{d}{dx}(3x - x^2 - 15 + 5x)$$
$$\frac{dy}{dx} = \frac{d}{dx}(-x^2 + 8x - 15)$$
$$\frac{dy}{dx} = -\frac{d}{dx}x^2 + 8\frac{d}{dx}x - \frac{d}{dx}(15)$$

$\frac{dy}{dx}=-\frac{d}{dx}x^2+8\frac{d}{dx}x-\frac{d}{dx}(15)$

$\frac{dy}{dx}=-2x+8+0$

$\boldsymbol{\frac{dy}{dx}=-2x+8}$

Which is required.

Question#5. $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$

Solution.

Let $y=\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$

Differentiate w.r.t x

$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$

$\frac{dy}{dx}=\frac{d}{dx}\left(\left(\sqrt{x}\right)^2+\left(\frac{1}{\sqrt{x}}\right)^2-2(\sqrt{x})(\frac{1}{\sqrt{x}})\right)$

$\frac{dy}{dx}=\frac{d}{dx}\left(x+\frac{1}{x}-2\right)$

$\frac{dy}{dx}=\frac{d}{dx}(x+x^{-1}-2)$

$\frac{dy}{dx}=\frac{d}{dx}x+\frac{d}{dx}x^{-1}-\frac{d}{dx}(2)$

$\frac{dy}{dx}=1-x^{-2}-0$

$\frac{dy}{dx}=1-\frac{1}{x^2}$

$\boldsymbol{\frac{dy}{dx}=\frac{x^2-1}{x^2}}$

Which is required.

Question#7. $\frac{(1+\sqrt{x})(x-x^{\frac{3}{2}})}{\sqrt{x}}$

Solution.

Let $y=\frac{(1+\sqrt{x})\left(x-x^{\frac{3}{2}}\right)}{\sqrt{x}}$

$y=\frac{(1+\sqrt{x})x\left(1-x^{\frac{1}{2}}\right)}{\sqrt{x}}$

$y=\frac{x\left(1-\left(\sqrt{x}\right)^2\right)}{\sqrt{x}}$

$y=\sqrt{x}(1-x)$

$y=\sqrt{x}-x\sqrt{x}$

$y=x^{\frac{1}{2}}-x^{\frac{3}{2}}$

Differentiate w.r.t x

$\frac{dy}{dx}=\frac{d}{dx}(x^{\frac{1}{2}}-x^{\frac{3}{2}})$

$\frac{dy}{dx}=\frac{d}{dx}x^{\frac{1}{2}}-\frac{d}{dx}x^{\frac{3}{2}}$

$\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}-\frac{3}{2}x^{\frac{1}{2}}$

$\frac{dy}{dx}=\frac{1}{2}(\frac{1}{\sqrt{x}}-3\sqrt{x})$

$\boldsymbol{\frac{dy}{dx}=\frac{1}{2}(\frac{1-3x}{\sqrt{x}})}$

Which is required.

Question#8. $\frac{(x^2+1)^2}{x^2-1}$

Solution.

Let $y=\frac{(x^2+1)^2}{x^2-1}$

Differentiate w.r.t x

$\frac{dy}{dx}=\frac{d}{dx}(\frac{(x^2+1)^2}{x^2-1})$

$\frac{dy}{dx}=\frac{(x^2-1)\frac{d}{dx}((x^2+1)^2)-(x^2+1)^2\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$

$\frac{dy}{dx}=\frac{(x^2-1)(2(x^2+1)^1)\frac{d}{dx}(x^2+1)-(x^2+1)^2(2x)}{(x^2-1)^2}$

$\frac{dy}{dx}=\frac{(x^2-1)(2(x^2+1)^1)(2x)-(x^2+1)^2(2x)}{(x^2-1)^2}$

$\frac{dy}{dx}=\frac{2x(x^2+1))(2(x^2-1)-x^2-1)}{(x^2-1)^2}$

$\frac{dy}{dx}=\frac{2x(x^2+1))(2x^2-2-x^2-1)}{(x^2-1)^2}$

$\boldsymbol{\frac{dy}{dx}=\frac{2x(x^2+1))(x^2-3)}{(x^2-1)^2}}$

Which is required.

Question # 9. $\frac{x^2+1}{x^2-3}$

Solution.

Let $y=\frac{x^2+1}{x^2-3}$

Differentiate w.r.t x

$\frac{dy}{dx}=\frac{d}{dx}(\frac{x^2+1}{x^2-3})$

$\frac{dy}{dx}=\frac{(x^2-3)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-3)}{(x^2-3)^2}$

$\frac{dy}{dx}=\frac{(x^2-3)(2x)-(x^2+1)(2x)}{(x^2-3)^2}$

$\frac{dy}{dx}=\frac{2x((x^2-3)-(x^2+1))}{(x^2-3)^2}$

$\frac{dy}{dx}=\frac{2x(x^2-3-x^2-1)}{(x^2-3)^2}$

$\frac{dy}{dx}=\frac{2x(-4)}{(x^2-3)^2}$

$\boldsymbol{\frac{dy}{dx}=\frac{-8x}{(x^2-3)^2}}$

Which is required.

Question#10. $\frac{\sqrt{1+x}}{\sqrt{1-x}}$

Solution.

Let $y=\frac{\sqrt{1+x}}{\sqrt{1-x}}=\sqrt{\frac{1+x}{1-x}}=\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$

Differentiate w.r.t x

$\frac{dy}{dx}=\frac{d}{dx}(\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}})$

$\frac{dy}{dx}=\frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\frac{d}{dx}(\frac{1+x}{1-x})$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}\left(\frac{(1-x)\frac{d}{dx}(1+x)-(1+x)\frac{d}{dx}(1-x)}{(1-x)^2}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}\left(\frac{1-x+1+x}{(1-x)^2}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}\left(\frac{2}{(1-x)^2}\right)$$

$$\frac{dy}{dx}=\left(\frac{1}{(1-x)^{2-\frac{1}{2}}(1+x)^{\frac{1}{2}}}\right)$$

$$\boldsymbol{\frac{dy}{dx}=\left(\frac{1}{(1-x)^{\frac{3}{2}}(1+x)^{\frac{1}{2}}}\right)}$$

Which is required.

Question # 11. $\frac{2x-1}{\sqrt{x^2+1}}$

Solution.

Let $y=\frac{2x-1}{\sqrt{x^2+1}}$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x-1}{\sqrt{x^2+1}}\right)$$

$$\frac{dy}{dx}=\left(\frac{(\sqrt{x^2+1})\frac{d}{dx}(2x-1)-(2x-1)\frac{d}{dx}(\sqrt{x^2+1})}{(\sqrt{x^2+1})^2}\right)$$

$$\frac{dy}{dx}=\frac{(\sqrt{x^2+1})(2)-(2x-1)\frac{1}{2\sqrt{x^2+1}}(2x)}{x^2+1}$$

$$\frac{dy}{dx}=\frac{(x^2+1)(2)-(2x-1)(x)}{(x^2+1)(x^2+1)^{\frac{1}{2}}}$$

$$\frac{dy}{dx}=\frac{2x^2+2-2x^2+x}{(x^2+1)^{1+\frac{1}{2}}}$$

$$\boldsymbol{\frac{dy}{dx}=\frac{x+2}{(x^2+1)^{\frac{3}{2}}}}$$

Which is required.

Question # 12. $\frac{\sqrt{a-x}}{\sqrt{a+x}}$

Solution.

Let $y=\frac{\sqrt{a-x}}{\sqrt{a+x}}=\sqrt{\frac{a-x}{a+x}}$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{\frac{a-x}{a+x}}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{a-x}{a+x}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\frac{a-x}{a+x}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}\left(\frac{(a+x)\frac{d}{dx}(a-x)-(a-x)\frac{d}{dx}(a+x)}{(a+x)^2}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}\left(\frac{(a+x)(-1)-(a-x)(1)}{(a+x)^2}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}\left(\frac{-a-x-a+x}{(a+x)^2}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{-2a}{(a+x)^{2-\frac{1}{2}}(a-x)^{\frac{1}{2}}}\right)$$

$$\boldsymbol{\frac{dy}{dx}=\left(\frac{-a}{(a+x)^{\frac{3}{2}}(a-x)^{\frac{1}{2}}}\right)}$$

Which is required.

Question # 13. $\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$

Solution.

Let $y=\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}=\sqrt{\frac{x^2+1}{x^2-1}}$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{\frac{x^2+1}{x^2-1}}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{2}}\left(\frac{(x^2-1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{2}}\left(\frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{2}}\left(\frac{2x(x^2-1-x^2-1)}{(x^2-1)^2}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{2}}\left(\frac{2x(-2)}{(x^2-1)^2}\right)$$

$$\frac{dy}{dx}=\left(\frac{x(-2)}{(x^2-1)^{2-\frac{1}{2}}(x^2+1)^{\frac{1}{2}}}\right)$$

$$\boldsymbol{\frac{dy}{dx}=\frac{-2x}{(x^2-1)^{\frac{3}{2}}\sqrt{x^2+1}}}$$

Which is required.

Question # 14. $\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

Solution.

Let $y=\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

$$y=\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\cdot\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$$

$$y=\frac{(\sqrt{1+x}-\sqrt{1-x})^2}{(\sqrt{1+x})^2-(\sqrt{1-x})^2}$$

$$y=\frac{(\sqrt{1+x})^2+(\sqrt{1-x})^2-2(\sqrt{1+x})(\sqrt{1-x})}{(\sqrt{1+x})^2-(\sqrt{1-x})^2}$$

$$y=\frac{1+x+1-x-2\sqrt{1-x^2}}{1+x-1+x}$$
$$y=\frac{2-2\sqrt{1-x^2}}{2x}$$
$$y=\frac{1-\sqrt{1-x^2}}{x}$$
Differentiate w.r.t x
$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1-\sqrt{1-x^2}}{x}\right)$$
$$\frac{dy}{dx}=\frac{x\frac{d}{dx}\left(1-\sqrt{1-x^2}\right)-\left(1-\sqrt{1-x^2}\right)\frac{d}{dx}x}{x^2}$$
$$\frac{dy}{dx}=\frac{x\left(-\frac{-2x}{2\sqrt{1-x^2}}\right)-\left(1-\sqrt{1-x^2}\right)}{x^2}$$
$$\frac{dy}{dx}=\frac{x^2-\sqrt{1-x^2}+1-x^2}{x^2\sqrt{1-x^2}}$$
$$\boldsymbol{\frac{dy}{dx}=\frac{1-\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}}$$
Which is required.

Question # 15. $\frac{x\sqrt{a+x}}{\sqrt{a-x}}$

Solution.

Let $y=\frac{x\sqrt{a+x}}{\sqrt{a-x}}=x\sqrt{\frac{a+x}{a-x}}$

Differentiate w.r.t x
$$\frac{dy}{dx}=\frac{d}{dx}\left(x\sqrt{\frac{a+x}{a-x}}\right)$$
$$\frac{dy}{dx}=\sqrt{\frac{a+x}{a-x}}\frac{d}{dx}(x)+x\frac{d}{dx}\sqrt{\frac{a+x}{a-x}}$$
$$\frac{dy}{dx}=\sqrt{\frac{a+x}{a-x}}+x\frac{1}{2}\left(\frac{a+x}{a-x}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\frac{a+x}{a-x}\right)$$
$$\frac{dy}{dx}=\sqrt{\frac{a+x}{a-x}}+x\frac{1}{2}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}\left(\frac{(a-x)(1)-(a+x)(-1)}{(a-x)^2}\right)$$
$$\frac{dy}{dx}=\sqrt{\frac{a+x}{a-x}}+x\frac{1}{2}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}\left(\frac{a-x+a+x}{(a-x)^2}\right)$$
$$\frac{dy}{dx}=\sqrt{\frac{a+x}{a-x}}+x\frac{1}{2}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}\left(\frac{2a}{(a-x)^2}\right)$$
$$\frac{dy}{dx}=\sqrt{\frac{a+x}{a-x}}+\left(\frac{ax}{(a-x)^{\frac{3}{2}}((a+x)^{\frac{1}{2}}}\right)$$
$$\frac{dy}{dx}=\frac{(a+x)(a-x)+ax}{(a-x)^{\frac{3}{2}}(a+x)^{\frac{1}{2}}}$$
$$\boldsymbol{\frac{dy}{dx}=\frac{a^2-x^2+ax}{(a-x)^{\frac{3}{2}}(a+x)^{\frac{1}{2}}}}$$

Which is required.

Question # 16. $\boldsymbol{If\ y=\sqrt{x}-\frac{1}{\sqrt{x}}\ ,Show\ that\ 2x\frac{dy}{dx}+y=2\sqrt{x}.}$

Solution.

Since $y=\sqrt{x}-\frac{1}{\sqrt{x}}$
$$y=x^{\frac{1}{2}}-x^{-\frac{1}{2}}$$
Differentiating w.r.t "x"
$$\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{-\frac{3}{2}}$$
Multiplying by 2x both sides
$$2x\frac{dy}{dx}=2x\left(\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{-\frac{3}{2}}\right)$$
$$2x\frac{dy}{dx}=x^{\frac{1}{2}}+x^{-\frac{1}{2}}$$
Adding u on both sides, we have
$$2x\frac{dy}{dx}+y=x^{\frac{1}{2}}+x^{-\frac{1}{2}}+y$$
$$2x\frac{dy}{dx}+y=x^{\frac{1}{2}}+x^{-\frac{1}{2}}+x^{\frac{1}{2}}-x^{-\frac{1}{2}}$$
$$2x\frac{dy}{dx}+y=2x^{\frac{1}{2}}$$
$$\boldsymbol{2x\frac{dy}{dx}+y=2\sqrt{x}}$$
Hence Proved.

Question # 17. $\boldsymbol{If\ y=x^4+2x^2+2\ then\ prove\ that\ \frac{dy}{dx}=4x\sqrt{y-1}.}$

Solution.

Since $y=x^4+2x^2+2$

$Differentiate\ w.r.t."x"$
$$\frac{dy}{dx}=4x^3+4x$$
$$\frac{dy}{dx}=4x(x^2+1)$$
$$\frac{dy}{dx}=4x\sqrt{(x^2+1)^2}$$
$$\frac{dy}{dx}=4x\sqrt{x^4+2x^2+1}$$
$$\frac{dy}{dx}=4x\sqrt{x^4+2x^2+2-1}$$
$$\boldsymbol{\frac{dy}{dx}=4x\sqrt{y-1}\qquad \because\ y=x^4+2x^2+1}$$
Hence Proved. Which is require

The Chain Rule:

$if\ y=(fog)(x)or\ y=f(g(x))let\ u=g(x)then\ y=f(x)\ and\ u=g(x)\ so\ we\ find\ \frac{dy}{du}\ and\ \frac{du}{dx}\ then$

$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}\ is\ called\ chain\ rule.$

Derivative of a function given in the form of parametric equation.

Differentiation of implicit Relations:

Exercise 2.4

Question # 1.

Find $\frac{dy}{dx}$ by making the suitable substitution in the following functions defined as:

Differentiate the following w.r.t "x".

(i) $y = \sqrt{\frac{1-x}{1+x}}$

Solution.

Given $y = \sqrt{\frac{1-x}{1+x}}$

Put $u = \frac{1-x}{1+x}$

So $y = \sqrt{u}$

$$y = u^{\frac{1}{2}}$$

Differentiate "u" w.r.t . x

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

$$\frac{du}{dx} = \frac{(1+x)\frac{d}{dx}(1-x) - (1-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{(1+x)(-1) - (1-x)\,(1)}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{-1-x-1+x}{(1+x)^2}$$

$$\frac{du}{dx} = \frac{-2}{(1+x)^2}$$

Now differentiate y w.r.t. u

$$\frac{dy}{du} = \frac{d}{du}u^{\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}.\frac{-2}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{-1}{(1-x)^{\frac{1}{2}}(1+x)^{2-\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x}\,(1+x)^{\frac{3}{2}}}$$

Which is required.

(ii) $y = \sqrt{x+\sqrt{x}}$

Solution.

Given $y = \sqrt{x+\sqrt{x}}$

Put $u = x+\sqrt{x}$

So $y = \sqrt{u}$

$$y = u^{\frac{1}{2}}$$

Differentiate "u" w.r.t . x

$$\frac{du}{dx} = \frac{d}{dx}(x+\sqrt{x})$$

$$\frac{du}{dx} = 1 + \frac{1}{2}x^{\frac{-1}{2}}$$

$$\frac{du}{dx} = 1 + \frac{1}{2\sqrt{x}}$$

$$\frac{du}{dx} = \frac{2\sqrt{x}+1}{2\sqrt{x}}$$

Now differentiate y w.r.t. u

$$\frac{dy}{du} = \frac{d}{du}u^{\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2}(x+\sqrt{x})^{-\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2\sqrt{x+\sqrt{x}}}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+\sqrt{x}}}.\frac{2\sqrt{x}+1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{2\sqrt{x}+1}{4\sqrt{x}.\sqrt{x+\sqrt{x}}}$$

Which is required.

(iii) $y = x\sqrt{\frac{a+x}{a-x}}$

Solution.

Given $y = x\sqrt{\frac{a+x}{a-x}}$

Put $u = \frac{a+x}{a-x}$

So $y = x\sqrt{u}$

$$y = x.u^{\frac{1}{2}}$$

Differentiate "u" w.r.t . " x"

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{a+x}{a-x}\right)\frac{du}{dx}$$

$$= \frac{(a-x)\frac{d}{dx}(a+x) - (a+x)\frac{d}{dx}(a-x)}{(a-x)^2}$$

$$\frac{du}{dx} = \frac{(a-x)(1) - (a+x)\,(-1)}{(a-x)^2}$$

$$\frac{du}{dx} = \frac{a-x+a+x}{(a-x)^2}$$

$$\frac{du}{dx} = \frac{2a}{(a-x)^2}$$

Now differentiate y w.r.t. x

$$\frac{dy}{du} = \frac{d}{du}\left(xu^{\frac{1}{2}}\right)$$

$$\frac{dy}{du} = x.\frac{d}{du}\left(u^{\frac{1}{2}}\right) + u^{\frac{1}{2}}\frac{d}{du}(x)$$

$$\frac{dy}{du} = x.\frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dx} + u^{\frac{1}{2}}(1)$$

$$\frac{dy}{du}=x.\frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dx}+u^{\frac{1}{2}}$$

Now Using Above values

$$\frac{dy}{dx}=x.\frac{1}{2}\left(\frac{a+x}{a-x}\right)^{-\frac{1}{2}}.\frac{2a}{(a-x)^2}+\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{ax\,(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}(a-x)^2}+\frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}$$

$$\frac{dy}{dx}=\frac{ax}{(a+x)^{\frac{1}{2}}(a-x)^{2-\frac{1}{2}}}+\frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}$$

$$\frac{dy}{dx}=\frac{ax}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}}+\frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}$$

$$\frac{dy}{dx}=\frac{ax}{(a+x)^{\frac{1}{2}}(a-x)^{1}(a-x)^{\frac{1}{2}}}+\frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}$$

$$\frac{dy}{dx}=\frac{ax+(a+x)(a-x)}{(a+x)^{\frac{1}{2}}(a-x)^{1}(a-x)^{\frac{1}{2}}}$$

$$\frac{dy}{dx}=\frac{ax+a^2-x^2}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}}$$

Which is required.

(iv) $\boldsymbol{y=\left(3x^2-2x+7\right)^6}$

Solution.

Given $y=(3x^2-2x+7)^6$

Put $u=3x^2-2x+7$

So $y=u^6$

Differentiate ''u'' w.r.t . x

$$\frac{du}{dx}=\frac{d}{dx}(3x^2-2x+7)$$

$$\frac{du}{dx}=3(2x)-2(1)+0$$

$$\frac{du}{dx}=6x-2$$

Now differentiate y w.r.t. u

$$\frac{dy}{du}=\frac{d}{du}u^6$$

$$\frac{dy}{du}=6u^5$$

$$\frac{dy}{du}=6(3x^2-2x+7)^5$$

Now by chain rule

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$$

$$\frac{dy}{dx}=6(3x^2-2x+7)^5\,.(6x-2)$$

$$\frac{dy}{dx}=12(3x-1)(3x^2-2x+7)^5$$

Which is required.

(v) $\boldsymbol{y=\sqrt{\frac{a^2+x^2}{a^2-x^2}}}$

Solution.

Given $y=\sqrt{\frac{a^2+x^2}{a^2-x^2}}$

Put $u=\frac{a^2+x^2}{a^2-x^2}$

So $y=\sqrt{u}$

$$y=u^{\frac{1}{2}}$$

Differentiate ''u'' w.r.t . x

$$\frac{du}{dx}=\frac{d}{dx}\left(\frac{a^2+x^2}{a^2-x^2}\right)$$

$$\frac{du}{dx}=\frac{(a^2-x^2)\frac{d}{dx}(a^2+x^2)-(a^2+x^2)\frac{d}{dx}(a^2-x^2)}{(a^2-x^2)^2}$$

$$\frac{du}{dx}=\frac{(a^2-x^2)(2x)-(a^2+x^2)(-2x)}{(a^2-x^2)^2}$$

$$\frac{du}{dx}=\frac{2x(a^2-x^2+a^2+x^2)}{(a^2-x^2)^2}$$

$$\frac{du}{dx}=\frac{2a^2x}{(a^2-x^2)^2}$$

Now differentiate y w.r.t. u

$$\frac{dy}{du}=\frac{d}{du}u^{\frac{1}{2}}$$

$$\frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{dy}{du}=\frac{1}{2}\left(\frac{a^2+x^2}{a^2-x^2}\right)^{-\frac{1}{2}}$$

$$\frac{dy}{du}=\frac{1}{2}\left(\frac{a^2-x^2}{a^2+x^2}\right)^{\frac{1}{2}}$$

Now by chain rule

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{a^2-x^2}{a^2+x^2}\right)^{\frac{1}{2}}.\frac{2a^2x}{(a^2-x^2)^2}$$

$$\frac{dy}{dx}=\frac{a^2x}{(a^2+x^2)^{\frac{1}{2}}(a^2-x^2)^{2-\frac{1}{2}}}$$

$$\frac{dy}{dx}=\frac{a^2x}{(a^2+x^2)^{\frac{1}{2}}(a^2-x^2)^{\frac{3}{2}}}$$

Which is required.

Question # 2 Find $\frac{dy}{dx}$ *if* :

(i) $\boldsymbol{3x+4y+7=0}$

Solution:

Given $3x+4y+7=0$

Differentiate w.r.t. ''x''

$$\frac{d}{dx}(3x+4y+7)=\frac{d}{dx}(0)$$

$$3+4\frac{dy}{dx}=0$$

$\frac{dy}{dx}=-\frac{3}{4}.$

Which is required.

(ii) $\boldsymbol{xy+y^2=2}$

Solution:

Given $xy + y^2 = 2$

Differentiate w.r.t. ''x''

$$\frac{d}{dx}(xy + y^2) = \frac{d}{dx}(2)$$

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$

$$y.(1) + x.\frac{dy}{dx} + 2y.\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{(x + 2y)}$$

Which is required.

(iii) $\boldsymbol{x^2 - 4xy - 5y = 0}$

Solution:

Given $x^2 - 4xy - 5y = 0$

Differentiate w.r.t. ''x''

$$\frac{d}{dx}(x^2 - 4xy - 5y) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(x^2) - 4\frac{d}{dx}(xy) - 5\frac{d}{dx}(y) = 0$$

$$2x - 4\left(y.(1) + x.\frac{dy}{dx}\right) - 5\frac{dy}{dx} = 0$$

$$2x - 4y - 4x\frac{dy}{dx} - 5\frac{dy}{dx} = 0$$

$$-(4x + 5)\frac{dy}{dx} = -(2x - 4y)$$

$$\frac{dy}{dx} = \frac{2x - 4y}{4x + 5}$$

Which is required.

(iv) $\boldsymbol{4x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.}$

Solution:

Given $4x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Differentiate w.r.t. ''x''

$$\frac{d}{dx}(4x^2 + 2hxy + by^2 + 2gx + 2fy + c) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(4x^2) + 2h\frac{d}{dx}(xy) + b\frac{d}{dx}(y^2) + 2g\frac{d}{dx}(x) + 2f\frac{d}{dx}(y) + \frac{d}{dx}(c) = 0$$

$$4(2x) + 2h\left(y.(1) + x.\frac{dy}{dx}\right) + 2by\frac{dy}{dx} + 2g(1) + 2f\frac{dy}{dx} + 0 = 0$$

$$8x + 2hy + 2hx.\frac{dy}{dx} + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$2(hx + by + f)\frac{dy}{dx} = -2(4x + hy + g)$$

$$\frac{dy}{dx} = -\frac{4x + hy + g}{(hx + by + f)}$$

Which is required.

(v) $\boldsymbol{x\sqrt{1+y} + y\sqrt{1+x} = 0}$

Solution:

Given $x\sqrt{1+y} + y\sqrt{1+x} = 0$

Differentiate w.r.t. ''x''

$$\frac{d}{dx}(x\sqrt{1+y} + y\sqrt{1+x}) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(x\sqrt{1+y}) + \frac{d}{dx}(y\sqrt{1+x}) = 0$$

$$\sqrt{1+y}.\frac{d}{dx}(x) + x.\frac{d}{dx}\left(\sqrt{1+y}\right) + y\frac{d}{dx}(\sqrt{1+x}) + \sqrt{1+x}\frac{d}{dx}(y) = 0$$

$$\sqrt{1+y}.(1) + \frac{1}{2}x(1+y)^{-\frac{1}{2}}\frac{dy}{dx} + y\frac{1}{2}(1+x)^{-\frac{1}{2}}(1) + \sqrt{1+x}\frac{dy}{dx} = 0$$

$$\sqrt{1+y} + \frac{1}{2}\frac{x}{\sqrt{1+y}}\frac{dy}{dx} + \frac{y}{2\sqrt{1+x}} + \sqrt{1+x}\frac{dy}{dx} = 0$$

$$\left(\frac{x}{2\sqrt{1+y}} + \sqrt{1+x}\right)\frac{dy}{dx} = -\left(\frac{y}{2\sqrt{1+x}} + \sqrt{1+y}\right)$$

$$\left(\frac{x + 2\sqrt{1+y}\sqrt{1+x}}{2\sqrt{1+y}}\right)\frac{dy}{dx} = -\left(\frac{y + 2\sqrt{1+y}.\sqrt{1+x}}{2\sqrt{1+x}}\right)$$

$$\frac{dy}{dx} = -\left(\frac{\sqrt{1+y}\left(y + 2\sqrt{1+y}.\sqrt{1+x}\right)}{\sqrt{1+x}\left(x + 2\sqrt{1+y}\sqrt{1+x}\right)}\right)$$

Which is required.

(vi) $\boldsymbol{y(x^2 - 1) = x\sqrt{x^2 + 4} = 0}$

Solution:

Given $y(x^2 - 1) = x\sqrt{x^2 + 4}$

Differentiate w.r.t. ''x''

$$\frac{d}{dx}y(x^2 - 1) = \frac{d}{dx}(x\sqrt{x^2 + 4})$$

$$(x^2 - 1)\frac{d}{dx}(y) + y\frac{d}{dx}(x^2 - 1) = x\frac{d}{dx}\left(\sqrt{x^2 + 4}\right) + \sqrt{x^2 + 4}\frac{d}{dx}(x)$$

$$(x^2 - 1)\frac{dy}{dx} + y(2x) = x\frac{1}{2}(x^2 + 4)^{-\frac{1}{2}}(2x) + \sqrt{x^2 + 4}(1)$$

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{x^2}{\sqrt{x^2 + 4}} + \sqrt{x^2 + 4}$$

$$(x^2 - 1)\frac{dy}{dx} = \frac{x^2}{\sqrt{x^2 + 4}} + \sqrt{x^2 + 4} - 2xy$$

$$(x^2 - 1)\frac{dy}{dx} = \frac{x^2 + x^2 + 4 - 2xy\sqrt{x^2 + 4}}{\sqrt{x^2 + 4}}$$

$$\frac{dy}{dx} = \frac{2x^2 + 4 - 2xy\sqrt{x^2 + 4}}{(x^2 - 1)\sqrt{x^2 + 4}}$$

Which is required.

Question # 3 Find $\frac{dy}{dx}$ of the following parametric functions:

(i) $x = \theta + \frac{1}{\theta}$ **and** $y = \theta + 1$

Solution.

Since $x = \theta + \frac{1}{\theta}$

$$x = \theta + \theta^{-1}$$

Differentiate "x" w.r.t. θ, We have

$$\frac{dx}{d\theta} = 1 + (-1.\theta^{-2})$$
$$\frac{dx}{d\theta} = 1 - \frac{1}{\theta^2}$$
$$\frac{dx}{d\theta} = \frac{\theta^2 - 1}{\theta^2}$$
$$\frac{d\theta}{dx} = \frac{\theta^2}{\theta^2 - 1}$$

Now $y = \theta + 1$

Differentiate "y" w.r.t. θ, We have

$$\frac{dy}{d\theta} = 1 + 0$$
$$\frac{dy}{d\theta} = 1$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx}$$
$$\frac{dy}{dx} = 1.\frac{\theta^2}{\theta^2 - 1}$$
$$\frac{dy}{dx} = \frac{\theta^2}{\theta^2 - 1}$$

Which is required.

(ii) $x = \frac{a(1-t^2)}{1+t^2}$ and $y = \frac{2bt}{1+t^2}$

Solution.

Since $x = \frac{a(1-t^2)}{1+t^2}$

Differentiate "x" w.r.t. t, We have

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{a(1-t^2)}{1+t^2}\right)$$
$$\frac{dx}{dt} = a\frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$
$$\frac{dx}{dt} = a\frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}$$
$$\frac{dx}{dt} = -2at\frac{1+t^2+1-t^2}{(1+t^2)^2}$$
$$\frac{dx}{dt} = \frac{-4at}{(1+t^2)^2}$$
$$\frac{dt}{dx} = \frac{-(1+t^2)^2}{4at}$$

Now $y = \frac{2bt}{1+t^2}$

Differentiate "y" w.r.t. t, We have

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{2bt}{1+t^2}\right)$$
$$\frac{dy}{dt} = 2b\frac{(1+t^2)\frac{d}{dt}(t) - (t)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$
$$\frac{dy}{dt} = 2b\frac{(1+t^2)(1) - (t)(2t)}{(1+t^2)^2}$$
$$\frac{dy}{dt} = 2b\frac{(1+t^2-2t^2)}{(1+t^2)^2}$$
$$\frac{dy}{dt} = 2b\frac{(1-t^2)}{(1+t^2)^2}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$
$$\frac{dy}{dx} = 2b\frac{(1-t^2)}{(1+t^2)^2}.\frac{-(1+t^2)^2}{4at}$$
$$\frac{dy}{dx} = -\frac{b(1-t^2)}{2at}$$

Which is required.

Question # 4. Prove that $y\frac{dy}{dx} + x = 0$ if $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$.

Solution.

Since $x = \frac{(1-t^2)}{1+t^2}$

Differentiate "x" w.r.t. t, We have

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{(1-t^2)}{1+t^2}\right)$$
$$\frac{dx}{dt} = \frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$
$$\frac{dx}{dt} = \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}$$
$$\frac{dx}{dt} = -2t\frac{1+t^2+1-t^2}{(1+t^2)^2}$$
$$\frac{dx}{dt} = \frac{-4t}{(1+t^2)^2}$$
$$\frac{dt}{dx} = \frac{-(1+t^2)^2}{4t}$$

Now $y = \frac{2t}{1+t^2}$

Differentiate "y" w.r.t. t, We have

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$$
$$\frac{dy}{dt} = 2\frac{(1+t^2)\frac{d}{dt}(t) - (t)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$
$$\frac{dy}{dt} = 2\frac{(1+t^2)(1) - (t)(2t)}{(1+t^2)^2}$$
$$\frac{dy}{dt} = 2\frac{(1+t^2-2t^2)}{(1+t^2)^2}$$
$$\frac{dy}{dt} = 2\frac{(1-t^2)}{(1+t^2)^2}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = 2\frac{(1-t^2)}{(1+t^2)^2} \cdot \frac{-(1+t^2)^2}{4t}$$

$$\frac{dy}{dx} = -\frac{(1-t^2)}{2t}$$

$$L.H.S = y\frac{dy}{dx} + x$$

$$L.H.S = \frac{2t}{1+t^2} \cdot -\frac{(1-t^2)}{2t} + \frac{(1-t^2)}{1+t^2}$$

$$L.H.S = -\frac{(1-t^2)}{1+t^2} + \frac{(1-t^2)}{1+t^2}$$

$$L.H.S = 0$$

$$L.H.S = R.H.S$$

Hence Proved.

Question # 5. Differentiate

(i) $x^2 - \frac{1}{x^2}$ $w.r.t.$ x^4

Solution.

Suppose $y = x^2 - \frac{1}{x^2}$ and $u = x^4$

Differentiate "y" w.r.t. "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 - \frac{1}{x^2}\right)$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 - x^{-2})$$

$$\frac{dy}{dx} = 2x + 2x^{-3}$$

$$\frac{dy}{dx} = 2x + \frac{2}{x^3}$$

$$\frac{dy}{dx} = \frac{2(x^4+1)}{x^3}$$

Differentiate u w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx}(x^4)$$

$$\frac{du}{dx} = 4x^3$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{1}{\frac{dx}{du}}$$

$$\frac{dy}{du} = \frac{2(x^4+1)}{x^3} \cdot \frac{1}{4x^3}$$

$$\frac{dy}{du} = \frac{(x^4+1)}{2x^6}$$

Which is required.

(ii) $(1+x^2)^n$ $w.r.t.$ x^2

Solution.

Suppose $y = (1+x^2)^n$ and $u = x^2$

Differentiate "y" w.r.t. "x"

$$\frac{dy}{dx} = \frac{d}{dx}((1+x^2)^n)$$

$$\frac{dy}{dx} = n(1+x^2)^{n-1}(2x)$$

$$\frac{dy}{dx} = 2nx(1+x^2)^{n-1}$$

Differentiate u w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx}(x^2)$$

$$\frac{du}{dx} = 2x$$

Now by chain rule

$$\frac{dy}{du} = 2nx(1+x^2)^{n-1} \cdot \frac{1}{2x}$$

$$\frac{dy}{du} = n(1+x^2)^{n-1}$$

Which is required.

(iii) $\frac{x^2+1}{x^2-1}$ $w.r.t.$ $\frac{x-1}{x+1}$

Solution.

Suppose $y = \frac{x^2+1}{x^2-1}$ and $u = \frac{x-1}{x+1}$

Differentiate "y" w.r.t. "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$\frac{dy}{dx} = \frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{2x(-2)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{(x^2-1)^2}$$

Differentiate u w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{x-1}{x+1}\right)$$

$$\frac{du}{dx} = \frac{(x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$\frac{du}{dx} = \frac{(x+1) - (x-1)}{(x+1)^2}$$

$$\frac{du}{dx} = \frac{x+1-x+1}{(x+1)^2}$$

$$\frac{du}{dx} = \frac{2}{(x+1)^2}$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{1}{\frac{du}{dx}}$$

$$\frac{dy}{du} = \frac{-4x}{(x^2-1)^2} \cdot \frac{1}{\frac{2}{(x+1)^2}}$$

$$\frac{dy}{du} = \frac{-2x(x+1)^2}{(x^2-1)^2}$$

$$\frac{dy}{du} = \frac{-2x(x+1)^2}{(x-1)^2(x+1)^2}$$

$$\frac{dy}{du} = \frac{-2x}{(x-1)^2}$$

Which is required.

(iv) $\frac{ax+b}{cx+d}$ $w.r.t.$ $\frac{ax^2+b}{ax^2+d}$

Solution.

Suppose $y = \frac{ax+b}{cx+d}$ and $u = \frac{ax^2+b}{ax^2+d}$

Differentiate "y" w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{ax+b}{cx+d}\right)$$
$$\frac{dy}{dx} = \frac{(cx+d)\frac{d}{dx}(ax+b) - (ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$
$$\frac{dy}{dx} = \frac{(cx+d)(a) - (ax+b)(c)}{(cx+d)^2}$$
$$\frac{dy}{dx} = \frac{acx+ad-acx-bc}{(cx+d)^2}$$
$$\frac{dy}{dx} = \frac{ad-bc}{(cx+d)^2}$$

Differentiate u w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{ax^2+b}{ax^2+d}\right)$$
$$\frac{du}{dx} = \frac{(ax^2+d)\frac{d}{dx}(ax^2+b) - (ax^2+b)\frac{d}{dx}(ax^2+d)}{(ax^2+d)^2}$$
$$\frac{du}{dx} = \frac{(ax^2+d)(2ax) - (ax^2+b)(2ax)}{(ax^2+d)^2}$$
$$\frac{du}{dx} = \frac{2ax(ax^2+d-ax^2-b)}{(ax^2+d)^2}$$
$$\frac{du}{dx} = \frac{2ax(d-b)}{(ax^2+d)^2}$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$
$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{1}{\frac{du}{dx}}$$
$$\frac{dy}{du} = \frac{ad-bc}{(cx+d)^2}\cdot\frac{1}{\frac{2ax(d-b)}{(ax^2+d)^2}}$$
$$\frac{dy}{du} = \frac{(ad-bc)(ax^2+d)^2}{2ax(d-b)(cx+d)^2}$$

Which is required.

(v) $\frac{x^2+1}{x^2-1}$ $w.r.t. x^3$

Solution.

Suppose $y = \frac{x^2+1}{x^2-1}$ and $u = x^3$

Differentiate "y" w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$
$$\frac{dy}{dx} = \frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$
$$\frac{dy}{dx} = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2}$$
$$\frac{dy}{dx} = 2x\frac{x^2-1-x^2-1}{(x^2-1)^2}$$
$$\frac{dy}{dx} = 2x\frac{-2}{(x^2-1)^2}$$
$$\frac{dy}{dx} = \frac{-4x}{(x^2-1)^2}$$

Differentiate u w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx}(x^3)$$
$$\frac{du}{dx} = 3x^2$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$
$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{1}{\frac{du}{dx}}$$
$$\frac{dy}{du} = \frac{-4x}{(x^2-1)^2}\cdot\frac{1}{3x^2}$$
$$\frac{dy}{du} = \frac{-4}{3x(x^2-1)^2}$$

Which is required.

Derivative of Trigonometric function:

1. $\frac{d}{dx}(\sin x) = \cos x$

Proof: $let\ y = \sin x \rightarrow (i)$

⇨ $y + \delta y = \sin(x+\delta x) \rightarrow (ii)$

$by\ (ii)-(i) \Rightarrow y+\delta y - y = \sin(x+\delta x) - \sin x$

$$(\because \sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

⇨ $\delta y = 2\cos\left(\frac{x+\delta x+x}{2}\right)\sin\frac{(x+\delta x-x)}{2}$

⇨ $\delta y = 2\cos\left(\frac{2x+\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)$

dividing by δx and take limit δx → 0

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[2\cos\left(x+\frac{\delta x}{2}\right)\cdot\frac{\left(\sin\left(\frac{\delta x}{2}\right)\right)}{\delta x}\right]$$

⇨ $\frac{dy}{dx} = 2\lim_{\delta x\to 0}\left[\cos\left(x+\frac{\delta x}{2}\right)\cdot\frac{\left(\sin\left(\frac{\delta x}{2}\right)\right)}{2\times\frac{\delta x}{2}}\right]$

⇨ $\frac{dy}{dx} = \cos x$ $\qquad (\because)\lim_{\theta\to 0}\frac{\sin\theta}{2} = 1$

Thus $\frac{d}{dx}\sin x = \cos x$

2. $\frac{d}{dx}(\cos x) = -\sin x$

Proof: $let\ y = \cos x \rightarrow (i)$

⇨ $y+\delta y = \cos(x+\delta x) \rightarrow (ii)$

$by\ (ii)-(i) \Rightarrow y+\delta y - y = \cos(x+\delta x) - \cos x$

⇨ $\delta y = \cos(x+\delta x) - \cos x$

$$(\because \cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

⇨ $\delta y = -2\sin\left(\frac{x+\delta x+x}{2}\right)\sin\frac{(x+\delta x-x)}{2}$

⇨ $\delta y = -2\sin\left(\frac{2x+\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)$

dividing by δx and take limit δx → 0

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[-2\sin\left(x+\frac{\delta x}{2}\right).\frac{\left(\sin\left(\frac{\delta x}{2}\right)\right)}{\delta x}\right]$$

$\Rightarrow \frac{dy}{dx}=-2\lim_{\delta x\to 0}\left[-2\left(x+\frac{\delta x}{2}\right).\frac{\left(\sin\left(\frac{\delta x}{2}\right)\right)}{2\times\frac{\delta x}{2}}\right]$

$\Rightarrow \frac{dy}{dx}=-sinx \qquad (\because)\lim_{\theta\to 0}\frac{sin\theta}{2}=1$

Thus $\frac{d}{dx}cosx=-sinx$

3. $\boldsymbol{\frac{d}{dx}(Tanx)=sec^2x}$

Proof: $let\ y=Tanx\to(i)$

$\Rightarrow y+\delta y=Tan(x+\delta x)\to(ii)$

$by\ (ii)-(i)\Rightarrow y+\delta y-y=\text{Tan}(x+\delta x)-Tanx$

$\Rightarrow \delta y=\frac{\sin(x+\delta x)}{\cos(x+\delta x)}-\frac{sinx}{cosx}$

$\Rightarrow \frac{\sin(x+\delta x-x)cosx-\cos(x+\delta x).sinx}{\cos(x+\delta x).cosx}$

$(\because sin\alpha cos\beta-cos\alpha sin\beta=\sin(\alpha-\beta)$

$\Rightarrow \delta y=\frac{\sin(x+\delta s-x)}{\cos(x+\delta x).cosx}=\frac{sin\delta x}{\cos(x+\delta x).cosx}$

$dividing\ by\ \delta x\ and\ take\ limit\ \delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{1}{\cos(x+\delta x).cosx}.\lim_{\delta x\to 0}\left(\frac{sin\delta x}{\delta x}\right)$$

$\Rightarrow \frac{dy}{dx}=\frac{1}{cosx.cosx}(1)=\frac{1}{\cos^2 x}=\sec^2 x$

$\Rightarrow \frac{dy}{dx}=\sec^2 x$

$\Rightarrow$ Thus $\frac{d}{dx}Tanx=\sec^2 x$

4. $\boldsymbol{\frac{d}{dx}(secx)=secxtanx}$

Proof:

$let\ y=secx\to(i)$

$y+\delta x=\sec(x+\delta x)\to(ii)$

$by\ (ii)-(i)\quad\Rightarrow y+\delta y-y=secx(x+\delta x)-secx$

$$\Rightarrow \delta y=\frac{1}{\cos(x+\delta x)}-\frac{1}{cosx}=\frac{cosx-\cos(x+\delta x)}{\cos(x+\delta x).cosx}$$

$$\left(\because\ cosA-cosB=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right)$$

$$=\frac{-2\sin\left(\frac{x+x+\delta x}{2}\right)\sin\left(\frac{x-x-\delta x}{2}\right)}{\cos(x+\delta x).\delta x}$$

$$=\frac{-2}{\cos(x+\delta x).cosx}\left[\sin\left(x+\frac{\delta x}{2}\right)\sin\left(-\frac{\delta x}{2}\right)\right]$$

$$=\frac{2}{\cos(x+\delta x).cosx}\left[\sin\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)\right]$$

$\because \sin(-\theta)-sin\theta$

Dividing by $\delta x\ and\ take\ limit\ \delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{2}{\cos(x+\delta x).cosx}\left[\lim_{\delta x\to 0}\sin\left(x+\frac{\delta x}{2}\right)\lim_{\delta x\to 0}\sin\frac{\left(\frac{\delta x}{2}\right)}{2\times\frac{\delta x}{2}}\right]$$

$\Rightarrow \frac{dy}{dx}=-\frac{1}{cosx.cosx}(sinx.1)=\frac{1}{\cos x}.\frac{sinx}{\cos x}$

$\Rightarrow \frac{dy}{dx}=secxtanx \qquad hence\ \frac{d}{dx}(secx)=secxtanx$

5. $\boldsymbol{\frac{d}{dx}(cosecx)-cosecxcotx}$

Proof:

$let\ y=cosecx\to(i)$

$y+\delta x=co\sec(x+\delta x)\to(ii)$

$by\ (ii)-(i)\quad\Rightarrow y+\delta y-y=cosecx(x+\delta x)-cosecx$

$$\Rightarrow\ \delta y=\frac{1}{\sin(x+\delta x)}-\frac{1}{sinx}=\frac{sinx-\sin(x+\delta x)}{\sin(x+\delta x).sinx}$$

$$\left(\because\ sinA-sinB=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right)$$

$$=\frac{2\cos\left(\frac{x+x+\delta x}{2}\right)\sin\left(\frac{x-x-\delta x}{2}\right)}{\sin(x+\delta x).sinx}$$

$$=\frac{1}{\sin(x+\delta x).sinx}\left[2\cos\left(x+\frac{\delta x}{2}\right)\sin\left(-\frac{\delta x}{2}\right)\right]$$

$$=\frac{1}{\cos(x+\delta x).cosx}\left[\cos\left(x+\frac{\delta x}{2}\right)\sin\left(-\frac{\delta x}{2}\right)\right]$$

$\because \sin(-\theta)-sin\theta$

Dividing by $\delta x\ and\ take\ limit\ \delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{2}{\sin(x+\delta x).sinx}\left[\lim_{\delta x\to 0}2\cos\left(x+\frac{\delta x}{2}\right)\lim_{\delta x\to 0}\sin\frac{\left(\frac{\delta x}{2}\right)}{2\times\frac{\delta x}{2}}\right]$$

$\Rightarrow \frac{dy}{dx}=-\frac{1}{sinx.sinx}(cosx.1)=\frac{-1}{sinx}.\frac{cosx}{sinx}$

$\Rightarrow \frac{dy}{dx}=-cosecxcotx$

$\Rightarrow hence\ \frac{d}{dx}(cosecx)=-cosecxcotx$

6. $\boldsymbol{\frac{d}{dx}(cotx)=cosec^2x}$

Proof: $let\ y=cotx\to(i)$

$\Rightarrow \boldsymbol{y+\delta y=cot(x+\delta x)\to(ii)}$

$by\ (ii)-(i)\Rightarrow y+\delta y-y=\cot(x+\delta x)-cotx$

$\Rightarrow \delta y=\frac{\cos(x+\delta x)}{\sin(x+\delta x)}-\frac{cosx}{sinx}$

$\Rightarrow \frac{\cos(x+\delta x-x)sinx-\sin(x+\delta x).cosx}{\sin(x+\delta x).sinx}$

$(\because sin\alpha cos\beta-cos\alpha sin\beta=\sin(\alpha-\beta)$

$\Rightarrow \delta y=\frac{\sin(x-\delta s-x)}{\sin(x+\delta x).sinx}=\frac{\sin(-\delta x)}{\sin(x+\delta x).sinx}$

$dividing\ by\ \delta x\ and\ take\ limit\ \delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{-1}{\sin(x+\delta x).sinx}.\lim_{\delta x\to 0}\left(\frac{sin\delta x}{\delta x}\right)$$

$\Rightarrow \frac{dy}{dx}=\frac{1}{sinx.sinx}(1)=\frac{1}{\sin^2 x}=-co\sec^2 x$

$\Rightarrow \frac{dy}{dx}=-\text{cosec}^2 x$

$\Rightarrow$ Thus $\frac{d}{dx}cotx=cosec^2x$

Derivative of inverse trigonometric function:

1. $\boldsymbol{\frac{d}{dx}[sin^{-1}x]=\frac{1}{\sqrt{1-x^2}} \qquad x\in(-1,1)\ or\ -1<x<1}$

Proof: $let\ y=sin^{-1}x$

$\Rightarrow siny=x\Rightarrow\frac{d}{dx}(siny)=\frac{d}{dx}(x)$

$\Rightarrow cosy.\frac{dy}{dx}=1\Rightarrow\frac{dy}{ds}=\frac{1}{cosy}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-sin^2y}} \qquad \because cos^2\theta=1-sin^2\theta$

$\Rightarrow cos\theta=\sqrt{1-sin^2\theta}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}} \qquad \because siny=x$

Hence $\frac{d}{dx}(sin^{-1})=\frac{1}{\sqrt{1-x^2}}$

2. $\boldsymbol{\frac{d}{dx}(cox^{-1}x)=-\frac{1}{\sqrt{1-x^2}} \qquad x\in(-1,1)}$

Proof: $let\ y = cos^{-1}x$

⇨ $cosy = x \Rightarrow \frac{d}{dx}(cosy) = \frac{d}{dx}(x)$

⇨ $-siny.\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{ds} = \frac{-1}{siny}$

⇨ $\frac{dy}{dx} = \frac{-1}{\sqrt{1-cos^2y}}$ $\because sin^2\theta = 1 - cos^2\theta$

⇨ $sin\theta = \sqrt{1-cos^2\theta}$

⇨ $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$ $\because cosy = x$

Hence $\frac{d}{dx}(cos^{-1}) = \frac{-1}{\sqrt{1-x^2}}$

3. $\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}$ $,x \in R$

Proof: $lety = Tan^{-1}x$

$Tany = x \Rightarrow \frac{d}{dx}(tany) = \frac{d}{dx}(x)$

$\Rightarrow sec^2y.\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{sec^2y}$

⇨ $\frac{dy}{dx} = \frac{1}{1+tan^2y}$ $\because 1 + tan^2\theta = sec^2\theta$

⇨ $\frac{dy}{dx} = \frac{1}{1+x^2}$ $\because tany = x$

⇨ Hence $\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}$

4. $\frac{d}{dx}(cot^1x) = \frac{-1}{1+x^2}$ $,x \in [-1,1]',$ $[-1,1] = (-\infty,-1) \cup (1,+\infty)$

Proof: $lety = cot^{-1}x$

$coty = x \Rightarrow \frac{d}{dx}(coty) = \frac{d}{dx}(x)$

$\Rightarrow -cosec^2y.\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{cosec^2y}$

⇨ $\frac{dy}{dx} = \frac{-1}{1+cot^2y}$ $\because cosec^2\theta = 1 + cot^2$

⇨ $\frac{dy}{dx} = \frac{-1}{1+x^2}$ $\because coty = x$

⇨ Hence $\frac{d}{dx}(cot^{-1}x) = \frac{-1}{1+x^2}$

5. $\frac{d}{dx}(sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$ $,x \in [-1,1]$

Proof: $let\ y = sec^{-1}x \Rightarrow secy = x$

$\Rightarrow \frac{d}{dx}(secy) = \frac{d}{dx}(x) \Rightarrow secytany.\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{secytany} = \frac{1}{secy\sqrt{sec^{2y}-1}}$ $\because 1 + tan^2\theta = sec^2\theta$

$\Rightarrow tan^2\theta = sec^2 - 1$

$\Rightarrow tan\theta = \sqrt{sec^2\theta - 1}$

$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$ $\because secy = x$

$hence\ \frac{d}{dx}(sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

6. $\frac{d}{dx}(cosec^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$ $,x \in R$

Proof: $let\ y = cosec^{-1}x \Rightarrow cosecy = x$

$\Rightarrow \frac{d}{dx}(cosecy) = \frac{d}{dx}(x) \Rightarrow cosecycoty.\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{cosecycoty} = \frac{1}{cosecy\sqrt{cosec^{2y}-1}}$

$\because 1 + cot^2\theta = cosec^2\theta$

$\Rightarrow cot^2\theta = cosec^2 - 1$

$\Rightarrow cot\theta = \sqrt{cosec^2\theta - 1}$

$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$ $\because cosecy = x$

$hence\ \frac{d}{dx}(cosec^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$

Exercise 2.5

QUESTION NO.1:

Differentiate the following trigonometric functions from the first principals.

- **i)** $sin2x$
- **ii)** Tan3x
- **iii)** $sin2x + cos2x$
- **iv)** $cosx^2$
- **v)** tan^2x
- **vi)** $\sqrt{tanx}$
- **vii)** $cos\sqrt{x}$

Solution:

(i)

Let $y = sin2x \rightarrow (1)$

$y + \delta y = sin2(x + \delta x) \rightarrow (2)$

$Eq\ 2 - eq1$

$y + \delta y - y = sin2(x + \delta x) - sin2x$

$\delta y = sin(2x + 2\delta x) - sin2x$

$using\ sinP - sinQ = 2cos\left(\frac{P+Q}{2}\right)sin\left(\frac{P-Q}{2}\right)$

$\delta y = 2cos\left(\frac{2x + 2\delta x + 2x}{2}\right)sin\left(\frac{2x + 2\delta x - 2x}{2}\right)$

$= 2cos\left(\frac{4x + 2\delta x}{2}\right)sin\left(\frac{2\delta x}{2}\right)$

$\delta y = 2cos(2x + \delta x)sin\delta x$

Dividing both side by δx

$\frac{\delta y}{\delta x} = \frac{2cos(2x + \delta x)sin\delta x}{\delta x}$

$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = 2\lim_{\delta x \to 0}cos(2x + \delta x).\lim_{\delta x \to 0}\frac{sin\delta x}{\delta x}$

$\frac{dy}{dx} = 2cos(2x + 0).1$

$\frac{dy}{dx} = 2cos2x$

(ii) $let\ \ y = tan3x \rightarrow (1)$

$y + \delta y = tan3(x + \delta x) \rightarrow (2)$

Eq2-eq1

$y + \delta y - y = tan(3x + 3\delta x) - tan3x$

$\delta y = \frac{sin(3x + 3\delta x)}{cos(3x + 3\delta x)} - \frac{sin3x}{cos3x}$

$= \frac{sin(3x + 3\delta x).cos3x - cos(3x + 3\delta x).sin3x}{cos(3x + 3\delta x).cos3x}$

$using\ sin\alpha cos\beta - c0s\alpha sin\beta = sin(\alpha - \beta)$

$= \frac{sin(3x + 3\delta x - 3x)}{cos(3x + 3\delta x).cos3x}$

$dividing\ both\ sides\ by\ \delta x$

$\frac{\delta y}{\delta x} = \frac{1}{cos(3x + 3\delta x).cos3x}.\frac{sin3\delta x}{\delta x}$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{3}{\cos(3x+3\delta x).\cos 3x}.\lim_{\delta x\to 0}\frac{\sin 3\delta x}{\delta x}$$

$$\frac{dy}{dx}=\frac{3}{\cos(3x+0)\cos 3x}.1$$

$$=\frac{3}{\cos 3x.\cos 3x}$$

$$=\frac{3}{\cos^2 3x}$$

$$\frac{dy}{dx}=3\sec^2 3x$$

(iii) Let $y=\sin 2x+\cos 2x\to(1)$

$$y+\delta y=\sin 2(x+\delta x)+\cos 2(x+\delta x)\to(2)$$

$$Eq2-eq1$$

$$y+\delta y-y=\sin(2x+2\delta x)+\cos(2x+2\delta x)-\sin 2x-\cos 2x$$

$$\delta y=[\sin(2x+2\delta x)-\sin 2x]+[\cos(2x+2\delta x)-\cos 2x]$$

$$=2\cos\left(\frac{2x+2\delta x+2x}{2}\right).\sin\left(\frac{2x+2\delta x-2x}{2}\right)-2\sin\left(\frac{2x+2\delta x+2x}{2}\right)\sin\left(\frac{2x+2\delta x-2x}{2}\right)$$

$$=2\cos\left(\frac{4x+2\delta x}{2}\right).\sin\delta x-2\sin\left(\frac{4x+2\delta x}{2}\right).\sin\delta x$$

$$\delta y=[2\cos(2x+2\delta x)-2\sin(2x+2\delta x)].\sin\delta x$$

dividing both sides by δx

$$\frac{\delta y}{\delta x}=[2\cos(2x+2\delta x)-2\sin(2x+2\delta x)].\frac{\sin\delta c}{\delta x}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}[2\cos(2x+2\delta x)-2\sin(2x+2\delta x)]\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}$$

$$=\{2\cos 2x-2\sin 2x\}.1$$

$$\frac{dy}{dx}=2\cos 2x-2\sin 2$$

(iv) $\cos x^2$

Solution:

$$let\ y=\cos$$

$$Let\ \ y=\cos x^2\to(1)$$

$$y+\delta y=\cos(x+\delta x)^2\to(2)$$

$$Eq(2)-eq(1)$$

$$y+\delta y-y=\cos(x+\delta x)^2-\cos x^2$$

$$\delta y=-2\sin\left(\frac{x^2+\delta x^2+2x\delta x+x^2}{2}\right).\sin\left(\frac{x^2+\delta x^2+2x\delta x-x^2}{2}\right)$$

$$\delta y=\frac{-2\sin(2x^2+2x\delta x+\delta x^2).\sin\left(\frac{\delta x(\delta x+2x)}{2}\right)}{\delta x}$$

$$\frac{\delta y}{\delta x}=-2\sin(2x^2+2x\delta x+\delta x^2).\frac{\sin(\delta x(\delta x+2x))}{\delta x.\frac{\delta x+2x}{2}}.\frac{\delta x+2x}{2}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}-2\sin(2x^2+2x\delta x+\delta x^2).\frac{\sin(\delta x(\delta x+2x))}{\delta x.\frac{\delta x+2x}{2}}.\frac{\delta x+2x}{2}$$

$$\frac{dy}{dx}=-2\sin\left(\frac{2x^2}{2}\right).1.\frac{2x}{2}$$

$$\frac{dy}{dx}=-2x\sin x^2$$

(v)

$$let\ \ \ y=\tan^2 x\to(1)$$

$$y+\delta y=\tan^2(x+\delta x)\to(2)$$

$$eq2-eq1$$

$$y+\delta y-y=\tan^2(x+\delta x)-\tan^2 x$$

$$\delta y=\{\tan(x+\delta x)-\tan x\}.\{\tan(x+\delta x)+\tan x\}$$

$$\delta y=\{\tan(x+\delta x)-\tan x\}.\left[\frac{\sin(x+\delta x)}{\cos(x+\delta x)}-\frac{\sin x}{\cos x}\right]$$

$$\delta y=[\tan(x+\delta x)-\tan x].\left[\frac{\sin(x+\delta x).\cos x-\cos(x+\delta x).\sin x}{\cos(x+\delta x).\cos x}\right]$$

$$\delta y=[\tan(x+\delta x)-\tan x].\left[\frac{\sin(x+\delta x-x)}{\cos(x+\delta x).\cos x}\right]$$

$$\delta y=\frac{[\tan(x+\delta x)-\tan x]}{\cos(x+\delta x).\cos x}.\sin\delta x$$

dividing both sides by δx

$$\frac{\delta y}{\delta x}=\frac{[\tan(x+\delta x)-\tan x]}{\cos(x+\delta x).\cos x}.\frac{\sin\delta x}{\delta x}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{[\tan(x+\delta x)-\tan x]}{\cos(x+\delta x).\cos x}.\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}$$

$$\frac{dy}{dx}=\frac{\tan x+\tan x}{\cos x.\cos x}.1$$

$$\frac{dy}{dx}=\frac{2\tan x}{\cos^2 x}$$

$$\frac{dy}{dx}=2\tan x\sec^2$$

(vi)

$$let\ \ \ y=\sqrt{\tan x}------1$$

$$y+\delta y=\sqrt{\tan(x+\delta x)}-----2$$

$$eq2-eq1$$

$$\delta y=\sqrt{\tan(x+\delta x)}-\sqrt{\tan x}$$

$$\delta y=\sqrt{\tan(x+\delta x)}-\sqrt{\tan x}\times\frac{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}$$

$$\delta y=\frac{\left(\sqrt{\tan(x+\delta x)}\right)^2-\left(\sqrt{\tan x}\right)^2}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}$$

$$\delta y=\frac{\tan(x+\delta x)-\tan x}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}$$

$$\delta y=\frac{1}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}\left[\frac{\sin(x+\delta x)}{\cos(x+\delta x)}-\frac{\sin x}{\cos x}\right.$$

$$\delta y=\frac{1}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}\left[\frac{\sin(x+\delta x)\cos x-\cos(x+\delta x).\sin s}{\cos(x+\delta x)\cos x}\right]$$

$$\delta y=\frac{\sin(x+\delta x-x)}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}.\cos(x+\delta x)\cos x}.$$

$$\delta y=\frac{1}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}.\cos(x+\delta x)\cos x}.\sin\delta x$$

dividing both sides by δx *and take limit on both sides*

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{1}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}.\cos(x+\delta x)\cos x}.\lim_{\delta x\to 0}\sin\delta x$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{\tan(x)}+\sqrt{\tan x}.\cos(x)\cos x}.1$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\tan x}\cos^2 x}$$

$$\frac{dy}{dx}=\frac{\sec^2 x}{2\sqrt{\tan x}}$$

(vii)

$$\cos\sqrt{x}$$

Solution:

$$Let\ y=\cos\sqrt{x}\rightarrow(i)$$

$$\Rightarrow y+\delta y=\cos\sqrt{x+\delta x}\rightarrow(ii)$$

$$by\ (ii)-(i)\Rightarrow y+\delta y-y=\cos\sqrt{x+\delta x}-\cos\sqrt{x}$$

$$\Rightarrow \delta y=\cos\sqrt{x+\delta x}-\cos\sqrt{x}$$

$$\because \cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$=-2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)$$

dividing by δx

$$\Rightarrow\frac{\delta y}{\delta x}=\frac{-2\sin\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{\delta x}$$

$$\because(\sqrt{x+\delta x}+\sqrt{x})(\sqrt{x+\delta x}-\sqrt{x})=\delta x$$

$$\Rightarrow\frac{\delta y}{\delta x}=\frac{-2\sin\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}}{(\sqrt{x+\delta x}+\sqrt{x})}\times\frac{\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{2\times\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}$$

Take limit $\delta x\rightarrow 0$

$$\Rightarrow\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{-\sin\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}}{(\sqrt{x+\delta x}+\sqrt{x})}\times\lim_{\delta x\to 0}\frac{\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}$$

$$\Rightarrow\frac{dy}{dx}=\frac{-\sin\frac{\sqrt{x}+\sqrt{x}}{2}}{\sqrt{x}+\sqrt{x}}(1)$$

$$=\frac{-\sin\left(\frac{2\sqrt{x}}{2}\right)}{2\sqrt{x}}$$

$$\Rightarrow\frac{dy}{dx}=\frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Question # 2.

Differentiate the following w.r.t the variable involved.

(i) $x^2\sec 4x$

Solution.

Let $y=x^2\sec 4x$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}(x^2\sec 4x)$$

$$\frac{dy}{dx}=(x^2)\frac{d}{dx}(\sec 4x)+(\sec 4x)\frac{d}{dx}(x^2)$$

$$\frac{dy}{dx}=(x^2)(\sec 4x\tan 4x\,(4x))+(\sec 4x)(2x)$$

$$\frac{dy}{dx}=4x^2\,(\sec 4x Tan 4x)+2x\sec 4x$$

Which is required.

(ii) $y=\tan^3\theta.\sec^2\theta$

Solution.

Let $y=\tan^3\theta\,.\sec^2\theta$

Differentiate w.r.t θ

$$\frac{dy}{d\theta}=\frac{d}{d\theta}(\tan^3\theta.\sec^2\theta)$$

$$\frac{dy}{d\theta}=(\sec^2\theta)\frac{d}{\theta}(\tan^3\theta)+(\tan^3\theta)\frac{d}{d\theta}(\sec^2\theta)$$

$$\frac{dy}{d\theta}=(\sec^2\theta)(3\tan^2\theta(\sec^2\theta))+(\tan^3\theta)(2\sec\theta\,(\sec\theta\,\tan\theta))$$

$$\frac{dy}{d\theta}=(3\sec^4\theta\tan^2\theta)+(2\tan^4\theta\sec^2\theta)$$

$$\frac{dy}{d\theta}=\sec^2\theta\tan^2\theta(3\sec^2\theta\tan^2\theta+2\tan^2\theta\sec^2\theta)$$

(iii) $y=(\sin 2\theta-\cos 3\theta)^2$

Solution.

Let $y=(\sin 2\theta-\cos 3\theta)^2$

Differentiate w.r.t θ

$$\frac{dy}{d\theta}=\frac{d}{d\theta}(\sin 2\theta-\cos 3\theta)^2$$

$$\frac{dy}{d\theta}=2(\sin 2\theta-\cos 3\theta)\left(\frac{d}{d\theta}\right)(\sin 2\theta-\cos 3\theta)$$

$$\frac{dy}{d\theta}=2(\sin 2\theta-\cos 3\theta)\left[\cos\theta\frac{d}{d\theta}(2\theta)-(-\sin 3\theta\frac{d}{d\theta}(3\theta)\right]$$

$$\frac{dy}{dx}=2(\sin 2\theta-\cos 3\theta)(2\cos 2\theta+3\sin 3\theta)$$

Which is required.

(iv) $y=\cos\sqrt{x}+\sqrt{\sin x}$

Solution.

Let $y=\cos\sqrt{x}+\sqrt{\sin x}$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}(\cos\sqrt{x}+\sqrt{\sin x})$$

$$\frac{dy}{dx}=\frac{-\sin\sqrt{x}}{2\sqrt{x}}+\frac{1}{2\sqrt{\sin x}}(\cos x)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{-\sin\sqrt{x}}{\sqrt{x}}\frac{\cos x}{\sqrt{\sin x}}\right)$$

Which is required.

Question # 3. Find $\frac{dy}{dx}$ if:

(i) $y=x\cos y$

Solution.

Since $y=x\cos y$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}(x\cos y)$$

$$\frac{dy}{dx}=(\cos y)\frac{d}{dx}(x)+(x)\frac{d}{dx}(\cos y)$$

$$\frac{dy}{dx}=\cos y(1)+x(-\sin y)\frac{dy}{dx}$$

$$\frac{dy}{dx}=\cos y-x\sin y\frac{dy}{dx}$$

$$\frac{dy}{dx}+x\sin y\frac{dy}{dx}=\cos y$$

$$\frac{dy}{dx}(1+x\sin y)=\cos y$$

$$\frac{dy}{dx}=\frac{cosy}{1+x\,siny}$$

Which is required.

(ii) $\boldsymbol{x=y\,siny}$

Solution.

Since $x=y\,siny$

Differentiate w.r.t x

$$\frac{dx}{dx}=\frac{d}{dx}(y\,siny)$$

$$1=(siny)\frac{d}{dx}(y)+(y)\frac{d}{dx}(siny)$$

$$1=(siny)\left(\frac{dx}{dx}\right)+(y)(cosy)\left(\frac{dx}{dx}\right)$$

$$1=\frac{dx}{dx}(siny+y\,cosy)$$

$$\frac{dy}{dx}=\frac{1}{siny+y\,cosy}$$

Question # 4. Find derivative w.r.t x

(i) $\boldsymbol{y=cos\sqrt{\frac{1+x}{1+2x}}}$

Solution.

Let $y=cos\sqrt{\frac{1+x}{1+2x}}$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\left(cos\sqrt{\frac{1+x}{1+2x}}\right)$$

$$\frac{dy}{dx}=\left(-sin\sqrt{\frac{1+x}{1+2x}}\right)\frac{d}{dx}\left(\sqrt{\frac{1+x}{1+2x}}\right)$$

$$\frac{dy}{dx}=\left(-sin\sqrt{\frac{1+x}{1+2x}}\right)\frac{1}{2}\left(\frac{1+x}{1+2x}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\frac{1+x}{1+2x}\right)$$

$$\frac{dy}{dx}=\left(-sin\sqrt{\frac{1+x}{1+2x}}\right)\frac{1}{2}\left(\frac{1+2x}{1+x}\right)^{\frac{1}{2}}\left(\frac{(1+2x)\frac{d}{dx}(1+x)-(1+x)\frac{d}{dx}(1+2x)}{(1+2x)^2}\right)$$

$$\frac{dy}{dx}=\left(-sin\sqrt{\frac{1+x}{1+2x}}\right)\frac{1}{2}\left(\frac{1+2x}{1+x}\right)^{\frac{1}{2}}\left(\frac{(1+2x)(1)-(1+x)(2)}{(1+2x)^2}\right)$$

$$\frac{dy}{dx}=\left(-sin\sqrt{\frac{1+x}{1+2x}}\right)\frac{1}{2}\left(\frac{1+2x}{1+x}\right)^{\frac{1}{2}}\left(\frac{1+2x-2-2x}{(1+2x)^2}\right)$$

$$\frac{dy}{dx}=\left(-sin\sqrt{\frac{1+x}{1+2x}}\right)\frac{1}{2}\left(\frac{1}{1+x}\right)^{\frac{1}{2}}\left(\frac{-1}{(1+2x)^{2-\frac{1}{2}}}\right)$$

$$\frac{dy}{dx}=\left(\frac{sin\sqrt{\frac{1+x}{1+2x}}}{2(1+2x)^{\frac{3}{2}}\sqrt{1+x}}\right)$$

Which is required.

(i) $\boldsymbol{y=sin\sqrt{\frac{1+2x}{1+x}}}$

Solution.

Let $y=sin\sqrt{\frac{1+2x}{1+x}}$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\left(sin\sqrt{\frac{1+2x}{1+x}}\right)$$

$$\frac{dy}{dx}=\left(cos\sqrt{\frac{1+2x}{1+x}}\right)\frac{d}{dx}\left(\sqrt{\frac{1+2x}{1+x}}\right)$$

$$\frac{dy}{dx}=\left(cos\sqrt{\frac{1+2x}{1+x}}\right)\frac{1}{2}\left(\frac{1+2x}{1+x}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\frac{1+2x}{1+x}\right)$$

$$\frac{dy}{dx}=\left(cos\sqrt{\frac{1+2x}{1+x}}\right)\frac{1}{2}\left(\frac{1+2x}{1+x}\right)^{\frac{1}{2}}\left(\frac{(1+x)\frac{d}{dx}(1+2x)-(1+2x)\frac{d}{dx}(1+x)}{(1+x)^2}\right)$$

$$\frac{dy}{dx}=\left(cos\sqrt{\frac{1+2x}{1+x}}\right)\frac{1}{2}\left(\frac{1+x}{1+2x}\right)^{\frac{1}{2}}\left(\frac{(1+x)(2)-(1+2x)(1)}{(1+x)^2}\right)$$

$$\frac{dy}{dx}=\left(cos\sqrt{\frac{1+2x}{1+x}}\right)\frac{1}{2}\left(\frac{1+x}{1+2x}\right)^{\frac{1}{2}}\left(\frac{2+2x-1-2x}{(1+2x)^2}\right)$$

$$\frac{dy}{dx}=\left(cos\sqrt{\frac{1+2x}{1+x}}\right)\frac{1}{2}\left(\frac{1}{1+2x}\right)^{\frac{1}{2}}\left(\frac{1}{(1+x)^{2-\frac{1}{2}}}\right)$$

$$\frac{dy}{dx}=\left(\frac{cos\sqrt{\frac{1+2x}{1+x}}}{2(1+x)^{\frac{3}{2}}\sqrt{1+2x}}\right)$$

Which is required.

Question # 5 Differentiate

(i) $\boldsymbol{sinx\ w.r.t.\ cotx}$

Solution.

Let $y=$ sinx and $u=cotx$

we find $\frac{dy}{du}$

Now $y=$ sinx

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}(\text{sinx})$$

$$\frac{dy}{dx}=cosx$$

Now $u=$ cotx

Differentiate w.r.t x

$$\frac{du}{dx}=-Cosec^2x$$

$$\frac{dx}{du}=-\frac{1}{Cosec^2x}$$

$$\frac{dx}{du} = -Sin^2x$$

Now using chain rule

$$\frac{dy}{du} = \frac{dy}{dx}.\frac{dx}{du}$$

$$\frac{dy}{du} = (Cosx)(-Sin^2x) = -Sin^2x\ Cosx.$$

(ii). $\boldsymbol{Sin^2x\ w.r.t\ \ Cos^4x}$

Solution.

Let $y = \ Sin^2x$ and $u = Cos^4x$

we find $\frac{dy}{du}$

Now $y = \ Sin^2x$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\ (Sin^2x)$$

$$\frac{dy}{dx} = 2Sinx\ cosx$$

Now $u = Cos^4x$

Differentiate w.r.t x

$$\frac{du}{dx} = 4Cos^3x(-Sinx) = -4SinxCos^3x$$

$$\frac{dx}{du} = -\frac{1}{4SinxCos^3x}$$

Now using chain rule

$$\frac{dy}{du} = \frac{dy}{dx}.\frac{dx}{du}$$

$$\frac{dy}{du} = (2Sinx\ cosx)\left(-\frac{1}{4SinxCos^3x}\right)$$

$$\frac{dy}{du} = -\frac{1}{2Cos^2x}$$

$$\frac{dy}{du} = -\frac{1}{2}Sec^2x$$

Question.6.

If $tany\ (1 + tanx) = 1 - tanx$, *show that* $\frac{dy}{dx} = -1.$

Solution.

Since

$$tany\ (1 + tanx) = 1 - tanx$$

$$tany = \frac{1 - tanx}{1 + tanx}$$

$$tany = \frac{\tan\frac{\pi}{4} - tanx}{\tan\frac{\pi}{4} + tanx}$$

$$tany = \tan\left(\frac{\pi}{4} - x\right)$$

$$y = \frac{\pi}{4} - x$$

Differentiate w.r.t "x" , we have

$$\frac{dy}{dx} = 0 - 1$$

$$\frac{dy}{dx} = -1$$

Hence proved.

Question.7.

If $\boldsymbol{y = \sqrt{tanx + \sqrt{tanx + \sqrt{tanx + \cdots\infty}}}}$, $\boldsymbol{Prove\ that\ (2y - 1)\frac{dy}{dx} = Sec^2x}$

Solution.

Since

$$y = \sqrt{tanx + \sqrt{tanx + \sqrt{tanx + \cdots\infty}}} - - - (i)$$

Squarring on both sides, we have

$$y^2 = tanx + \sqrt{tanx + \sqrt{tanx + \cdots\infty}}$$

$$y^2 = tanx + \sqrt{tanx + \sqrt{tanx + \sqrt{tanx + \cdots\infty}}}$$

$$y^2 = tanx + y$$

Differentiate w.r.t "x" , we have

$$2y\frac{dy}{dx} = Sec^2x + \frac{dy}{dx}$$

$$2y\frac{dy}{dx} - \frac{dy}{dx} = Sec^2x$$

$$(2y - 1)\frac{dy}{dx} = Sec^2x$$

Hence Proved.

Question.8

If $\boldsymbol{x = aCos^3\theta}$, $\boldsymbol{y = b\ Sin^3\theta}$ **, Show that** $\boldsymbol{a\frac{dy}{dx} + b\ tanx = 0}$

Solution

$$x = aCos^3\theta\ \ ,y = b\ Sin^3\theta$$

Diff. $''x''$ w r.t.$''\theta''$, *we have*

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(aCos^3\theta)$$

$$\frac{dx}{d\theta} = a.3Cos^2\frac{d}{d\theta}(Cos\theta)$$

$$\frac{dx}{d\theta} = a.3Cos^2(-Sin\theta)$$

$$\frac{dx}{d\theta} = -3aSin\theta Cos^2$$

$$\frac{d\theta}{dx} = \frac{-1}{3aSin\theta Cos^2}$$

Diff. $''y''$ w r.t.$''\theta''$, *we have*

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\ Sin^3\theta)$$

$$\frac{dy}{d\theta} = b.3Sin^2\frac{d}{d\theta}(Sin\theta)$$

$$\frac{dy}{d\theta} = b.3Sin^2Cos\theta$$

$$\frac{dy}{d\theta} = 3bSin^2Cos\theta$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx}$$

$$\frac{dy}{dx} = 3bSin^2Cos\theta.\frac{-1}{3aSin\theta Cos^2}$$
$$\frac{dy}{dx} = -\frac{b}{a}\ tan\theta$$
$$a\frac{dy}{dx} = -b\ tan\theta$$
$$a\frac{dy}{dx} + b\ tan\theta = 0$$

Hence proved.

Question.10. Differentiate $''x''$, *we have*

(i) $Cos^{-1}\frac{x}{a}$

Solution.

Let $y = Cos^{-1}\frac{x}{a}$

Diff.w.r.t $''x''$, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(Cos^{-1}\frac{x}{a}\right)$$
$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}.\frac{d}{dx}\left(\frac{x}{a}\right)$$
$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}.\left(\frac{1}{a}\right)$$
$$\frac{dy}{dx} = -\frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}}.\left(\frac{1}{a}\right)$$
$$\frac{dy}{dx} = -\frac{1}{\frac{\sqrt{a^2-x^2}}{a}}.\left(\frac{1}{a}\right)$$
$$\frac{dy}{dx} = -\frac{a}{\sqrt{a^2-x^2}}.\left(\frac{1}{a}\right)$$
$$\frac{dy}{dx} = -\frac{1}{\sqrt{a^2-x^2}}.$$

Which is required.

(ii) $Cot^{-1}\frac{x}{a}$

Solution.

Let $y = Cot^{-1}\frac{x}{a}$

Diff.w.r.t $''x''$, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(Cot^{-1}\frac{x}{a}\right)$$
$$\frac{dy}{dx} = -\frac{1}{1+\left(\frac{x}{a}\right)^2}.\frac{d}{dx}\left(\frac{x}{a}\right)$$
$$\frac{dy}{dx} = -\frac{1}{1+\frac{x^2}{a^2}}.\left(\frac{1}{a}\right)$$
$$\frac{dy}{dx} = -\frac{1}{\frac{a^2+x^2}{a^2}}.\left(\frac{1}{a}\right)$$
$$\frac{dy}{dx} = -\frac{1}{\frac{a^2+x^2}{a^2}}.\left(\frac{1}{a}\right)$$
$$\frac{dy}{dx} = -\frac{a^2}{a^2+x^2}.\left(\frac{1}{a}\right)$$
$$\frac{dy}{dx} = -\frac{a}{a^2+x^2}.$$

Which is required.

(iii) $\frac{1}{a}Sin^{-1}\frac{a}{x}$

Solution.

Let $y = \frac{1}{a}Sin^{-1}\frac{a}{x}$

Diff.w.r.t $''x''$, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{a}Sin^{-1}\frac{a}{x}\right)$$
$$\frac{dy}{dx} = \frac{1}{a}.\frac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}}.\frac{d}{dx}\left(\frac{a}{x}\right)$$
$$\frac{dy}{dx} = \frac{1}{a}.\frac{1}{\sqrt{1-\frac{a^2}{x^2}}}.\left(\frac{-a}{x^2}\right)$$
$$\frac{dy}{dx} = \frac{1}{a}.\frac{1}{\sqrt{\frac{x^2-a^2}{x^2}}}.\left(\frac{-a}{x^2}\right)$$
$$\frac{dy}{dx} = \frac{1}{a}.\frac{1}{\frac{\sqrt{x^2-a^2}}{x}}.\left(\frac{-a}{x^2}\right)$$
$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2-a^2}}.\left(\frac{-1}{x^2}\right)$$
$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-a^2}}$$

Which is required.

(iv) $Sin^{-1}\sqrt{1-x^2}$

Solution.

Let $y = Sin^{-1}\sqrt{1-x^2}$

Diff.w.r.t $''x''$, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(Sin^{-1}\sqrt{1-x^2}\right)$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}.\frac{d}{dx}\left(\sqrt{1-x^2}\right)$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(1-x^2)}}.\left(\frac{1}{2\sqrt{1-x^2}}\frac{d}{dx}(1-x^2)\right)$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-1+x^2}}.\left(\frac{1}{2\sqrt{1-x^2}}(-2x)\right)$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2}}.\left(\frac{-x}{\sqrt{1-x^2}}\right)$$
$$\frac{dy}{dx} = \frac{1}{x}.\left(\frac{-x}{\sqrt{1-x^2}}\right)$$
$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

Which is required.

(v) $Sec^{-1}\left(\frac{x^2-1}{x^2+1}\right)$

Solution.

Let $y = Sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Diff.w.r.t $''x''$, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(Sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)\right)$$

$$\frac{dy}{dx}=\frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2-1}}.\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$\frac{dy}{dx}=\frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\frac{\sqrt{(x^2+1)^2-(x^2-1)^2}}{(x^2-1)}}.$$

$$\left(\frac{(x^2-1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}\right)$$

$$\frac{dy}{dx}=\frac{1}{(x^2+1)\frac{\sqrt{4.x^2.1}}{(x^2-1)^2}}.\left(\frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}\right)$$

$$\frac{dy}{dx}=\frac{1}{(x^2+1).2x}.(2x(x^2-1-x^2-1)$$

$$\frac{dy}{dx}=\frac{1}{(x^2+1)}.(-2)$$

$$\frac{dy}{dx}=\frac{-2}{x^2+1}$$

Which is required.

(vi) $Cot^{-1}\left(\frac{2x}{1-x^2}\right)$

Solution.

Let $y=Cot^{-1}\left(\frac{2x}{1-x^2}\right)$

Diff.w.r.t $''x''$, *we have*

$$\frac{dy}{dx}=\frac{d}{dx}\left(Cot^{-1}\left(\frac{2x}{1-x^2}\right)\right)$$

$$\frac{dy}{dx}=-\frac{1}{1+\left(\frac{2x}{1-x^2}\right)^2}.\frac{d}{dx}\left(\frac{2x}{1-x^2}\right)$$

$$\frac{dy}{dx}=-\frac{1}{1+\frac{4x^2}{(1-x^2)^2}}.\left(2.\frac{(1-x^2)\frac{d}{dx}(x)-x\frac{d}{dx}(1-x^2)}{(1-x^2)^2}\right)$$

$$\frac{dy}{dx}=-\frac{1}{\frac{(1-x^2)^2+4x^2}{(1-x^2)^2}}.\left(2.\frac{(1-x^2).1-x(-2x)}{(1-x^2)^2}\right)$$

$$\frac{dy}{dx}=-\frac{1}{1+x^4-2x^2+4x^2}.(2(1-x^2+2x^2))$$

$$\frac{dy}{dx}=-\frac{1}{1+x^4+2x^2}.(2(1+x^2))$$

$$\frac{dy}{dx}=-\frac{1}{(1+x^2)^2}.(2(1+x^2))$$

$$\frac{dy}{dx}=\frac{-2}{1+x^2}$$

Which is required.

(vii) $Cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution.

Let $y=Cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Diff.w.r.t $''x''$, *we have*

$$\frac{dy}{dx}=\frac{d}{dx}\left(Cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right)$$

$$\frac{dy}{dx}=-\frac{x^2-1}{x^2-1\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}.\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\frac{dy}{dx}=-\frac{x^2-1}{x^2-1\sqrt{1-\frac{(1-x^2)^2}{(1+x^2)^2}}}.\left(\frac{(1+x^2)\frac{d}{dx}(1-x^2)-(1-x^2)\frac{d}{dx}(1+x^2)}{(x^2+1)^2}\right)$$

$$\frac{dy}{dx}=-\frac{x^2-1}{x^2-1\sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}}}.\left(\frac{(x^2+1)(-2x)-(1-x^2)(2x)}{(x^2+1)^2}\right)$$

$$=\frac{x^2-1}{\frac{(x^2-1)\sqrt{x^4+2x^2+1-(x^4+1-2x)}}{x^2-1}}.\frac{2x(x^2-1-x^2-1)}{(x^2-1)^2}$$

$$=\frac{(x^2-1)^2}{(x^2+1)\sqrt{2x^2+2x^2}}.\frac{2x(-2)}{(x^2-1)^2}$$

$$\frac{-4x}{(x^2\mp1)\sqrt{4x^2}}=\frac{-4x}{(x^2-1).2x}=\frac{-2x}{(x^2+1)}$$

$$\Rightarrow\frac{dy}{dx}=-\frac{2}{(x^2+1)}$$

Which is required.

Question.11.

Show that $\frac{dy}{dx}=\frac{y}{x}$ ***if*** $\frac{y}{x}=\tan^{-1}\frac{x}{y}$

Solution.

Since

$$\frac{y}{x}=\tan^{-1}\frac{x}{y}$$

$$y=x\tan^{-1}\frac{x}{y}$$

Diff. w.r.t $''x''$, *we have*

$$\frac{dy}{dx}=x\frac{d}{dx}\tan^{-1}\frac{x}{y}+\tan^{-1}\frac{x}{y}\frac{d}{dx}(x)$$

$$\frac{dy}{dx}=x\frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{d}{dx}\left(\frac{x}{y}\right)+\tan^{-1}\frac{x}{y}.1$$

$$\frac{dy}{dx}=x\frac{1}{1+\frac{x^2}{y^2}}\left(\frac{y\frac{d}{dx}x-x\frac{dy}{dx}}{y^2}\right)+\frac{y}{x}$$

$$\frac{dy}{dx}=x\frac{1}{\frac{y^2+x^2}{y^2}}\left(\frac{y-x\frac{dy}{dx}}{y^2}\right)+\frac{y}{x}$$

$$\frac{dy}{dx}=x\frac{1}{y^2+x^2}\left(y-x\frac{dy}{dx}\right)+\frac{y}{x}$$

$$\frac{dy}{dx}=\frac{x}{x^2+y^2}\left(y-x\frac{dy}{dx}\right)+\frac{y}{x}$$

$$\frac{dy}{dx}=\frac{xy}{x^2+y^2}-\frac{x^2}{x^2+y^2}\frac{dy}{dx}+\frac{y}{x}$$

$$\frac{dy}{dx}+\frac{x^2}{x^2+y^2}\frac{dy}{dx}=\frac{xy}{x^2+y^2}+\frac{y}{x}$$

$$\left(1+\frac{x^2}{x^2+y^2}\right)\frac{dy}{dx}=\frac{y}{x}\left(\frac{x^2}{x^2+y^2}+1\right)$$

$$\left(1+\frac{x^2}{x^2+y^2}\right)\frac{dy}{dx}=\frac{y}{x}\left(1+\frac{x^2}{x^2+y^2}\right)$$
$$\frac{dy}{dx}=\frac{y}{x}$$

Hence Proved.

Question.12.

If $y=\tan(p\tan^{-1}x)$, Show that $(1+x^2)y_1-p(1+y^2)$

Solution.

Since

$$y=\tan(p\tan^{-1}x)$$
$$\tan^{-1}y=p\tan^{-1}x$$

Differentiate with respect to $''x''$, we have

$$\frac{1}{1+y^2}\frac{dy}{dx}=p\frac{1}{1+x^2}$$

Since $\frac{dy}{dx}=y_1$

$$\frac{1}{1+y^2}y_1=p\frac{1}{1+x^2}$$
$$(1+x^2)y_1=p(1+y^2)$$
$$(1+x^2)y_1-p(1+y^2)=0$$

Hence Proved.

Derivative of exponential Functions:

if $f(x)=a^x$ where $a>0$ then $f(x)$ is called general expontial function. here the base a is constant and exponent x is variable.

if $f(x)=e^x$ wher $e\cong 2.71$ then $f(x)$ is called natural expontial function here base e is constant and expontent x is variable.

1. $\frac{d}{dx}(e^x)=e^x$

Proof:

Let $y=e^x\rightarrow(i)$

$\Rightarrow y+\delta y=e^{x+\delta x}\rightarrow(ii)$

by $(ii)-(i)\Rightarrow y+\delta y-y=e^{x+\delta x}-e^x$

$\Rightarrow \delta y=e^x.e^{\delta x}-e^x=e^x(e^{\delta x}-1)$

Dividing by δx and take limit $\delta x\rightarrow 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[e^x\left(\frac{e^{\delta x}-1}{\delta x}\right)\right]$$
$$\frac{dy}{dx}=e^x.\lim_{\delta x\to 0}\frac{e^{\delta x}-1}{\delta x}$$
$$\frac{dy}{dx}=e^x.lne \qquad \because \lim_{\delta x\to 0}\frac{a^x-1}{x}=lna$$
$$\frac{dy}{dx}=e^x \qquad and\ \lim_{\delta x\to 0}\frac{e^x-1}{x}=lne=1$$
$$\frac{d}{dx}(e^x)=e^x$$

2. $\frac{d}{dx}(a^x)=a^x lna$

Proof:

$$Let\ y=a^x\rightarrow(i)$$
$$\Rightarrow y+\delta y=a^{x+\delta x}\rightarrow(ii)$$
$$by\ (ii)-(i)\Rightarrow y+\delta y-y=a^{x+\delta x}-a^x$$
$$\Rightarrow \delta y=a^x.a^{\delta x}-a^x=a^x(a^x-1)$$

Divivding by δx and take limit $\delta x\rightarrow 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[a^x\frac{a^{\delta x}-1}{\delta x}\right]$$
$$\frac{dy}{dx}=a^x\lim_{\delta x\to 0}\frac{a^x-1}{\delta x}=a^x lna$$
$$Hence\ \frac{d}{dx}(a^x)=a^x lna \qquad (\because \lim_{\delta x\to 0}\frac{a^x-1}{x}=lna)$$

Note: as $a^x=a^x \Rightarrow lna^x=lna^x$

$$\Rightarrow lna^x=xlna$$
$$=xlna.lne \qquad \because lna^b=blna$$
$$=xlna \qquad \because lne=1$$
$$ln^x=lne^{xlna}$$

Derivative of the logarithmic function:

logarithmic function: if $a>0$ $a\neq 1$ and $x=a^y$ the the functioon defined by $y=log_a x$. $(x>0)$ is called the logarthmic of x to the base a.

The logarithmic functions $log_e x$ and $log_{10}x$ are called natural and common logarthmic resp. $y=log_e x$ is written as $y=lnx$

1. $\frac{d}{dx}(lnx)=\frac{1}{x}$

Proof:

Let $y=lnx\rightarrow(i)$

$$\Rightarrow y+\delta y=\ln(x+\delta x)-(ii)$$
$$\Rightarrow (ii)-(i)\Rightarrow y+\delta y-y=\ln(x+\delta x)-lnx$$
$$\delta y=\ln\frac{(x+\delta x)}{x} \qquad (\because lnx-lny=\ln\frac{x}{y})$$
$$\delta y=\ln\left(\frac{x}{x}+\frac{\delta x}{x}\right)=\ln\left(1+\frac{\delta x}{x}\right)$$

Dividing by δx and take limit $\delta x\rightarrow o$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[\frac{1}{\delta x}\ln\left(1+\frac{\delta x}{x}\right)\right]$$
$$\frac{dy}{dx}=\frac{1}{x}\lim_{\delta x\to 0}\ln\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}} \qquad \left(\because \lim_{\delta x\to 0}\left(1+\frac{1}{x}\right)^n=e\right)$$
$$\frac{dy}{dx}=\frac{1}{x}lne=\frac{1}{x}$$
$$Hence\ \frac{d}{dx}(lnx)=\frac{1}{x}$$

2. $\frac{d}{dx}[\log_a x]=\frac{1}{x}\frac{1}{lna}$

Proof:

$$Let\ u=\log_a x\rightarrow(i)$$
$$\Rightarrow y+\delta y=\log_a(x+\delta x)-(ii)$$
$$\Rightarrow (ii)-(i)\Rightarrow y+\delta y-y=\log_a(x+\delta x)-\log_a x$$
$$\delta y=\log_a\frac{(x+\delta x)}{x} \qquad (\because \log_a x-lny=\log_a\frac{x}{y})$$
$$\delta y=\log_a\left(\frac{x}{x}+\frac{\delta x}{x}\right)=\log_a\left(1+\frac{\delta x}{x}\right)$$

Dividing by δx and take limit $\delta x\rightarrow o$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[\frac{1}{\delta x}\log_a\left(1+\frac{\delta x}{x}\right)\right]$$

$$\frac{dy}{dx}=\frac{1}{x}\lim_{\delta x\to 0}\log_a\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}} \quad \left(\because \lim_{\delta x\to 0}\left(1+\frac{1}{x}\right)^n = e\right)$$

$$\frac{dy}{dx}=\frac{1}{x}\log_a e=\frac{1}{x}.\frac{1}{\log_e a}=\frac{1}{x}.\frac{1}{lna}$$

$$Hence\ \frac{d}{dx}(\log_a x)=\frac{1}{xlna}$$

Logarithmic differentiation:

Derivative of hyperbolic function:

1. $\frac{d}{dx}(sinhx)=coshx$

Proof:

$we\ know\ thta\ sinhx=\frac{e^x-e^{-x}}{2}$

$$so\ \frac{d}{dx}(sinhx)=\frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right)$$

$$\frac{1}{2}\frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right)=\frac{1}{2}\left\{\frac{d}{dx}(e^x)-\frac{d}{dx}(e^{-x})\right\}$$

$$\frac{1}{2}(e^x-e^{-x})\frac{d}{dx}(-x)=\frac{1}{2}(e^x-e^{-x}(-1))$$

$$\frac{1}{2}(e^x+e^{-x})=\frac{e^x+e^{-x}}{2}=coshx$$

$\frac{d}{dx}(sinhx)=coshx \quad \because\ coshx=\frac{e^x+e^{-x}}{2}$

2. $\frac{d}{dx}(coshx)=sinhx$

Proof:

$we\ know\ thta\ coshx=\frac{e^x+e^{-x}}{2}$

$$so\ \frac{d}{dx}(coshx)=\frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right)$$

$$\frac{1}{2}\frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right)=\frac{1}{2}\left\{\frac{d}{dx}(e^x)+\frac{d}{dx}(e^{-x})\right\}$$

$$\frac{1}{2}(e^x+e^{-x})\frac{d}{dx}(-x)=\frac{1}{2}(e^x+e^{-x}(-1))$$

$$\frac{1}{2}(e^x-e^{-x})=\frac{e^x-e^{-x}}{2}=sinhx$$

$\frac{d}{dx}(coshx)=sinhx \quad \because\ sinhx=\frac{e^x-e^{-x}}{2}$

3. $\frac{d}{dx}(tanhx)=\operatorname{sech}^2 x$

Proof:

We know that $tanhx=\frac{e^x-e^x}{e^x+e^x}$

So $\frac{d}{dx}(tanhx)=\frac{d}{dx}\left(\frac{e^x-e^x}{e^x+e^x}\right)$

$$=\frac{(e^x+e^{-x})\frac{d}{dx}(e^x-e^{-x})-(e^x-e^{-x})\frac{d}{dx}(e^x+e^{-x})}{(e^x+e^{-x})^2}$$

$$=\frac{(e^x+e^{-x})(e^x-e^{-x}(-1))-(e^x-e^{-x})(e^x+e^{-x}(-1))}{(e^x+e^{-x})^2}$$

$$=\frac{(e^x+e^{-x})(e^x+e^{-x})-(e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$$

$$=\frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}$$

$$=\frac{e^{2x}+e^{-2x}+2e^xe^{-x}-(e^{2x}+e^{-2x}-2e^xe^{-x})}{(e^x+e^{-x})^2}$$

$$=\frac{e^{2x}+e^{-2x}+2-e^{2x}-e^{-2x}+2}{(e^x+e^{-x})^2}=\frac{4}{(e^x+e^{-x})^2}$$

$$\frac{d}{dx}(tanhx)=\left(\frac{2}{e^x+e^{-x}}\right)^2=(sechx)^2=\operatorname{sech}^2 x$$

$$\Rightarrow \frac{d}{dx}(tanhx)=\operatorname{sech}^2 x \quad \because sechx=\frac{2}{e^x+e^{-x}}$$

4.

$$\frac{d}{dx}(cosechx)=-cosechxcothx$$

Proof:

we know that $cosechx=\frac{1}{sinhx}$

$$so\ \frac{d}{dx}(cosechx)=\frac{d}{dx}\left(\frac{1}{sinhx}\right)$$

$$=\frac{sinhx\frac{d}{dx}(1)-(1)\frac{d}{dx}(sinhx)}{\sinh^2 x}$$

$$=\frac{sinhx(0)-coshx}{\sinh^2 x}$$

$$\frac{d}{dx}(cosechx)=\frac{-coshx}{sinhx.sinhx}=-cothxcosechx$$

$$\frac{d}{dx}(cosechx)=-cosechxcothx$$

5.

$$\frac{d}{dx}(sechx)=-sechxtanhx$$

Proof:

$$we\ know\ that\ sechx=\frac{1}{coshx}$$

$$\Rightarrow \frac{d}{dx}(sechx)=\frac{d}{dx}\left(\frac{1}{coshx}\right)$$

$$\Rightarrow =\frac{coshx\frac{d}{dx}(1)-(1)\frac{d}{dx}(coshx)}{\cosh^2 x}$$

$$=\frac{coshx(0)-(1)(sinhx)}{\cosh^2 x}$$

$$=\frac{-sinhx}{coshx.coshx}=-tanhx.sechx$$

$$\frac{d}{dx}(sechx)=-sechxtanhx$$

6.

$$\frac{d}{dx}(cothx)=-cosech^2x$$

Proof:

$$we\ konw\ that\ cothx=\frac{1}{tanhx}$$

$$\Rightarrow \frac{d}{dx}(cothx)=\frac{d}{dx}\left(\frac{1}{tanhx}\right)$$

$$=\frac{tanhx\frac{d}{dx}(1)-(1)\frac{d}{dx}(tanhx)}{\tanh^2 x}$$

$$=\frac{tanhx(o)-(1)(\operatorname{sech}^2 x)}{\tanh^2 x}$$

$$= \frac{-\operatorname{sech}^2 x}{\frac{\sinh^2 x}{\cosh^2 x}} = \frac{-\operatorname{sech}^2 x}{\sinh^2 x}\cosh^2 x$$

$$= \frac{-\operatorname{sech}^2 x}{\sinh^2 x} \times \frac{1}{\operatorname{sech}^2 x} = -\frac{1}{\sinh^2 x}$$

Hence $\frac{d}{dx}(cothx) = -cosech^2 x$

Derivative of inverse hyperbolic function:

The inverse hyperbolic functions are

1. $y = \sinh^{-1} x \quad iff \quad x = sinhy \quad ; x, y \in R$
2. $y = \cosh^{-1} x \quad iff\ x = coshy; \quad x \in [1, \infty), y \in [0, \infty)$
3. $y = \tanh^{-1} x \quad iff\ x = tanhy\ ; x \in (-1,1), y \in R$
4. $y = \coth^{-1} x \quad iff\ x = cothy \ ; \ x \in [-1,1]', y \in R - \{0\}$
5. $y = \operatorname{sech}^{-1} x \ \ iff\ x = sechy; \ \ x \in (0,1], y \in [0, \infty)$
6. $y = cosech^{-1}x \ \ iff \ \ x = cosechy\ ; x \in R - \{0\}, y \in R - \{0\}$

Prove that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

Proof:

$$Let\ y = \sinh^{-1} x \Rightarrow x = sinhy$$

$$\Rightarrow x = \frac{e^y - e^{-y}}{2} \qquad \because sinhx = \frac{e^x - e^{-x}}{2}$$

$$\Rightarrow 2x = e^y - e^{-y} = e^y - \frac{1}{e^y} = \frac{(e^y)^2 - 1}{e^y}$$

$$\Rightarrow 2xe^y = e^{2y} - 1 \Rightarrow e^{2y} - 2xe^y - 1 = 0$$

$$\Rightarrow (e^y)^2 - 2xe^y - 1 = 0$$

$$\Rightarrow e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)} \quad using\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2x \pm \sqrt{4x^2 + 4}}{2} = \frac{2 \pm \sqrt{4(x^2 + 1)}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2 + 1}}{2} = 2\left(\frac{x \pm \sqrt{x^2 + 1}}{2}\right)$$

$$e^y = x \pm \sqrt{x^2 + 1}$$

$$\Rightarrow e^y = x + \sqrt{x^2 + 1} \quad , e^y = x - \sqrt{x^2 + 1}$$

As e^y is + ve for $y \in R$, so (rejected being − ve for $x \in R$)

$$\Rightarrow e^y = x + \sqrt{x^2 + 1}\ ,$$

$$\Rightarrow y = \ln(x + \sqrt{x^2 + 1})$$

Hence $\boldsymbol{\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)}$

Note:
$\because e^y = x$
$lne^y = lnx$
$ylne = lnx$
$y(1) = lnx$
$y = lnx$

Prove that $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$

Proof:

$$Let\ y = \cosh^{-1} x \Rightarrow x = coshy$$

$$\Rightarrow x = \frac{e^y + e^{-y}}{2} \qquad \because coshx = \frac{e^x + e^{-x}}{2}$$

$$\Rightarrow 2x = e^y + e^{-y} = e^y + \frac{1}{e^y} = \frac{(e^y)^2 + 1}{e^y}$$

$$\Rightarrow 2xe^y = e^{2y} + 1 \Rightarrow e^{2y} - 2xe^y + 1 = 0$$

$$\Rightarrow (e^y)^2 - 2xe^y + 1 = 0$$

$$\Rightarrow e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)} \quad using\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2x \pm \sqrt{4x^2 - 4}}{2} = \frac{2 \pm \sqrt{4(x^2 - 1)}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2 - 1}}{2} = 2\left(\frac{x \pm \sqrt{x^2 - 1}}{2}\right)$$

$$e^y = x \pm \sqrt{x^2 - 1}$$

$$\Rightarrow e^y = x + \sqrt{x^2 - 1} \quad , e^y = x - \sqrt{x^2 - 1}$$

As e^y is + ve for $y \in R$, so (rejected being − ve for $x > 1$

$$\Rightarrow e^y = x + \sqrt{x^2 - 1}\ ,$$

$$\Rightarrow y = \ln(x + \sqrt{x^2 - 1})$$

Hence $\boldsymbol{\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)}$

Note:
$\because e^y = x$
$lne^y = lnx$
$ylne = lnx$
$y(1) = lnx$
$y = lnx$

1. $\boldsymbol{\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}} \qquad \boldsymbol{x, y \in R}$

Proof:

$$Let\ y = \sinh^{-1} x$$

$$\Rightarrow \frac{d}{dx}(x) = \frac{d}{dx}(sinhy) = coshy\frac{dy}{dx}$$

$$\Rightarrow \cosh\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{coshy}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{coh^2 y}} = \frac{1}{\sqrt{1 + \sinh^2 y}}$$

$$\because \cosh^2 x - \sinh^2 x = 1$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\cosh^2 y}} = \frac{1}{\sqrt{1 + \sin^2 y}}$$

$$(\because sinhy = x)$$

2. $\boldsymbol{\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}} \qquad \boldsymbol{x \in [1, \infty), y \in [o, \infty)}$

Proof:

$$Let\ y = \cosh^{-1} x$$

$$\Rightarrow \frac{d}{dx}(x) = \frac{d}{dx}(coshy) = sinhy\frac{dy}{dx}$$

$$\Rightarrow \sinh\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{sinhy}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{sinh^2 y}} = \frac{1}{\sqrt{cosh^2 y - 1}}$$

$$\because \cosh^2 x - \sinh^2 x = 1$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

3.

$$\boldsymbol{\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2}}$$

Proof:

$$Let\ \ y=\tanh^{-1}x\ ;x\in(-1,1),y\in R$$
$$\Rightarrow tanhy=x$$
$$\Rightarrow \frac{d}{dx}(tany)=\frac{d}{dx}(x)$$
$$\Rightarrow \operatorname{sech}^2 y\frac{dy}{dx}=1$$
$$\frac{dy}{dx}=\frac{1}{\operatorname{sech}^2 y}=\frac{1}{1-\tanh^2 y}$$
$$\frac{dy}{dx}=\frac{1}{1-x^2}\qquad (\because \operatorname{sech}^2 x+\tanh^2 x=1)$$

4.

$$\frac{d}{dx}(\coth^{-1}x)=\frac{1}{1-x^2}$$

Proof:

$$Let\ \ y=\coth^{-1}x\ ;\ \ |x|>1$$
$$\Rightarrow cothy=x$$
$$\Rightarrow \frac{d}{dx}(tany)=\frac{d}{dx}(x)$$
$$\Rightarrow -cosech^2 y\frac{dy}{dx}=1$$
$$\frac{dy}{dx}=\frac{1}{co\operatorname{sech}^2 y}=\frac{1}{1-\coth^2 y}$$
$$\frac{dy}{dx}=\frac{1}{1-x^2}\qquad (\because -co\operatorname{sech}^2 x+\coth^2 x=1)$$

5. $\frac{d}{dx}(\operatorname{sech}^{-1}x)=-\frac{1}{x\sqrt{1-x^2}}\ \ ,0<x<1$

Proof:

$$Let\ \ y=\operatorname{sech}^{-1}x$$
$$\Rightarrow sechy=x\Rightarrow \frac{d}{dx}(sechy)=\frac{d}{dx}(x)$$
$$\Rightarrow -sechytanhy\frac{dy}{dx}=1$$
$$\Rightarrow \frac{dy}{dx}=\frac{-1}{sechytanhy}=\frac{-1}{sechy\sqrt{\tanh^2 y}}$$
$$\frac{dy}{dx}=\frac{-1}{sechy\sqrt{1-\operatorname{sech}^2 y}}\ (\because \operatorname{sech}^2 y+\tanh^2 y=1)$$

$$\frac{d}{dx}(\operatorname{sech}^{-1}x)=-\frac{1}{x\sqrt{1-x^2}}$$

6. $\frac{d}{dx}(\operatorname{cosech}^{-1}x)=-\frac{1}{x\sqrt{1+x^2}}\ \ ,x\in R-\{0\}$

Proof:

$$Let\ \ y=co\operatorname{sech}^{-1}x$$
$$\Rightarrow cosechy=x\Rightarrow \frac{d}{dx}(cosechy)=\frac{d}{dx}(x)$$
$$\Rightarrow -cosechycothy\frac{dy}{dx}=1$$
$$\Rightarrow \frac{dy}{dx}=\frac{-1}{cosechycothy}=\frac{-1}{cosechy\sqrt{\coth^2 y}}$$
$$\begin{pmatrix}\because \coth^2 x-cosec^2x=1\\ \coth^2 x=1+cosech^2x\end{pmatrix}$$
$$\frac{dy}{dx}=-\frac{1}{cosechy\sqrt{1+cosech^2x}}=\frac{-1}{x\sqrt{1+x^2}}$$

Exercise 2.6

Question # 1) Find $f'(x)$ if

i) $f(x)=e^{\sqrt{x}-1}$

Solution:

Let $f(x)=e^{\sqrt{x}-1}$

Differentiate w.r.t x

$$\frac{d(f(x))}{dx}=\frac{d}{dx}\left(e^{\sqrt{x}-1}\right)$$
$$f'(x)=e^{\sqrt{x}-1}\frac{d}{dx}(\sqrt{x}-1)$$
$$\frac{dy}{dx}=\frac{e^{\sqrt{x}-1}}{2\sqrt{x}}$$

Which is required.

ii) $f(x)=x^3e^{\frac{1}{x}}$

Solution:

Let $f(x)=x^3e^{\frac{1}{x}}$

Differentiate w.r.t x

$$\frac{d(f(x))}{dx}=\frac{d}{dx}\left(x^3e^{\frac{1}{x}}\right)$$
$$f'(x)=e^{\frac{1}{x}}\frac{d}{dx}(x^3)+x^3\frac{d}{dx}\left(e^{\frac{1}{x}}\right)$$
$$f'(x)=e^{\frac{1}{x}}(3x^2)+x^3\left(e^{\frac{1}{x}}\right)\frac{d}{dx}\left(\frac{1}{x}\right)$$
$$f'(x)=e^{\frac{1}{x}}(3x^2)+x^3\left(e^{\frac{1}{x}}\right)\frac{d}{dx}\left(\frac{1}{x}\right)$$
$$f'(x)=e^{\frac{1}{x}}(3x^2)+x^3\left(e^{\frac{1}{x}}\right)\left(\frac{-1}{x^2}\right)$$
$$f'(x)=xe^{\frac{1}{x}}(3x-1)$$

Which is required.

iii) $f(x)=e^x(1+lnx)$

Solution:

Let $f(x)=e^x(1+lnx)$

Differentiate w.r.t x

$$\frac{d(f(x))}{dx}=\frac{d}{dx}(e^x(1+lnx))$$
$$f'(x)=e^x\frac{d}{dx}(1+lnx)+(1+lnx)\frac{d}{dx}(e^x)$$
$$f'(x)=e^x(0+\frac{1}{x})+(1+lnx)(e^x)$$
$$f'(x)=\frac{e^x+x(1+lnx)(e^x)}{x}$$
$$f'(x)=\frac{(1+x(1+lnx))e^x}{x}$$

Which is required.

iv) $f(x)=\frac{e^x}{e^{-x}+1}$

Solution.

Let $f(x)=\frac{e^x}{e^{-x}+1}$

Differentiate w.r.t x

$$\frac{d(f(x))}{dx}=\frac{d}{dx}\left(\frac{e^x}{e^{-x}+1}\right)$$
$$f'(x)=\frac{(e^{-x}+1)\frac{d}{dx}(e^x)-(e^x)\frac{d}{dx}(e^{-x}+1)}{(e^{-x}+1)^2}$$

$$f'(x)=\frac{(e^{-x}+1)(e^{x})-(e^{x})(e^{-x}(-1))}{(e^{-x}+1)^2}$$
$$f'(x)=\frac{(1+e^{x})-(-1)}{(e^{-x}+1)^2}$$
$$f'(x)=\frac{2+e^{x}}{(e^{-x}+1)^2}$$
Which is required.

v) $\boldsymbol{f(x)=ln(e^{x}+e^{-x})}$

Solution:

Let $f(x)=ln(e^{x}+e^{-x})$

Differentiate w.r.t x

$$\frac{d(f(x))}{dx}=\frac{d}{dx}(ln(e^{x}+e^{-x}))$$
$$f'(x)=\frac{1}{e^{x}+e^{-x}}\frac{d}{dx}(e^{x}+e^{-x})$$
$$f'(x)=\frac{1}{e^{x}+e^{-x}}(e^{x}+e^{-x}(-1))$$
$$f'(x)=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$
$$f'(x)=\frac{e^{x}-\frac{1}{e^{x}}}{e^{x}+\frac{1}{e^{x}}}$$
$$f'(x)=\frac{e^{2x}-1}{e^{2x}+1}$$
Which is required.

vi) $\boldsymbol{f(x)=\frac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}}}$

Solution.

Let $f(x)=\frac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}}$

Differentiate w.r.t x

$$\frac{d(f(x))}{dx}=\frac{d}{dx}\left(\frac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}}\right)$$
$$f'(x)=\frac{(e^{ax}+e^{-ax})\frac{d}{dx}(e^{ax}-e^{-ax})-(e^{ax}-e^{-ax})\frac{d}{dx}(e^{ax}+e^{-ax})}{(e^{ax}+e^{-ax})^2}$$
$$f'(x)=\frac{(e^{ax}+e^{-ax})(e^{ax}(a)-e^{-ax}(-a))-(e^{ax}-e^{-ax})(e^{ax}(a)+e^{-ax}(-))}{(e^{ax}+e^{-ax})^2}$$
$$f'(x)=\frac{a[(e^{ax}+e^{-ax})^2-(e^{ax}-e^{-ax})^2]}{(e^{ax}+e^{-ax})^2}$$
$$f'(x)=\frac{a[(e^{ax})^2+(e^{-ax})^2+2(e^{ax})(e^{-ax})-(e^{ax})^2-(e^{-ax})^2-2(e^{ax})(e^{-ax})]}{(e^{ax}+e^{-ax})^2}$$
$$f'(x)=\frac{a[4e^{0}]}{(e^{ax}+e^{-ax})^2}$$
$$f'(x)=\frac{4a}{(e^{ax}+e^{-ax})^2}$$
Which is required.

vii) $\boldsymbol{f(x)=\sqrt{ln(e^{2x}+e^{-2x})}}$

Solution:

Let $f(x)=\sqrt{ln(e^{2x}+e^{-2x})}$

Differentiate w.r.t x

$$\frac{d(f(x))}{dx}=\frac{d}{dx}\left(\sqrt{ln(e^{2x}+e^{-2x})}\right)$$
$$f'(x)=\frac{1}{2\sqrt{ln(e^{2x}+e^{-2x})}}\frac{d}{dx}(ln(e^{2x}+e^{-2x}))$$
$$f'(x)=\frac{1}{2\sqrt{ln(e^{2x}+e^{-2x})}}[\frac{1}{e^{2x}+e^{-2x}}\frac{d}{dx}(e^{2x}+e^{-2x})]$$
$$f'(x)=\frac{2(e^{2x}-e^{-2x})}{2\sqrt{ln(e^{x}+e^{-x})}(e^{2x}+e^{-2x})}$$
$$f'(x)=\frac{(e^{2x}-e^{-2x})}{\sqrt{ln(e^{x}+e^{-x})}(e^{2x}+e^{-2x})}$$
Which is required.

viii) $\boldsymbol{f(x)=ln\sqrt{(e^{2x}+e^{-2x})}}$

Solution:

Let $f(x)=ln\sqrt{(e^{2x}+e^{-2x})}=\frac{1}{2}(\ln(e^{2x}+e^{-2x}))$

Differentiate w.r.t x

$$\frac{d(f(x))}{dx}=\frac{1}{2}\frac{d}{dx}(\ln(e^{2x}+e^{-2x}))$$
$$f'(x)=\frac{2e^{2x}-2e^{-2x}}{2(e^{2x}+e^{-2x})}$$
$$f'(x)=\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}]$$
$$f'(x)=tanh2x$$

Question # 2. Find $\frac{dy}{dx}$ if

Solution.

i) $y=x^2\ln\sqrt{x}$

Let $y=x^2\ln\sqrt{x}=\frac{1}{2}x^2lnx$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{2}x^2lnx\right)$$
$$\frac{dy}{dx}=\frac{1}{2}\left[(x^2)\frac{d}{dx}(lnx)+(lnx)\frac{d}{dx}(x^2)\right]$$
$$\frac{dy}{dx}=\frac{1}{2}\left[x^2\left(\frac{1}{x}\right)+lnx\,(2x)\right]$$

ii) $\boldsymbol{y=x\sqrt{lnx}}$

Solution.

Let $y=x\sqrt{lnx}$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}(x\sqrt{lnx})$$
$$\frac{dy}{dx}=(x)\frac{d}{dx}\left(\sqrt{lnx}\right)+\left(\sqrt{lnx}\right)\frac{d}{dx}(x)$$
$$\frac{dy}{dx}=x\frac{1}{2\sqrt{lnx}}\frac{d}{dx}(lnx)+\sqrt{lnx}\,(1)$$
$$\frac{dy}{dx}=x\frac{1}{2\sqrt{lnx}}\left(\frac{1}{x}\right)+\sqrt{lnx}\,(1)$$
$$\frac{dy}{dx}=\frac{1+2\left(\sqrt{lnx}\right)^2}{2\sqrt{lnx}}$$
$$\frac{dy}{dx}=\frac{1+2lnx}{2\sqrt{lnx}}$$
Which is required.

iii) $\boldsymbol{y=\frac{x}{lnx}}$

Solution.

Let $y=\frac{x}{lnx}$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x}{lnx}\right)$$

$$\frac{dy}{dx}=\left(\frac{(lnx)\frac{d}{dx}(x)-(x)\frac{d}{dx}(lnx)}{(lnx)^2}\right)$$

$$\frac{dy}{dx}=\left(\frac{(lnx)(1)-(x)\left(\frac{1}{x}\right)}{(lnx)^2}\right)$$

$$\frac{dy}{dx}=\frac{lnx-1}{(lnx)^2}$$

iv) $\boldsymbol{y=x^2\ln\left(\frac{1}{x}\right)}$

Solution.

Let $y=\ x^2\ln\left(\frac{1}{x}\right)=x^2\ln(x^{-1})=-x^2\ln(x)$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\ (-x^2\ln(x))$$

$$\frac{dy}{dx}=-\left[(x^2)\frac{d}{dx}(lnx)+(lnx)\frac{d}{dx}(x^2)\right]$$

$$\frac{dy}{dx}=-\left[(x^2)\left(\frac{1}{x}\right)+(lnx)(2x)\right]$$

$$\frac{dy}{dx}=-[x+2xlnx]$$

$$\frac{dy}{dx}=-x(1+2lnx)$$

Which is required.

v) $\boldsymbol{y=ln\sqrt{\frac{x^2-1}{x^2-1}}}$

Solution.

Let $y=\ \ ln\sqrt{\frac{x^2-1}{x^2+1}}=\frac{1}{2}\ln\left(\frac{x^2-1}{x^2+1}\right)$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{2}\ln\left(\frac{x^2-1}{x^2+1}\right)\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{\frac{x^2-1}{x^2+1}}\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)\right]$$

$$\frac{dy}{dx}=\frac{1(x^2+1)}{2(x^2-1)}\left(\frac{(x^2+1)\frac{d}{dx}(x^2-1)-(x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}\right)$$

$$\frac{dy}{dx}=\left(\frac{(x^2+1)(2x)-(x^2-1)(2x)}{2(x^2-1)(x^2+1)}\right)$$

$$\frac{dy}{dx}=\left(\frac{2x(x^2+1-x^2+1}{2(x^2-1)(x^2+1)}\right)$$

$$\frac{dy}{dx}=\left(\frac{2x}{(x^4-1)}\right)$$

Which is required.

vi) $\boldsymbol{y=\ln(x+\sqrt{x^2+1})}$

Solution.

Let $y=\ \ln(x+\sqrt{x^2+1})$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right)$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{x^2+1}}$$

Which is required.

vii) $\boldsymbol{y=\ln(9-x^2)}$

Solution.

Let $y=\ \ln(9-x^2)$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\ (\ln(9-x^2)$$

$$\frac{dy}{dx}=\frac{1}{9-x^2}\frac{d}{dx}(9-x^2)$$

$$\frac{dy}{dx}=\frac{1}{9-x^2}\ \ (-2x)$$

$$\frac{dy}{dx}=\frac{-2x}{9-x^2}$$

Which is required.

viii) $\boldsymbol{y=e^{-2x}.sin2x}$

Solution.

Let $y=\ \ e^{-2x}.sin2x$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\ (e^{-2x}.sin2x)$$

$$\frac{dy}{dx}=(e^{-2x})(cos2x\ (2))+(sin2x)(e^{-2x}(-2))$$

$$\frac{dy}{dx}=2e^{-2x}(cos2x-sin2x)$$

Which is required.

ix) $\boldsymbol{y=e^{-x}(x^3+2x^2+1\,)\,.}$

Solution.

Since $y=e^{-x}(x^3+2x^2+1\,)$

Differentiating w.r.t "x"

$$\frac{dy}{dx}=(x^3+2x^2+1\,)\frac{d}{dx}(e^{-x})+e^{-x}\frac{d}{dx}(x^3+2x^2+1\,)$$

$$\frac{dy}{dx}=(x^3+2x^2+1\,)(e^{-x}(-1))+e^{-x}(3x^2+4x\,)$$

$$\frac{dy}{dx}=e^{-x}\,(-x^3-2x^2-1+3x^2+4x)$$

$$\frac{dy}{dx}=-e^{-x}\,(x^3-x^2-4x+1)$$

x) $\boldsymbol{y=xe^{sinx}}$

Solution.

Since $y=xe^{sinx}$

Differentiate w.r.t."x"

$$\frac{dy}{dx}=\frac{d}{dx}(xe^{sinx})$$

$$\frac{dy}{dx}=(e^{sinx})\frac{d}{dx}(x)+(x)\frac{d}{dx}(e^{sinx})$$

$$\frac{dy}{dx}=(e^{sinx})(1)+(x)(e^{sinx}(cosx))$$

$$\frac{dy}{dx}=e^{sinx}(1+xcosx)$$

Which is required.

xi) $\boldsymbol{y=5e^{3x-4}}$

Solution.

Let $y=5e^{3x-4}$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\ (5e^{3x-4})$$

$$\frac{dy}{dx} = 5e^{3x-4}\frac{d}{dx}(3x-4)$$
$$\frac{dy}{dx} = 5e^{3x-4}\ (3)$$
$$\frac{dy}{dx} = 15e^{3x-4}$$
Which is required.

xii) $\boldsymbol{y = (x+1)^x}$

Solution.

Let $y = (x+1)^x$

Differentiate w.r.t x

Taking ln on both sides

$$lny = ln(x+1)^x$$
$$lny = xln(x+1)$$
$$\frac{1}{y}\frac{dy}{dx} = (\ln(x+1))\frac{d}{dx}(x) + x\frac{d}{dx}(ln(x+1))$$
$$\frac{1}{y}\frac{dy}{dx} = (\ln(x+1))(1) + (x)\frac{1}{(x+1)}\left(\frac{1}{x}\right)$$
$$\frac{dy}{dx} = y\left(\frac{\ln(x+1)+1}{x+1}\right)$$
$$\frac{dy}{dx} = (x+1)^x\left(\frac{\ln(x+1)+1}{x+1}\right)$$
$$\frac{dy}{dx} = (x+1)^{x-1}(\ln(x+1)+1)$$
Which is required.

xiii) $\boldsymbol{y = (lnx)^{lnx}}$

Solution.

Let $y = (lnx)^{lnx}$

Differentiate w.r.t x

Taking ln on both sides

$$lny = ln(lnx)^{lnx}$$
$$lny = lnx.\ln(lnx)$$
$$\frac{1}{y}\frac{dy}{dx} = (\ln(lnx))\frac{d}{dx}(lnx) + (lnx)\frac{d}{dx}(ln(lnx))$$
$$\frac{1}{y}\frac{dy}{dx} = (\ln(lnx))\left(\frac{1}{x}\right) + (lnx)\frac{1}{(lnx)}\left(\frac{1}{x}\right)$$
$$\frac{dy}{dx} = y\left(\frac{\ln(lnx)+1}{x}\right)$$
$$\frac{dy}{dx} = (lnx)^x\left(\frac{\ln(x+1)+1}{x}\right)$$
$$\frac{dy}{dx} = \frac{(x+1)^x}{x}(\ln(lnx)+1)$$

Q#3) Find $\frac{dy}{dx}$ if:

(i) $\boldsymbol{y = \cosh 2x}$

Solution.

Let $y = \cosh 2x$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\cosh 2x)$$
$$\frac{dy}{dx} = sinh2x\frac{d}{dx}(2x)$$
$$\frac{dy}{dx} = sinh2x\ (2)$$
$$\frac{dy}{dx} = 2sinh2x$$
Which is required.

(ii) $\boldsymbol{y = \sinh 3x}$

Let $y = \cosh 2x$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\sinh 3x)$$
$$\frac{dy}{dx} = cosh2x\frac{d}{dx}(3x)$$
$$\frac{dy}{dx} = cosh2x\ (3)$$
$$\frac{dy}{dx} = 3cosh3x$$

(iii) $\boldsymbol{y = \tanh^{-1}(sinx), -\frac{\pi}{2} < x < \frac{\pi}{2}}$

Sol:

Let $y = \tanh^{-1}(sinx)$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\tanh^{-1}(sinx),)$$
$$\frac{dy}{dx} = \frac{1}{1-(sinx)^2}\frac{d}{dx}(sinx)$$
$$\frac{dy}{dx} = \frac{1}{1-\sin^2 x}(cosx)$$
$$\frac{dy}{dx} = \frac{cosx}{\cos^2 x} = secx$$

(iv) $\boldsymbol{y = \sinh^{-1}(x^3)}$

Let $y = \sinh^{-1}(x^3)$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\sinh^{-1}(x^3))$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^3)^2}}\frac{d}{dx}(x^3)$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^6}}(3x^2)$$
$$\frac{dy}{dx} = \frac{3x^2}{\sqrt{1-x^6}}$$

(v) $\boldsymbol{y = \ln(\tanh x)}$

Let $y = \ln(\tanh x)$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\ln(\tanh x))$$
$$\frac{dy}{dx} = \frac{1}{tanhx}\frac{d}{dx}(tanhx)$$
$$\frac{dy}{dx} = \frac{coshx}{sinhx}(\text{sech}^2 x)$$
$$\frac{dy}{dx} = \frac{coshx}{sinhx\cosh^2 x}$$
$$\frac{dy}{dx} = \frac{1}{sinhx\ coshx}$$

(vi) $\boldsymbol{y = \sinh^{-1}\left(\frac{x}{2}\right)}$

Sol:

Let $y = \sinh^{-1}\left(\frac{x}{2}\right)$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\sinh^{-1}\left(\frac{x}{2}\right))$$

$$\frac{dy}{dx}=\frac{1}{1-(sinx)^2}\frac{d}{dx}(sinx)$$
$$\frac{dy}{dx}=\frac{1}{\sqrt{1+\left(\frac{x}{2}\right)^2}}\frac{d}{dx}\left(\frac{x}{2}\right)$$
$$\frac{dy}{dx}=\frac{1}{\sqrt{1+\frac{x^2}{4}}}\left(\frac{1}{2}\right)$$
$$\frac{dy}{dx}=\frac{1}{\sqrt{\frac{4+x^2}{4}}}\left(\frac{1}{2}\right)$$
$$\frac{dy}{dx}=\frac{2}{\sqrt{4+x^2}}\left(\frac{1}{2}\right)$$
$$\frac{dy}{dx}=\frac{1}{\sqrt{4+x^2}}$$

Successive differentiation (or higher derivative)

Let $y = f(x)$ be a function, then its successive or higher are given below.

1st derivative	2nd derivative	3rd derivative
y'	y''	y'''
$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$
y_1	y_2	y_3
D_y	D_y^2	D_y^3
$f'(x)$	$f''(x)$	$f'''(x)$
$\frac{df}{dx}$	$\frac{d^2y}{d^2x}$	$\frac{d^3y}{dx^3}$

Exercise 2.7

Question # 1

Find y_2 if:

(i) $\boldsymbol{y = 2x^5 - 3x^4 + 4x^3 + x - 2}$

Solution.

Let $y = 2x^5 - 3x^4 + 4x^3 + x - 2$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}(2x^5 - 3x^4 + 4x^3 + x - 2)$$
$$y_1 = 10x^4 - 12x^3 + 12x^2 + 1$$

Again differentiate w.r.t x

$$y_2=\frac{d}{dx}(10x^4 - 12x^3 + 12x^2 + 1)$$
$$y_2 = 40x^3 - 36x^2 + 24x$$

Which is required.

(ii) $\boldsymbol{y = (2x+5)^{\frac{3}{2}}}$

Solution.

$$\frac{dy}{dx}=\frac{d}{dx}\,(2x+5)^{\frac{3}{2}}$$
$$y_1=\frac{3}{2}(2x+5)^{\frac{1}{2}}(2)$$
$$y_1 = 3(2x+5)^{\frac{1}{2}} = 3\sqrt{2x+5}$$

Again differentiate w.r.t x

$$y_2 = 3\frac{d}{dx}\left(\sqrt{2x+5}\right)$$
$$y_2=\frac{3}{2\sqrt{2x+5}}(2)$$
$$y_2=\frac{3}{\sqrt{2x+5}}$$

Which is required.

(iii) $\boldsymbol{y=\sqrt{x}-\frac{1}{\sqrt{x}}}$

Sol: Since $y=\sqrt{x}-\frac{1}{\sqrt{x}}$

$$y = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$$

Differentiating w.r.t "x"

$$y_1=\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{-\frac{3}{2}}$$

Again differentiate w.r.t x

$$y_2=\frac{d}{dx}\left(\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{-\frac{3}{2}}\right)$$
$$y_2=\frac{-1}{4}x^{-\frac{3}{2}}+\frac{-3}{4}x^{-\frac{5}{2}}$$
$$y_2=\frac{-1}{4}x^{-\frac{3}{2}}+\frac{-3}{4}x^{-\frac{5}{2}}$$

Which is required.

Question # 2 Find y_2 if:

(i) $\boldsymbol{y = x^2.e^{-x}}$

Solution.

Let $y = x^2.e^{-x}$

Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}(x^2.e^{-x})$$
$$y_1 = (x^2)(e^{-x}(-1)) + (e^{-x})(2x)$$
$$y_1 = e^{-x}(2x - x^2)$$

Again differentiate w.r.t x

$$y_2=\frac{d}{dx}(e^{-x}(2x-x^2))$$
$$y_2 = ((2x-x^2))(e^{-x}(-1)) + (e^{-x})(2-2x)$$
$$y_2 = e^{-x}(-2x + x^2 + 2 - 2x)$$
$$y_2 = e^{-x}(x^2 - 4x + 2)$$

Which is required.

(ii) $y=\ln\left(\frac{2x+3}{3x+2}\right)$

$$\Rightarrow \frac{dy}{dx}=y_1=\frac{d}{dx}.\ln\left(\frac{2x+3}{3x+2}\right)$$
$$=\frac{d}{dx}[\ln(2x+3)-\ln(3x+2)]$$
$$y_1 = 2(2x+3)^{-1} - 3(3x+2)^{-1}$$
$$\Rightarrow \frac{d^2y}{dx^2}=y_2=2\frac{d}{dx}(2x+3)^{-1}-3\frac{d}{dx}(3x+2)^{-1}$$
$$=2(-1)(2x+3)^{-1}\frac{d}{dx}(2x+3)-3(-1)(3x+2)^{-2}\frac{d}{dx}(3x+2)$$
$$=-2(2x+3)^{-2}(2)+3(3x+2)^{-2})(3)$$
$$y_2=-\frac{4}{(2x+5)^2}+\frac{9}{(3x+2)^2}$$
$$=\frac{-4(3x+2)^2+9(2x+3)^2}{(2x+5)^2(3x+2)^2}$$
$$=\frac{-4(9x^2+4+12x)+9(4x^2+3+12x)}{(2x+5)^2(3x+2)^2}$$

$$= \frac{-36x^2 - 16 - 48x + 36x62 + 81 + 108x}{(2x+5)^2(3x+2)^2}$$

$$y_2 = \frac{60x + 65}{(2x+5)^2(3x+2)^2}$$

Q#3) Find y_2 if:

(i) $x^2 + y^2 = a^2$

Sol:

Differentiate w.r.t x

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(a^2)$$

$$2x + 2yy_1 = 0$$

$$2yy_1 = -2x$$

$$y_1 = -\frac{x}{y}$$

Differentiating again w.r.t "x"

$$\frac{dy_1}{dx} = -\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\frac{dy_1}{dx} = -\frac{(y)\frac{d}{dx}(x) - (x)\frac{d}{dx}(y)}{(y)^2}$$

$$y_2 = -\left(\frac{(y)(1) - (x)(y_1)}{(y)^2}\right)$$

Put value of "y_1"

$$y_2 = -\left(\frac{(y) - (x)\left(-\frac{x}{y}\right)}{(y)^2}\right)$$

$$y_2 = -\left(\frac{\left(\frac{y^2+x^2}{y}\right)}{(y)^2}\right)$$

$$y_2 = -\left(\frac{y^2+x^2}{y^3}\right)$$

$$y_2 = \frac{-a^2}{y^3}$$

(ii) $x^3 - y^3 = a^3$

Sol:

Differentiate w.r.t x

$$\frac{d}{dx}(x^3 - y^3) = \frac{d}{dx}(a^3)$$

$$3x^2 - 3y^2y_1 = 0$$

$$3y^2y_1 = 3x^2$$

$$y_1 = \frac{x^2}{y^2}$$

Differentiating again w.r.t "x"

$$\frac{dy_1}{dx} = \frac{d}{dx}\left(\frac{x^2}{y^2}\right)$$

$$\frac{dy_1}{dx} = \frac{(y^2)\frac{d}{dx}(x^2) - (x^2)\frac{d}{dx}(y^2)}{(y^2)^2}$$

$$y_2 = \frac{(y^2)(2x) - (x^2)(2yy_1)}{y^4}$$

Put value of "y_1"

$$y_2 = \frac{(y^2)(2x) - (x^2)\left(2y\left(\frac{x^2}{y^2}\right)\right)}{y^4}$$

$$y_2 = \frac{(2xy^2) - \left(\frac{2x^4}{y}\right)}{y^4}$$

$$y_2 = \frac{\left(\frac{2y^2x^2 - 2x^4}{y}\right)}{y^4}$$

$$y_2 = \frac{2x(y^3 - x^3)}{y^5}$$

$$= \frac{-2x}{y^5}(x^3 - y^3)$$

$$\frac{d^2y}{dx^2} = -\frac{2x}{y^5}(a^3) \qquad \because x^3 - y^3 = a^3$$

$$y_2 = -\frac{2xa^3}{y^5}$$

(iii) $x = acos\theta \; y = asin\theta$

Sol:

$$x = acos\theta \,, y = asin\theta$$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(acos\theta) = a(-sib\theta) = -asin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(asin\theta) = acos\theta$$

By chain rule, we have

$$\frac{dy}{dx} = \frac{d}{d\theta}\cdot\frac{d\theta}{dx} = (acos\theta)\left(\frac{1}{-asin\theta}\right)$$

$$\frac{dy}{dx} = -\frac{cox\theta}{sin\theta} = -cot\theta \Rightarrow y_1 = -cot\theta$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(-cot\theta) = -(-cosec^2\theta)\frac{d\theta}{dx}$$

$$= cosec^2\theta\left(\frac{1}{-asin\theta}\right) = \frac{1}{sin^2\theta}\left(\frac{1}{-asin\theta}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-1}{asin^3\theta}$$

$$y_2 = \frac{-1}{a\sin^3\theta}$$

Which is required

(iv) $x = at^2 \quad y = bt^4$

Sol:

$$x = at^2 \Rightarrow t^2 = \frac{x}{a}$$

$$y = bt^4 \Rightarrow y = b\left(\frac{x}{a}\right)^2$$

$$y = \frac{bx^2}{a^2}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{b}{a^2}\frac{d}{dx}(x^2)$$

$$y_1 = \frac{b}{a^2}(2x)$$

Again differentiate w.r.t x

$$y_2 = \frac{2b}{a^2}\frac{d}{dx}(x)$$

$y_2=\frac{2b}{a^2}$

(v) $x^2+y^2+2gx+2fy+c=0$

Sol: differentiate w.r.t "x"

$\frac{d}{dx}(x^2+y^2+2gx+2fy+c)=\frac{d}{dx}(0)$

$2x+2yy_1+2g+2fy_1=0$

$yy_1+fy_1=-x-g$

$y_1(y+f)=-(x+g)$

$y_1=-\frac{x+g}{y+f}$

Differentiating again w.r.t "x"

$\frac{dy_1}{dx}=-\frac{d}{dx}\left(\frac{x+g}{y+1}\right)$

$\frac{dy_1}{dx}=-\frac{(y+f)\frac{d}{dx}(x+g)-(x+g)\frac{d}{dx}(y+f)}{(y+f)^2}$

$y_2=-\left(\frac{(y+f)(1)-(x+g)(y_1)}{(y+f)^2}\right)$

Put value of "y_1"

$y_2=-\left(\frac{(y+f)(1)-(x+g)\left(-\frac{x+g}{y+1}\right)}{(y+f)^2}\right)$

$y_2=-\left(\frac{(y+f)^2+(x+g)^2}{(y+f)^3}\right)$

$y_2=-\left(\frac{y^2+f^2+2fy+x^2+g^2+2gx}{(y+f)^3}\right)$

$y_2=-\left(\frac{(x^2+y^2+2gx+2fy+c)+f^2+g^2-c}{(y+f)^3}\right)$

$y_2=-\left(\frac{(0)+f^2+g^2-c}{(y+f)^3}\right)$

$y_2=\frac{-f^2-g^2+c}{(y+f)^3}$

$\frac{d^2y}{dx^2}=\frac{-f^2-g^2+c}{(y+f)^3}$

$\frac{d^2y}{dx^2}=\frac{c-f^2-g^2}{(y+f)^3}$

Q#4 Find y_4 if:

(i) $y=sin3x$

Sol: Differentiate w.r.t x

$\frac{dy}{dx}=\frac{d}{dx}(sin3x)=cos3x\frac{d}{dx}(3x)$

$y_1=cos3x\,(3)$

$y_1=3cos3x$

Again differentiate w.r.t x

$\frac{d^2y}{dx^2}=y_2=\frac{d}{dx}(3cos3x)=3(-sin3x)\frac{d}{dx}(3x)$

$3\,(-sin3x\,(3))$

$y_2=-9sin3x$

Again differentiate w.r.t x

$\frac{d^3y}{dx^3}=y_3=\frac{d}{dx}(-9sin3x)=-9(cos3x)\frac{d}{dx}(3x)$

$=-9\,cos3x\,(3)$

$y_3=-27cos3x$

Again differentiate w.r.t x

$\frac{d^4y}{dx^4}=y_4=\frac{d}{dx}(-27cos3x)$

$=-27(-sin3x)\frac{d}{dx}(3x)$

$=-27\,(-sin3x\,(3))$

$y_4=81sin3x$

Which is required

(ii) $y=\cos^3 x$

Sol: $\left(\begin{array}{l}\because cos3x=4\cos^3 x-3cosx\\ \Rightarrow cos3x+3cosx=4\cos^3 x\\ \Rightarrow \frac{1}{4}(cos3x+3cosx=\cos^2 x\end{array}\right)$

Thus

$y=\frac{1}{4}[cos3x+3cosx]$

$\frac{dy}{dx}=y_1=\frac{1}{4}\left[\frac{d}{dx}(cos3x)+\frac{d}{dx}(3cosx)\right]$

$=\frac{1}{4}[(-sin3x(3)+3(-sinx)]$

$y_1=-\frac{3}{4}(sin3x+sinx)$

$\Rightarrow\frac{d^2y}{dx^2}=-\frac{3}{4}\left(\frac{d}{dx}(sin3x)+\frac{d}{dx}(sinx)\right)$

$=-\frac{3}{4}(cos3x\,(3)+cosx)$

$\Rightarrow\frac{d^2y}{dx^2}=y_2=-\frac{3}{4}[(3cos3x+cosx)]$

$\Rightarrow\frac{d^3y}{dx^3}=y_3=-\frac{3}{4}\left[3(\frac{d}{dx}(cos3x)+\frac{d}{dx}(cosx)\right]$

$=-\frac{3}{4}[3(-sin3x)(3)+(-sinx)]$

$=-\frac{3}{4}(-9sin3x-sinx)$

$y_3=\frac{3}{4}(9sin3x+sinx)$

$\Rightarrow\frac{d^4y}{dx^4}=y_4=\frac{3}{4}\left[9\frac{d}{dx}(sin3x)+\frac{d}{dx}(sinx)\right]$

$=\frac{3}{4}[9cos3x(3)+cosx]$

$=\frac{3}{4}[27cos3x+cosx]$

$=\frac{3}{4}(27(4\cos^3 x-3cosx)+cosx)$

$=\frac{3\times27\times4}{4}\cos^3 x-\frac{3\times27\times3}{4}cosx+\frac{3}{4}cosx$

$81\cos^3 x-\frac{243}{4}cosx+\frac{3}{4}cosx$

$y_4=81\cos^3+\left(\frac{3-243}{4}\right)cosx$

$\Rightarrow y_4=81\cos^3 x-\frac{240}{4}cosx$

$y_4=-60cosx+81\cos^3 x$

$y_4=81\cos^3 x-60cosx$

(iii) $y=\ln(x^2-9)=\ln(x-3)(x+3)$

$y=\ln(x-3)+\ln(x+3)$

$\Rightarrow \frac{dy}{dx}=y_1=\frac{d}{dx}[\ln(x-3)+\ln(x+3)]$

$y_1=\frac{1}{x-3}+\frac{1}{x+3}=(x-3)^{-1}+(x+3)^{-1}$

$\frac{d^2y}{d^2x}=y_2=\frac{d}{dx}(x-3)^{-1}+\frac{d}{dx}(x+3)^{-1}$

$y_2=(-1)(x-3)^{-2}+(-1)(x+3)^{-2}$

$\frac{d^3y}{dx^3}=y_3=\frac{d}{dx}(-1)(x-3)^{-2}+(-1)(x+3)^{-2}$

$y_3=(-1)(-2)(x-3)^{-3}+(-1)(-2)(x+3)^{-3}$

$\frac{d^4y}{dx^4}=y_4=\frac{d}{dx}((-1)(-2)(x-3)^{-3}+\frac{d}{dx}(-1)(-2)(x+3)^{-3}$

$=(-1)(-2)(-3)(x-3)^{-4}+(-1)(-2)(-3)(x+3)^{-4}$

$y_4=-\frac{6}{(x-3)^4}-\frac{6}{(x+3)^4}$

$=-6\left[\frac{1}{(x-3)^4}+1/(x+3)^4\right]$

Q#5)

***if $x=sinm\theta, y=cos\theta$*, show that $(1-x^2)y_2-xy_1+m^2y=0$**

Sol: Since $x=sinm\theta \Rightarrow \theta=\frac{1}{m}\sin^{-1}(x)$

And $y=cos\theta \Rightarrow \theta=\cos^{-1}(y)$

$\Rightarrow \frac{1}{m}\sin^{-1}(x)=\cos^{-1}(y)$

Differentiate w.r.t x

$\frac{1}{m}\frac{d}{dx}(\sin^{-1}(x))=\frac{d}{dx}(\cos^{-1}(y)$

$\frac{1}{m}\frac{1}{\sqrt{1-x^2}}=\frac{-1}{\sqrt{1-y^2}}(y_1)$

$\Rightarrow m\sqrt{1-y^2}=-\sqrt{1-x^2}y_1$

On squaring

$(1-x^2)y_1^2=m^2(1-y^2)$

Differentiating again w.r.t x

$(1-x^2)(2y_1y_2)+(-2x)(y_1^2)=m^2(-2yy_1)$

Taking common $2y_1$

$2y_1[(1-x^2)y_2-xy_1]=-2m^2yy_1$

$(1-x^2)y_2-xy_1=-m^2y$

$(1-x^2)y_2-xy_1+m^2y=0$

Hence proved.

Q#6) $f\ y=e^x sinx$, show that $\frac{d^2y}{dx^2}-\frac{2dy}{dx}+2y=0$

Sol: $y=e^x sinx$

Differentiate w.r.t x

$\frac{dy}{dx}=\frac{d}{dx}(e^x sinx)$

$\frac{dy}{dx}=(e^x)(cosx)+(sinx)(e^x)$

$\frac{dy}{dx}=e^x(cosx+sinx)$

Differentiating again w.r.t x

$\frac{d^2y}{dx^2}=(e^x)(-sinx+cosx)+(cosx+sinx)(e^x)$

$\frac{d^2y}{dx^2}=e^x(-sinx+cosx+cosx+sinx)$

$\frac{d^2y}{dx^2}=e^x(2cosx)$

Now consider

$\frac{d^2y}{dx^2}-\frac{2dy}{dx}+2y$

$=\left(e^x(2cosx)\right)-2\left(e^x(cosx+sinx)\right)+2(e^x sinx)$

$=e^x(2cosx-2sinx-2cosx+sinx)$

$=e^x(0)$

$\frac{d^2y}{dx^2}-\frac{2dy}{dx}+2y=0$

Hence proved.

Q#7) *If $y=e^{ax}sinbx$* , show that $\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+(a^2+b^2)y=0$

Sol: $y=e^{ax}sinbx$

differentiate w.r.t x

$\frac{dy}{dx}=\frac{d}{dx}(e^{ax}sinbx)$

$\frac{dy}{dx}=(e^{ax})(cosbx\ (b))+(sinbx)(e^{ax}(a))$

$\frac{dy}{dx}=e^{ax}(bcosbx+asinbx)$

Differentiating again w.r.t x

$\frac{d^2y}{dx^2}=(e^{ax})(-bsinbx\ (b)+a\ cosbx\ (a))+(bcosbx+asinbx)(e^{ax}(a))$

$\frac{d^2y}{dx^2}=e^{ax}(-b^2sinbx+abcosbx+abcosbx+a^2sinbx)$

$\frac{d^2y}{dx^2}=e^{ax}(-b^2sinbx+2abcosbx+a^2sinbx)$

Now consider

$\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+(a^2+b^2)y$

$=\left(e^{ax}(-b^2sinbx+2ab\ cosbx+a^2sinbx)\right)-2a(e^{ax}(bcosbx+asinbx))+a^2+b^2(e^{ax}sinbx)$

$=e^{ax}(-b^2sinbx+2ab\ cosbx+a^2sinbx-2ab\ cosx-2a^2sinbx+a^2\ sinbx+b^2sinbx)$

$=0$

$\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+(a^2+b^2)y=0$

Hence proved.

Q#8) *If $y=\left(cos^{-1}x\right)^2$*,

prove that $(1-x^2)y_2-xy_1-2=0$

Sol:

Let $y=(\cos^{-1}x)^2$

Differentiate w.r.t x

$\frac{dy}{dx} = \frac{d}{dx}((\cos^{-1}x)^2)$

$y_1 = 2\cos^{-1}x\left(\frac{-1}{\sqrt{1-x^2}}\right)$

On squaring

$y_1^2 = 4(\cos^{-1}x)^2\left(\frac{1}{1-x^2}\right)$

$(1-x^2)y_1^2 = 4(\cos^{-1}x)^2$

$(1-x^2)y_1^2 = 4y$

Differentiating again w.r.t x

$(1-x^2)(2y_1y_2) + (-2x)(y_1^2) = 4y_1$

Taking common $2y_1$

$2y_1[(1-x^2)y_2 - xy_1] = 4y_1$

$(1-x^2)y_2 - xy_1 = 2$

$(1-x^2)y_2 - xy_1 - 2 = 0$

Hence proved.

Q#9) If $y = a\cos(lnx) + b\sin(lnx)$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Solution.

Let $y = a\cos(lnx) + b\sin(lnx)$

Differentiate w.r.t x

$\frac{dy}{dx} = \frac{d}{dx}(a\cos(lnx) + b\sin(lnx))$

$\frac{dy}{dx} = a\left(-\sin(lnx)\left(\frac{1}{x}\right)\right) + b\left(\cos(lnx)\left(\frac{1}{x}\right)\right)$

Multiply by x

$x\frac{dy}{dx} = -a\sin(lnx) + b\cos(lnx)$

Differentiating again w.r.t x

$$(x)\left(\frac{d^2y}{dx^2}\right) + (1)\left(\frac{dy}{dx}\right) = -a\left(\cos(lnx)\left(\frac{1}{x}\right)\right) + b\left(-\sin(lnx)\left(\frac{1}{x}\right)\right)$$

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\left(\frac{1}{x}\right)(a\cos(lnx) + b\sin(lnx))$

Multiply by x

$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$

$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Hence proved

McLaurin Series: *The series*

$f(x) = f(0) + xf'(o) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^n(x)$ is known as "McLaurin Series expansion**.**

Proof:

We know that power series of $f(x) in\ x$

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_nx^n \rightarrow (i)$

$\Rightarrow f(o) = a_0 \Rightarrow = f(o)$

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + na_nx^{n-1}$$

$$\Rightarrow f'(0) = a_1 \Rightarrow a_1 = f'(0)$$

$$f''(x) = 2a_2 + 6a_3x + 12a_4x^2 + \cdots + n(n-1)a_nx^{n-2}$$

$$f''(x) = 2a_2 \Rightarrow a_2 = \frac{f''(0)}{2} = \frac{f''(0)}{3!}$$

$$f'''(x) = 6a_3 + 24a_4x + \cdots + n(n-1)(n-2)a_nx^{n-3}$$

$$\Rightarrow f'''(0) = 6a_3 \Rightarrow a_3 = \frac{f'''(0)}{6} = \frac{f'''(0)}{3!}$$

Similarly, $\Rightarrow a_n = \frac{f^n(0)}{n!}$

Putting valves of $a_1, a_2, a_3, .., a_n\ in\ (1)$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0$$

NOTE:

***A function f can be expanded in Machianrium series* If the function is defined in the interval containing 0 and its derivative exist at $x = 0$ the expansion is only valid if it is converge.**

The above expansion is also named as "McLaren's theorem"

Note:

McLaren's series for $\frac{1}{x+1}$ is the geometric series with first term 1 and common ratio x

$$\because S = \frac{a}{1-r}\ we\ have$$

$1 - x + x^2 - x^3 + \cdots + \frac{1}{1-(-x)} = \frac{1}{1+x}$

Taylor Series:

The series

$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots + \frac{f^n(x)}{n!}h^n$

Is known as **"Taylor series expansion"**

Proof:

Let the power series of $f(x) in\ (x-a)$

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \cdots + a_n(x-a)^n \rightarrow (1)$$

$$\Rightarrow f(a) = a_0 + a_1(a-a) + a_2(a-a)^2 + a_3(a-a) + \cdots + a_n(a-a)^n$$

$$\Rightarrow f(a) = a_0 \Rightarrow a_o = f(a)$$

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a(x-a)^3 + \cdots + na_n(x-a)^{n-1}$$

$$\Rightarrow \boldsymbol{f'(a) = a_1 \Rightarrow a_1 = f'(a)}$$

$$\boldsymbol{f''(x) = 2a_2 + 6a_3(x-a)2a_4(x-a)^2 + \cdots + n(n-1)a_n(x-a)^{n-2}}$$

$$\Rightarrow \boldsymbol{f''(a) = 6a_3 \Rightarrow a_3 = \frac{f'''(a)}{3!}}$$

Similarly, $\boldsymbol{a_n = \frac{f^n(a)}{n!}}$

Putting valves of $a_1, a_2, a_3, \dots, a_n$ in (1)

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(x)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^n(a)}{n!}(x-a)^n \to (2)$$

This expansion is the Taylor series $for\ f\ at\ x = a.$

Replace $x\ by\ x+h\ and\ a\ by\ x\ we\ get$

$$f(x+h) = f(x) + f'(x)(x+h-x) + \frac{f''(x)}{2!}(a+h-x)^2 + \frac{f'''(x)}{3!}(x+h-x)^3 + \dots + \frac{f^n(x)}{n!}(x+h-h)^n + \dots$$

$$f(a+h) = f(x) + f'(x)(h) + \frac{f''(x)}{2!}(h)^2 + \frac{f'''(x)}{3!}(h)^3 + \dots + \frac{f^n(x)}{n!}(h)^n + \dots$$

Note:

$if\ we\ put\ a = 0\ in (2) then\ f(x)$

$$= f(0) + f'(0)x + \frac{f'''(o)}{2!}x^2 + \dots + \frac{f^n(0)}{n!}x^n$$

Which is McLaurin's series.

The above expansion is also named as Taylor theorem

Exercise 2.8

Question # 1. Apply the Maclaurin series expansion to prove that:

(i) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{4} - \frac{x^4}{4} + \dots\dots\dots$

Solution. Let $f(x) = \ln(x+1)$

$$f(0) = \ln(0+1) = ln1 = 0$$
$$f'(x) = \frac{1}{x+1}$$
$$f'(0) = \frac{1}{0+1} = 1$$
$$f''(x) = \frac{-1}{(x+1)^2}$$
$$f''(0) = \frac{-1}{(0+1)^2} = -1$$
$$f'''(x) = \frac{2}{(x+1)^3}$$
$$f'''(0) = \frac{2}{(0+1)^3} = 2$$
$$f^{(4)}(x) = \frac{-6}{(x+1)^4}$$
$$f^{(4)}(0) = \frac{-6}{(0+1)^4} = -6$$

By Maclaurin Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^4(0) + \dots\dots\dots$$

$$\ln(x+1) = 0 + x(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6) + \dots\dots\dots$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots\dots$$

Hence Proved.

(ii) $\mathbf{Cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\dots\dots}$

Solution. Let $f(x) = \text{Cosx}$

$$f(0) = Cos0 = 0$$
$$f'(x) = -Sinx$$
$$f'(0) = -Sin0 = 0$$
$$f''(x) = -Cosx$$
$$f''(0) = -Cos0 = -1$$
$$f'''(x) = Sinx$$
$$f'''(0) = Sin0 = 0$$
$$f^{(4)}(x) = Cosx$$
$$f^{(4)}(0) = Cos0 = 1$$
$$f^{(5)}(x) = -Sinx$$
$$f^{(5)}(0) = -Sin0 = 0$$
$$f^{(6)}(x) = -Cosx$$
$$f^{(6)}(0) = -Cos0 = -1$$

By Maclaurin Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^4(0) + \frac{x^5}{5!}f^{(5)}(0) + \frac{x^6}{6!}f^{(6)}(0) \dots\dots\dots$$

$$Cosx = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(-1) \dots\dots\dots$$

$$Cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots\dots\dots$$

Hence Proved.

(iii) $\mathbf{\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots\dots\dots}$

Solution. Let $f(x) = \sqrt{1+x}$

$$f(0) = \sqrt{1+0} = 1$$
$$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$$
$$f'(0) = \frac{1}{2}(1+0)^{-\frac{1}{2}} = \frac{1}{2}$$
$$f''(x) = \frac{1}{2}.\frac{-1}{2}(1+x)^{-\frac{3}{2}}$$
$$f''(0) = \frac{1}{2}.\frac{-1}{2}(1+0)^{-\frac{3}{2}} = -\frac{1}{4}$$
$$f'''(x) = \frac{1}{2}.\frac{-1}{2}.\frac{-3}{2}(1+x)^{-\frac{5}{2}}$$
$$f'''(0) = \frac{1}{2}.\frac{-1}{2}.\frac{-3}{2}(1+0)^{-\frac{5}{2}} = \frac{3}{8}$$

By Maclaurin Series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ \dots\dots\dots$$

$$\sqrt{1+x}=1+x\left(\frac{1}{2}\right)+\frac{x^2}{2!}\left(-\frac{1}{4}\right)+\frac{x^3}{3!}\left(\frac{1}{8}\right)+\ \dots\dots\dots$$

$$Cosx=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}\ \dots\dots\dots$$

Hence Proved.

(iv) $\mathbf{e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ \dots\dots\dots}$

Solution. Let $f(x)=e^x$

$$f(0)=e^0=1$$
$$f'(x)=e^x$$
$$f'(0)=e^0=1$$
$$f''(x)=e^x$$
$$f''(0)=e^0=1$$
$$f'''(x)=e^x$$
$$f'''(0)=e^0=1$$

By Maclaurin Series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ \dots\dots\dots$$

$$e^x=1+x(1)+\frac{x^2}{2!}(1)+\frac{x^3}{3!}(1)+\ \dots\dots\dots$$

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\ \dots\dots\dots$$

Hence Proved.

(v) $e^{2x}=1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\ \dots\dots\dots$

Solution. Let $f(x)=e^{2x}$

$$f(0)=e^{2(0)}=1$$
$$f'(x)=2e^{2x}$$
$$f'(0)=2e^0=2$$
$$f''(x)=4e^{2x}$$
$$f''(0)=4e^0=4$$
$$f'''(x)=8e^{2x}$$
$$f'''(0)=8e^0=8$$

By Maclaurin Series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ \dots\dots\dots$$

$$e^{2x}=1+x(2)+\frac{x^2}{2!}(4)+\frac{x^3}{3!}(8)+\ \dots\dots\dots$$

$$e^{2x}=1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}\ \dots\dots\dots$$

Hence Proved.

Question # 2 Show that $\mathbf{Cos(x+h)=Cosx-hSinx-\frac{h^2}{2!}Cosx+\frac{h^3}{3!}Sinx+\ \dots\dots\dots}$

And evaluate $\mathbf{Cos61^0}$.

Solution.

Let $f(x+h)=Cos(x+h)$

$$f(x)=Cosx$$
$$f'(x)=-Sinx$$
$$f''(x)=-Cosx$$
$$f'''(x)=Sinx$$

By Taylor Series

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ \dots\dots\dots$$

$$Cos(x+h)=Cosx+h(-Sinx)+\frac{h^2}{2!}(-Cosx)+\frac{h^3}{3!}(Sinx)+\ \dots\dots\dots$$

$$Cos(x+h)=Cosx-hSinx-\frac{h^2}{2!}Cosx+\frac{h^3}{3!}Sinx+\ \dots\dots\dots$$

Hence Proved.

$$Now\ ,Put\ x=60^0\ and\ h=1^0=\frac{\pi}{180}\ rad$$
$$=0.01745\ rad$$

$$Cos(60^0+1^0)=Cos60^0-(0.01745)Sin60^0-\frac{(0.01745)^2}{2!}Cos60^0+\frac{(0.01745)^3}{3!}Sin60^0+\ \dots\dots\dots$$

$$Cos(61^0)=0.5-(0.01745)(0.866)-\frac{(0.01745)^2}{2!}(0.5)+\frac{(0.01745)^3}{3!}(0.866)+\ \dots\dots\dots$$

$$Cos(61^0)=0.5-0.0151117-0.000076125+0.000000072+\ \dots\dots....$$

$$Cos(61^0)=0.484812247\approx 0.4848$$

Which is required.

Question # 3 Show that $\mathbf{2^{x+h}=2^x\left[1+(ln2)h+(ln2)^2\frac{h^2}{2!}+(ln2)^3\frac{h^3}{3!}+\ \dots\dots....\right]}$

Solution.

Let $f(x+h)=2^{x+h}$

$$f(x)=2^x$$
$$f'(x)=2^x.lna$$
$$f''(x)=-Cosx$$
$$f'''(x)=Sinx$$

By Taylor Series

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ \dots\dots\dots$$

$$Cos(x+h)=Cosx+h(-Sinx)+\frac{h^2}{2!}(-Cosx)+\frac{h^3}{3!}(Sinx)+\ \dots\dots\dots$$

$$Cos(x+h)=Cosx-hSinx-\frac{h^2}{2!}Cosx+\frac{h^3}{3!}Sinx+\ \dots\dots\dots$$

Hence Proved

Geometrical meaning of a derivative:

Let us draw a curve of the function $y = f(x)$ take two neighboring points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ on the curve. Draw a tangent line to the curve $at\ pt.$
$p(x, y)$ **Such that it makes an angle** $\varphi\ with\ x - axis$.
$Also$ Draw a secant line to the curve passing through the pts. $p(x, y) and\ Q(x + \delta x, y + \delta y).the\ secent\ line\ makes\ angle\ \emptyset\ with\ x - axis.$
Draw $\perp\ PM\ and\ QN\ from\ P\ and\ Q\ on\ x - axis. Also\ draw\ perpendicular\ PR\ on\ QN. in\ \triangle PQR$

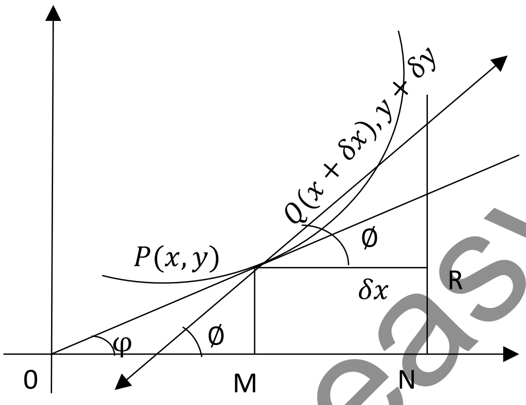
$tan\emptyset = \frac{|QR|}{|PR|} \quad |QR| = \delta y \quad |PR| = \delta x$

$$\Rightarrow Tan\emptyset = \frac{\delta y}{\delta x}$$

When $Q \to P\ \ then\ \delta x \to 0 \quad \delta y \to 0$
$\emptyset \to \varphi\ it\ means\ secent\ line\ becomes\ tangent\ line.$
Now,

$$\lim_{\emptyset \to \varphi} Tan\emptyset = \lim_{\emptyset \to \varphi} \frac{\delta y}{\delta x} \Rightarrow Tan\varphi = \frac{dy}{dx}$$

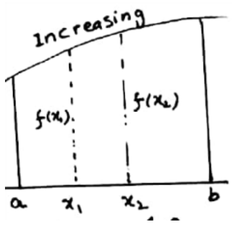
But $Tan\varphi$ is slope of tangent line. Thus geometrically derivative of
$y\ at\ x\ represent\ the\ slope\ of\ tangent\ line\ at\ p(x, y)$



Increasing and decreasing function:

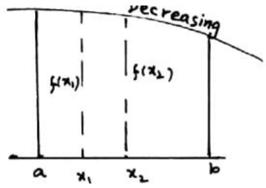
Increasing functions:

A function f is said to be increasing function on an interval (a, b) if for every $x_1, x_2 \in (a, b)\ then$
$f(x_1) < f(x_2)\ whenever\ x_1 < x_2$



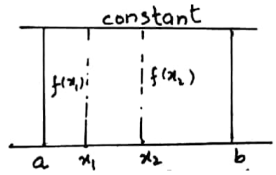
Decreasing function:

A function f is said to be decreasing function on an interval (a, b) if for every $x_1, x_2 \in (a, b)\ then$
$f(x_1) > f(x_2)\ whenever\ x_1 < x_2$



Constant function:

A function f is said to be constant function on an interval (a, b) if for every $x_1, x_2 \in (a, b)\ then$
$f(x_1) = f(x_2)\ whenever\ x_1 < x_2$



Alternative definitions:

Increasing functions:

A differentiable function f is said to be increasing function $if\ f'(x) > 0\ \forall x \in (a, b)$

Decreasing function:

A differentiable function f is said to be decreasing function $if\ f'(x) < 0\ \forall x \in (a, b)$

Constant function:

A differentiable function f is said to be constant function $if\ f'(x) = 0\ \forall x \in (a, b)$

Stationary point:

Any point where f is neither increasing nor decreasing is called stationary point.
Provided that $f'(x) = 0\ at\ that\ point.$

Relative Maxima:

A function $f(x)$ has relative maxima $f(c)$ at $x = c \in (a, b) if$

$(i) f(c - \delta x) > 0 \quad (ii) f'(c) = 0 \quad (iii) f'(c + \delta x) < 0$

$where\ \delta x\ is\ small + ve\ is\ small\ number.$

Relative Minima:

A function $f(x)$ has relative minima $f(c)\ at$ $x = \ c \in (a, b) if$

$(i) f(c - \delta x) < 0 \quad (ii) f'(c) = 0 \quad (iii) f'(c + \delta x) > 0$

$where\ \delta x\ is\ small + ve\ is\ small + ve\ number.$

Alternative definitions.

Relative maxima:

A function $f(x)$ has relative maxima $f(c) if\ f'(x)\ chnages\ sign\ from + ve\ t - ve\ as\ x\ change\ through\ C.$

Relative minima:

A function $f(x)$ has relative minima $f(c) if\ f'(x)\ changes\ sign\ from - ve\ t - ve\ as\ x\ change\ through\ C$

Relative Extrema:

Both relative maxima and relative minima are called in general "relative extreme"

Critical Valves and critical points:

if $c \in D_f$ and $f'(x)=0$ or $f'(c)$ does not exist,
Them the number c *is called a critical value for* f *while the point* $(c, f(x))$ *on the graph of* f *is* Named as a critical as a critical point.

Important note:

There are functions which have extrema (maxima or minima) at the points where there derivatives do not exist. For example

$$f(x)=|x| \quad and\ \emptyset(x)=\begin{cases}2-x & ,x>0\\ x+2, & x\le 0\end{cases}$$

***do not exist at* $(0,0)$ *and* $(0,20)$ *resp*.**

But

***f has minima at* $(0,0)$ *and* $\emptyset$ *has relative maxima at* $(0,2)$**

First derivative:

Let
f be diffrentiable in neighbourhood of C where $f'(c)=0$

1. A function $f(x)$ *has relative maxima at* $x=c\in(a,b)$ *if* $f'(x)$ *has realtive maxima at* $x=f'(c-\delta x)<0$
2. A function $f(x)$ *has relative minima at* $x=c\in(a,b)$ *if* $f'(c-\delta x)<0, f'(c)=0$ *and* $f'(c+\delta x)>0$

Second derivative:

Let
f be diffrentiable in neighbourhood of C where $f'(c)=0$
Then,

1. *f has relative maxima at c if* $f''(c)<0$
2. *f has relative minima at c if* $f''(c)>0$

Turning point:

A stationary point is called a turning point or a minimum point.

Point of inflexion:

A point at which the function has neither maximum nor minimum value is called point of inflexion.

Procedure for finding maxima and minima of a function:

Let $y=f(x)$ *be given function*

Step1. *Find* $\frac{dy}{dx}$ *or* $f'(x)$

Step11: put $\frac{dy}{dx}=0$ $f'(x)=0$

stepIII find $\frac{d^2y}{dx^2}$ *or* $f''(x)$ *at the roots.*

StepIV check the sign of $f''(x)$ *at the roots*

*if $f''(x)>0$ then $f(x)$ has minimum value

*if $f''(x)<0$ then $f(x)$ has maximum value.

*if $f''(x)=0$ then no information about the function then use first derivative rule"

Exercise 2.9

Question No.1

Determine the intervals in which f is increasing or decreasing for the domain mentioned in each case

(i) $\boldsymbol{f(x)=Sinx \quad ;x\in[-\pi,\pi]}$

Solution.

Given that

$$f(x)=Sinx \qquad ;x\in[-\pi,\pi]$$
$$f'(x)=Cosx$$
$$Cosx=0$$
$$x=-\frac{\pi}{2},\frac{\pi}{2}$$

So we have Sub-intervals
$\left(-\pi,-\frac{\pi}{2}\right),\left(-\frac{\pi}{2},\frac{\pi}{2}\right),\left(\frac{\pi}{2},\pi\right)$

$f'(x)=Cosx<0$ *whenever* $x\in\left(-\pi,-\frac{\pi}{2}\right)$

So $f(x)$ *is decreasing in the interval* $\left(-\pi,-\frac{\pi}{2}\right)$

$f'(x)=Cosx>0$ *whenever* $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

So $f(x)$ *is increasing in the interval* $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$f'(x)=Cosx<0$ *whenever* $x\in\left(\frac{\pi}{2},\pi\right)$

So $f(x)$ *is decreasing in the interval* $\left(\frac{\pi}{2},\pi\right)$

(ii) $\boldsymbol{f(x)=Cosx \quad ;x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)}$

Solution.

Given that

$$f(x)=Cosx \qquad ;x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$
$$f'(x)=-Sinx$$
$$Put\ f'(x)=0$$
$$-Sinx=0$$
$$x=0$$

So we have Sub-intervals $\left(-\frac{\pi}{2},0\right),\left(0,\frac{\pi}{2}\right)$

$f'(x)=-Sinx>0$ *whenever* $x\in\left(-\frac{\pi}{2},0\right)$

So $f(x)$ *is increasing in the interval* $\left(-\frac{\pi}{2},0\right)$

$f'(x)=-Sinx<0$ *whenever* $x\in\left(0,\frac{\pi}{2}\right)$

So $f(x)$ *is decreasing in the interval* $\left(0,\frac{\pi}{2}\right)$.

(iii) $\boldsymbol{f(x)=4-x^2 \quad ;x\in[-2,2]}$

Solution.

Given that

$$f(x)=4-x^2 \qquad ;x\in[-2,2]$$
$$f'(x)=-2x$$
$$Put\ f'(x)=0$$
$$-2x=0$$

$$x = 0$$

So we have Sub-intervals $(-2\ ,0\ )\ , (0\ ,2)$

$$f'(x) = -2x > 0\ whenever\ x \in (-2\ ,0\ )$$

$$So\ f(x) is\ increasing\ in\ the\ interval\ (-2\ ,0\ )$$

$$f'(x) = -2x < 0\ whenever\ x \in (0\ ,2)$$

$$So\ f(x) is\ decreasing\ in\ the\ interval\ (0\ ,2).$$

(iv) $\boldsymbol{f(x) = x^2 + 3x + 2 \quad ; x \in [-4, 1]}$

Solution.

$$\Rightarrow f'(x) = 2x + 3$$

$Put\ f'(x) = 0 \Rightarrow 2x + 3 = 0$

$$\Rightarrow x = -\frac{3}{2}$$

$$\because 2x + 3\ is + ve\ \ in \left(-\frac{3}{2}, 1\right)$$

So $f(x) is\ increasing\ on\ the\ interval\ \left(-\frac{3}{2}, 1\right)$

$\because 2x + 3\ is$

$- ve\ in\ \left(-4, -\frac{3}{2}\right)\ so\ f(x) is\ decreasing$

On the interval $\left(-4, -\frac{3}{2}\right)$

Question. 2.

Find the extreme values of the following functions defined as

(i) $\boldsymbol{f(x) = 1 - x^3}$

Solution.

Given that

$$f(x) = 1 - x^3$$

Differentiate w.r.t $''x''$

$$f'(x) = -3x^2 - - - (i)$$

For stationary points. Put $f'(x) = 0$

$$-3x^2 = 0$$

$$x = 0$$

Differentiate (i) w.r.t $''x'', we\ have$

$$f''(x) = -6x - - - (ii)$$

Now put $x = 0\ in\ (ii)$

$$f''(0) = -6(0) = 0$$

So second derivative test fails to determine the extreme points

Put $x = 0 - \varepsilon\ in\ (i)$

$$f'(x) = -3(-\varepsilon)^2 = -3\varepsilon^2 < 0$$

Put $x = 0 + \varepsilon\ in\ (i)$

$$f'(x) = -3(-\varepsilon)^2 = -3\varepsilon^2 < 0$$

As $f'(x)$ does not change its sign before and after $x = 0$.

Since $at\ x = 0\ , f(x) = 1$ therefore $(0,1)$ is the point inflection.

(ii) $\boldsymbol{f(x) = x^2 - x - 2}$

Solution.

Given that

$$f(x) = x^2 - x - 2$$

Differentiate w.r.t $''x''$

$$f'(x) = 2x - 1 \rightarrow (i)$$

For stationary points , Put $f'(x) = 0$

$$2x - 1 = 0$$

$$x = \frac{1}{2}$$

Differentiate (i) w.r.t $''x'', we\ have$

$$f''(x) = 2 - - - (ii)$$

Now put $x = \frac{1}{2}\ in\ (ii)$

$$f''\left(\frac{1}{2}\right) = 2$$

So second derivative test $f(x) is\ minimum\ at\ x = \frac{1}{2}$.

Now $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 2$

$$f\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} - 2 = -\frac{9}{4}.$$

(iii) $\boldsymbol{f(x) = 5x^2 - 6x + 2}$

Solution.

Given that

$$f(x) = 5x^2 - 6x + 2$$

Differentiate w.r.t $''x''$

$$f'(x) = 10x - 6 - - - (i)$$

For stationary points. Put $f'(x) = 0$

$$10x - 6 = 0$$

$$x = \frac{3}{5}$$

Differentiate (i) w.r.t $''x'', we\ have$

$$f''(x) = 10 - - - (ii)$$

Now put $x = \frac{3}{5}\ in\ (ii)$

$$f''\left(\frac{3}{5}\right) = 10$$

So second derivative test $f(x) is\ minimum\ at\ x = \frac{3}{5}$.

Now

$$f\left(\frac{3}{5}\right) = 5\left(\frac{3}{5}\right)^2 - 6\left(\frac{3}{5}\right) + 2$$

$$f\left(\frac{3}{5}\right) = \frac{9}{5} - \frac{18}{5} + 2 = \frac{1}{5}.$$

(iv) $\boldsymbol{f(x) = 3x^2}$

Solution.

Given that

$$f(x) = 3x^2$$

Differentiate w.r.t $''x''$

$$f'(x) = 6x - - - (i)$$

For stationary points. Put $f'(x) = 0$

$$6x = 0$$

$$x = 0$$

Differentiate (i) w.r.t $''x'', we\ have$

$$f''(x) = 6 - - - (ii)$$

Now put $x = 0\ in\ (ii)$

$$f''(0) = 6$$

So second derivative test $f(x) is\ minimum\ at\ x = 0$.

Now

$$f(0) = 3(0)^2$$

$$f(0) = 0.$$

(v) $\boldsymbol{f(x) = 3x^2 - 4x + 5}$

Solution.

$$\Rightarrow f'(x) = 6x - 4 \Rightarrow f''(x) = 6 \rightarrow (i)$$

$$Put\ f'(x) = 0 \Rightarrow 6x = 0 \Rightarrow x = 0\ put\ in (i)$$

$$\Rightarrow f''(o) = 6 > 0\ so\ f(x) hax\ minimum\ value\ \ at$$

$$x = 0\ and\ f(0) = 3(0)^2 = 0$$

(vi) $\boldsymbol{f(x) = 2x^3 - 2x^2 - 36x + 3}$

Solution.

Given that

$$f(x) = 2x^3 - 2x^2 - 36x + 3$$

Differentiate w.r.t $''x''$

$$f'(x) = 6x^2 - 4x - 36 - - - (i)$$

For stationary points . Put $f'(x) = 0$

$$6x^2 - 4x - 36 = 0$$

$$3x^2 - 2x - 18 = 0$$

$$x = \frac{2 \pm \sqrt{(2)^2 - 4(3)(-18)}}{2(3)}$$

$$x = \frac{2 \pm \sqrt{4 + 216}}{6}$$

$$x = \frac{2 \pm \sqrt{220}}{6}$$

$$x = \frac{2 \pm 2\sqrt{55}}{6}$$

$$x = \frac{1 \pm \sqrt{55}}{3}$$

Differentiate (i) w.r.t $''x'', we\ have$

$$f''(x) = 12x - 4 \rightarrow (ii)$$

Now put $x = \frac{1+\sqrt{55}}{3}\ in\ (ii)$

$$f''\left(\frac{1+\sqrt{55}}{3}\right) = 12\left(\frac{1+\sqrt{55}}{3}\right) - 4$$

$$f''\left(\frac{1+\sqrt{55}}{3}\right) = 4(1+\sqrt{55}) - 4$$

$$f''\left(\frac{1+\sqrt{55}}{3}\right) = 4 + 4\sqrt{55} - 4$$

$$f''\left(\frac{1+\sqrt{55}}{3}\right) = 4\sqrt{55} > 0$$

So second derivative test $f(x) is\ minimum\ at\ x = \frac{1+\sqrt{55}}{3}$.

Now

$$f\left(\frac{1+\sqrt{55}}{3}\right) = 2\left(\frac{1+\sqrt{55}}{3}\right)^3 - 2\left(\frac{1+\sqrt{55}}{3}\right)^2 - 36\left(\frac{1+\sqrt{55}}{3}\right) + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = \frac{2}{27}(1+\sqrt{55})^3 - \frac{2}{9}(1+\sqrt{55})^2 - \frac{36}{3}(1+\sqrt{55}) + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = \frac{2}{27}(1 + 3\sqrt{55} + 3.55 + 55\sqrt{55}) - \frac{2}{9}(1 + 2\sqrt{55} + 55) - 12(1+\sqrt{55}) + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = \frac{2}{27}(166 + 58\sqrt{55}) - \frac{2}{9}(56 + 2\sqrt{55}) - 12(1+\sqrt{55}) + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = \frac{332}{27} + \frac{116}{27}\sqrt{55} - \frac{112}{9} - \frac{4}{9}\sqrt{55} - 12 - 12\sqrt{55} + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = -\frac{1}{27}(247 + 220\sqrt{55})$$

Now put $x = \frac{1-\sqrt{55}}{3}\ in\ (ii)$

$$f''\left(\frac{1-\sqrt{55}}{3}\right) = 12\left(\frac{1-\sqrt{55}}{3}\right) - 4$$

$$f''\left(\frac{1-\sqrt{55}}{3}\right) = 4(1-\sqrt{55}) - 4$$

$$f''\left(\frac{1-\sqrt{55}}{3}\right) = 4 - 4\sqrt{55} - 4$$

$$f''\left(\frac{1-\sqrt{55}}{3}\right) = -4\sqrt{55} < 0$$

So second derivative test $f(x) is\ maximum\ at$ $x = \frac{1-\sqrt{55}}{3}$.

Now

$$f\left(\frac{1-\sqrt{55}}{3}\right) = 2\left(\frac{1-\sqrt{55}}{3}\right)^3 - 2\left(\frac{1-\sqrt{55}}{3}\right)^2 - 36\left(\frac{1-\sqrt{55}}{3}\right) + 3$$

$$f\left(\frac{1-\sqrt{55}}{3}\right) = \frac{2}{27}(1-\sqrt{55})^3 - \frac{2}{9}(1-\sqrt{55})^2 - \frac{36}{3}(1-\sqrt{55}) + 3$$

$$f\left(\frac{1-\sqrt{55}}{3}\right) = \frac{2}{27}(1 - 3\sqrt{55} + 3.55 - 5\sqrt{55}) - \frac{2}{9}(1 - 2\sqrt{55} + 55) - 12(1-\sqrt{55}) + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = \frac{2}{27}(166 - 58\sqrt{55}) - \frac{2}{9}(56 - 2\sqrt{55}) - 12(1-\sqrt{55}) + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = \frac{332}{27} - \frac{116}{27}\sqrt{55} - \frac{112}{9} + \frac{4}{9}\sqrt{55} - 12 + 12\sqrt{55} + 3$$

$$f\left(\frac{1+\sqrt{55}}{3}\right) = -\frac{1}{27}(247 - 220\sqrt{55})$$

(vii) $f(x) = x^4 - 4x^2$

Solution:

$\Rightarrow f'(x) = 4x^3 - 8x \Rightarrow f''(x) = 12x^2 - 8 \rightarrow (i)$

$Put\ f'(x) = 0\ so\ 4x^3 - 8x = 0$

$\Rightarrow 4x(x^2 - 8x) = 0$

$\Rightarrow 4x(x^2 - 2) =$

- $4x = 0 \qquad x^2 - 2 = 0$
- $x = 0 \qquad x = \pm\sqrt{2}$
- $thus\ x = 0 \quad ,x = \sqrt{2} \quad ,x = -\sqrt{2}$

$Put\ x = 0\ in\ (1)$

- $f''(0) = 12(0)^2 - 8 = -8 < 0$
- So $f(x) has\ maximam\ value\ at\ x = o\ and$
- $f(0) = (0)^4 - 4(0)^2 = 0$

$Put\ x = \sqrt{2}\ in\ (1)$

- $f''(\sqrt{2}) = 12\left(\sqrt{2}\right)^2 - 8 = 12(2) - 8 = 16 > 0$
- So $f(x) has\ minimum\ value\ at\ x = \sqrt{2}\ and$

$f(\sqrt{2}) = 12(\sqrt{2})^2$

$=\left[\left(\sqrt{2}\right)^2\right]^2 - 4(2) = (2)^2 - 8 = 4 - 8 = -4 < 0$

Put $x = -\sqrt{2}\ in\ (1)$

$f''(-\sqrt{2}) = 12(\sqrt{2})^2 - 8 = 12(2) - 8$

$= 24 - 8 = 16 > 0$

$so\ f(x) has\ minmum\ value\ at$

$x = -\sqrt{2}\ ans\ f(-\sqrt{2}) = (-\sqrt{2})^4 - 4(-\sqrt{2})^2$

$\left[(-\sqrt{2})^2\right]^2 - 4(2)$

$= (2)^2 - 8 = 4 - 8 = 4$

(viii) $f(x) = x^4 - 4x^2$

Solution.

- $f'(x) = \frac{d}{dx}\{(x-2)^2(x-1)\}$

$= (x-2)\frac{d}{dx}(x-1) + (x-1)\frac{d}{dx}(x-2)^2$

$= (x-2)^2(1(x-1).2(x-2)\frac{d}{dx}(x-2)$

$= (x-2)^2 + 2(x-1)(x-2)(1)$

$f'(x) = (x-2) + 2(x-1)(x-2)$

$= (x-2)\{x - 2 + 2x - 2\}$

$f'(x) = (x-2)(3x-4)$

$f''(x) = \frac{d}{dx}\{(x-2)(3x-4)\}$

$= (x-2)\frac{d}{dx}(3x-4)$

$+ (3x-4)\frac{d}{dx}(x-2)$

$(x-2)(3) + (3x-4)(1)$

$= 3x - 6 + 3x - 4$

$= 6x - 10$

$Put\ f'(x) = 0, \qquad so\ (x-2)(3x-4) = 0$

$x - 2 = 0 \quad 3x - 4 = 0$

$x = 2 \qquad x = \frac{4}{3}$

$Put\ x = 2\ in\ (1) f''(2) = 6(2) - 10 = 12 - 10 = 2 > 0$

$sp\ f(x) has\ minimum\ value\ at\ x = 2\ and$

$f(2) = (2-2)^2(2-1) = 0$

$put\ x = \frac{4}{3}\ in\ (1)\ f''\left(\frac{4}{3}\right) = 6\left(\frac{4}{3}\right) - 10 = 8 - 10 = -2 < 0$

$so\ f(x) has\ maximum\ value\ at\ x = \frac{4}{3}\ and$

$f\left(\frac{4}{5}\right) = \left(\frac{4}{3} - 2\right)^2\left(\frac{4}{3} - 1\right) = \left(\frac{4-6}{3}\right)^2\left(\frac{4-3}{3}\right)$

$= \left(-\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{4}{27}$

(ix) $f(x) = 5 + 3x - x^3$

- $f'(x) = 3 - 3x^2$
- $f''(x) = -6x \rightarrow (1)$

$put\ f'(x) = 0 \quad ,3 - 3x^2 = 0$

- $3(1 - x^2) = 0 \quad 3 \neq 0\ so\ 1 - x^2 = 0$
- $1 = x^2$
- $x = \pm 1$

$Put\ x = 1\ in\ (i) has\ maximum\ value\ at\ x$

$= 1\ and\ f(1) = 5 + 3(1) - (1)^3$

$= 5 + 3 - 1 = 7$

$put\ x = -1\ in\ (i) f''(-1) = -6(-1) = 6 > 0$

So f(x) $hax\ minimum\ value\ at\ x = 1$

$and\ f(-1) = 5 + 3(-1) - (-1)^3 = (5 - 3 - (-1)$

$f(-1) = 2 + 1 = 3$

Question.3.

Find the maximum and minimum values of the function defined by the following equation occurring in the interval [0,2π]

$$f(x) = Sinx + Cos\ x$$

Solution. Given function

$f(x) = Sinx + Cos\ x \qquad ; x \in [0,2\pi]$

Differentiate with respect to $''x''$.

$f'(x) = Cosx - Sinx --- (i)$

For Stationary points , $Put\ f'(x) = 0$

$Cosx - Sinx = 0$

$-Sinx = -Cosx$

$\frac{Sinx}{Cosx} = 1$

$tanx = 1$

$x = \tan^{-1} 1 = \frac{\pi}{4}, \pi + \frac{\pi}{4}$

$x = \frac{\pi}{4}, \frac{5\pi}{4} \quad when\ x \epsilon [0,2\pi]$

Now diff. $(i)\ w.r.t\ ''x''$

$f''(x) = -Sinx - Cosx$

Now put $x = \frac{\pi}{4}\ in\ (ii)$

$$f''\left(\frac{\pi}{4}\right) = -Sin\left(\frac{\pi}{4}\right) - Cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} < 0$$

So second derivative test $f(x) is\ MAximum\ at\ x = \frac{\pi}{4}$.

Now

$$f\left(\frac{\pi}{4}\right) = Sin\left(\frac{\pi}{4}\right) + Cos\left(\frac{\pi}{4}\right)$$

$$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Now put $x = \frac{5\pi}{4}$ $in\ (ii)$

$$f''\left(\frac{5\pi}{4}\right) = -Sin\left(\frac{5\pi}{4}\right) - Cos\left(\frac{5\pi}{4}\right)$$

$$= -\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right)$$

$$f''\left(\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} > 0$$

So second derivative test $f(x) is\ Minimum\ at\ x = \frac{5\pi}{4}$.

Now

$$f\left(\frac{5\pi}{4}\right) = Sin\left(\frac{5\pi}{4}\right) + Cos\left(\frac{5\pi}{4}\right)$$

$$f\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}.$$

Question.4.

Show that $y = \frac{lnx}{x}$ has maximum value at $x = e$.

Solution. Given that

$$y = \frac{lnx}{x}$$

Diff. w r .t $''x'', we\ have$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{lnx}{x}\right)$$

$$\frac{dy}{dx} = \frac{x.\frac{1}{x} - lnx.1}{x^2}$$

$$\frac{dy}{dx} = \frac{1 - lnx}{x^2} - - - (i)$$

For critical points. $Put\ \frac{dy}{dx} = 0$

$$\frac{1 - lnx}{x^2} = 0$$

$$1 - lnx = 0$$

$$lnx = 1$$

$$lnx = lne$$

$$x = e$$

Now diff. (i) with respect to $''x'', we\ have$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1 - lnx}{x^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{x^2.\left(-\frac{1}{x}\right) - (1 - lnx)(2x)}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{-x - 2x + 2xlnx}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{-3x + 2xlnx}{x^4}$$

$At\ x = e$, we have

$$\frac{d^2y}{dx^2}|_{x=e} = \frac{-3(e) + 2(e)\ln(e)}{(e)^4}$$

$$\frac{d^2y}{dx^2}|_{x=e} = \frac{-3e + 2e}{e^4}$$

$$\frac{d^2y}{dx^2}|_{x=e} = \frac{-e}{e^4} = -\frac{1}{e^3} < 0$$

Hence $y\ has\ a\ maximum\ value\ at\ x = e.$

Question.5.

Show that $y = x^x$ has maximum value at $x = \frac{1}{e}$.

Solution. Given that

$$y = x^x$$

Taking Log on both sides

$$lny = lnx^x = xlnx$$

Diff. w r .t $''x'', we\ have$

$$\frac{d}{dx}(lny) = \frac{d}{dx}(xlnx)$$

$$\frac{1}{y}.\frac{dy}{dx} = x\left(\frac{1}{x}\right) + lnx.1$$

$$\frac{dy}{dx} = y(1 + lnx)$$

$$\frac{dy}{dx} = x^x(1 + lnx) - - - (i).$$

For critical points , $Put\ \frac{dy}{dx} = 0$

$$x^x(1 + lnx) = 0$$

$$As\ \ x^x \neq 0\ \ then\ ,\ \ 1 + lnx = 0$$

$$1 + lnx = 0$$

$$lnx = -1$$

$$lnx = -lne$$

$$lnx = lne^{-1}$$

$$x = e^{-1}$$

$$x = \frac{1}{e}$$

Now diff. (i) with respect to $''x'', we\ have$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(x^x(1 + lnx)\right)$$

$$\frac{d^2y}{dx^2} = (1 + lnx)\frac{d}{dx}x^x + x^x\frac{d}{dx}\ (1 + lnx)$$

$$\frac{d^2y}{dx^2} = (1 + lnx)x^x(1 + lnx) + x^x\left(\frac{1}{x}\right)$$

$$\frac{d^2y}{dx^2} = x^x\left[(1 + lnx)^2 + \frac{1}{x}\right]$$

$$\frac{d^2y}{dx^2} = x^x\left[\frac{x(1 + lnx)^2 + 1}{x}\right]$$

$At\ x = \frac{1}{e}$, we have

$$\frac{d^2y}{dx^2}|_{x=\frac{1}{e}} = \left(\frac{1}{e}\right)^{\frac{1}{e}}\left[\frac{\frac{1}{e}\left(1 + \ln\left(\frac{1}{e}\right)\right)^2 + 1}{\frac{1}{e}}\right]$$

$$\frac{d^2y}{dx^2} > 0$$

Hence $y\ has\ a\ minimum\ value\ at\ x = \frac{1}{e}$.

Application of maxima and minima:

1. If we first from the relation of the form $y = f(x)$ from the given information.

2. Find the maximum or minimum value of f *as* required by using 1st or 2nd derivative value.

*mostly we will uses 2nd derivative rule.

Exercise 2.10

Question No1.find two integers whose sum is 20 and their product will be maximum.

Solution:

Let x *and* y be req. integer's product $f(x) = xy$

By given condition $x + y = 30$

⇨ $y = 30 - x \rightarrow (i)$

So,$f(x) = x(30 - x) = 30x - x^2$

$\Rightarrow f'(x) = 30 - 2x$

$\Rightarrow f''(x) = -2 \rightarrow (ii)$

Put $f'(x) = 0 \quad 30 - 2x = 0$

⇨ $2x = 30 \Rightarrow x = 15$ *put in* (ii)

⇨ $f''(15) = -2 <$

0 *thus* $f(x)$*has maximum value*

At $x = 15$ *and put in* $(i) \Rightarrow y = 30 - 15 = 15$

So *req. integer are* 15 *and* 15.

Question No: 2 divide 20 into two parts so that the sum of their squares will be minimum.

Solution:

let x *and* y *be req. two parts of* $20.$ *so* $x + y = 20$

$\Rightarrow y = 20 - x \rightarrow (i)$

by given condition $f(x) = x^2 + y^2$

⇨ $f(x) = x^2 + (20 - x)^2 = x^2 + 400 + x^2 - 40x$

⇨ $f'(x) = 4x - 40 \Rightarrow f''(x) = 4 \rightarrow (ii)$

⇨ *Put* $f'(x) = 0 \quad 4x - 40 = 0$

$x - 10 = 0 \Rightarrow x = 10$ *put in* (i)

$f''(10) = 4 > 0$ *so* $f(x)$*hax minimum value at* $x = 10$

and put in$(i)$$y = 20 - 10 = 10$

So req. two parts of 20 are 10 and 10.

Q3. Find two positive integers whose sum is 12 and the product of one with the square of the other will be maximum.

Solution:

Let x *and* y *be req. integers by given condition;*

$x + y = 12 \; y = 12 - x \rightarrow (i)$

Product $= f(x) = x^2y = x^2\,(12 - x)$

⇨ $f(x) = 12x^2 - x^3$

⇨ $f'(x) = 24x - 3x^2 \quad \Rightarrow f''(x) = 24 - 6x \rightarrow (ii)$

put $f'(x) = 0 \Rightarrow 24x - 30x^2 = 0 \Rightarrow 2(8 - x) = 0$

$\Rightarrow x = 0$ *and* $x = 8$

Put $x = 0$ *in* $(ii) \quad \Rightarrow f''(o) = 24 > 0$

So $f(x)$*has minimum value at* $x =$
o which is not required

So Put $x = 8$ *in* $(ii) f''(8) = 24 - 6(8) = 24 - 48$

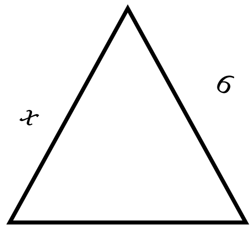
$f''(8) = -24 < 0$

So $f(x)$*hax maximum value at* $x =$
8 *and put in* (i)

$y = 12 - 8 = 4$ *so required are* 8 *and* $4.$

Question No.4 The perimeter of a triangle is 16cm. if one side of the other sides for maximum area of the triangle?

Solution:



Let x *and* y *be other sides.*

$\because$ *perimeter* $= x + y + 6$(*sum of all sides*)

⇨ $16 = x + y + 6$

⇨ $16 - 6 = x + y$

⇨ $x + y = 10$

⇨ And $y = 10 - x \rightarrow (i)$

$\because S = \dfrac{x + y + 6}{2} = \dfrac{16}{2}$ *or* $S = 8$

Also area $A^2 = f(x) = s(s - x)(s - y)(s - 6)$

⇨ $f(x) = 8(8 - x)(8 - y)(8 - 6)$

⇨ $f(x) = 16(8 - x)(8 - y)$

$= 16(8 - x)(8 - (10 - x))$

$f(x) = (8 - x)(x - 2)$

$f'(x) = 16\frac{d}{dx}\{(8 - x)(x - 2)\}$

$= 16\big((8 - x)(1) + (x - 2)(-1)\big)$

$= 16(8 - x - x + 2) = 16(10 - 2x)$

$f'(x) = 160 - 32x = 0$

$f''(X) = -32 \rightarrow (ii)$

Put $f'(x) = 0 \Rightarrow 160 - 32x = 0$

⇨ $32x = 160 \Rightarrow x = 5$

put $x = 5$ *in* $(ii) \;\Rightarrow f''(5) = -32 < 0$

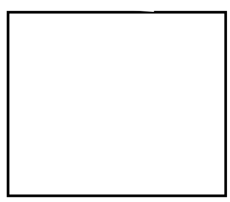
Thus $f(x)$*has maximum value at* $x =$
5 *and put in* (i)

$y = 10 - 5 = 5$

Hence req. sides are 5cm and 5cm.

Question No.5 find the dimension of a rectangular of largest area having perimeter 12cm.

Solution:



Let x *and* y *be required dimensions.*

Perimeter $= x + y + x + y$(*som of all sides*)

$\Rightarrow 120 = 2x + 2y$

$$\Rightarrow x + y = 60$$

And $y = 60 - x \rightarrow (i)$

$\because$ *Area of* $f(x) = xy$ *rectangle)*

$$\Rightarrow f(x) = x(60 - x)$$

$f(x) = 60x - x^2$

$\Rightarrow f'(x) = 60 - 2x$

$\Rightarrow f''(x) = -2 \rightarrow (ii)$

Put $f'(x) = 0$ *so* $60 - 2x = 0$

$f''(x) = -2 \rightarrow (ii)$

Put $f'(x) = 0$ *so* $60 - 2x = 0$

$2x = 60 \Rightarrow x = \frac{60}{2} = 30$ *put in* (ii)

$$\Rightarrow f''(30) = -2 < 0$$

so $f(x)$ *hax maximum (largest) value at* $x = 30$ *and*

put in $(i) y = 60 - 30 = 30$

Thus required dimensions are $30cm^2$ where its ?? erimter is minimum.

Question N0.6 find the length of sides of a variable rectangular having area $36cm^2$ where its permeter is minimum.

Solution:

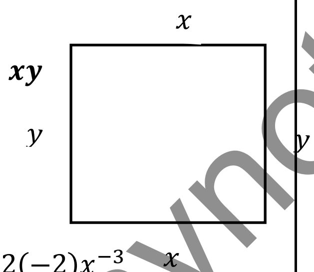
let x and y be required length

$$\boldsymbol{Perimeter = x + y + x + y}$$

- $f(x) = 2x + 2y$
- $f(x) = 2(x + y) \rightarrow (\boldsymbol{i})$

$\boldsymbol{area\ of\ reactangle = xy}$

- $\boldsymbol{36 = xy \Rightarrow y = \frac{36}{x}}$



so $(i) \Rightarrow f(x) = 2\left(x + \frac{36}{y}\right)$

$f(x) = 2x + 72^{-1}$

$f'(x) = 2 - \frac{72}{x^2}$, $\quad f''(x) = -72(-2)x^{-3}$

$f''(x) = \frac{144}{x^3} \rightarrow (iii)$

Put $f'(x) = 0 \Rightarrow 2 - \frac{72}{x^2} = 0$

$2 = \frac{72}{x^2} \Rightarrow x^2 - 36 \Rightarrow x = \pm 6$n

- $x = 6 \qquad x = -6$ *(rejected legth is always is + ve)*

- Put $x = 6$ *in* (iii) $f''(6) = \frac{144}{(6)^3} = \frac{144}{216} > 0$

So $f(x)$ has minimum value at $x = 6$ *and put in*(ii)

$$\Rightarrow y = \frac{36}{6} = 6$$

So sides of rectangle are 6cm and 6cm.

Question No.7 A box with a square base and open top is top is to have a volume of 4 cube dm. find the dimensions of the box which will require the least material.

Solution:

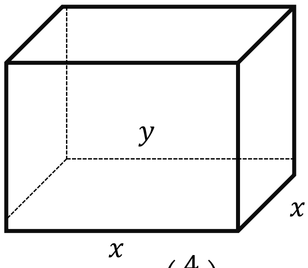
Let length, width, and hieght of box are x,x and y repectively. So

Volume=$x \times x \times y = x^2y \rightarrow (i)$

- $4 = x^2y \Rightarrow y = \frac{4}{x^2}$

$\because$ *surface area of the box =* *(area of the square + area of 4 verticle width)*

- $f(x) = x \times x \times 4(x \times y)$
- $f(x) = x^2 = 4xy$



$$x^2 + 4x\left(\frac{4}{x^2}\right) \qquad \because y = \frac{4}{x^2}$$

$f(x) = x^2 + 16x^{-1} \Rightarrow f'(x) = 2x + 16(-1)x^{-2}$

$$f'(x) = 2x - \frac{16}{x^2}$$

$\Rightarrow f''(x) = 2 + 16(-1)(-2)\,x^{-3}$

$$f''(x) = 2 + \frac{32}{x^3} \rightarrow (ii)$$

put $f'(x) = 0$ *so* $2x - \frac{16}{x^2} = 0 \Rightarrow \frac{2x^3 - 16}{x^2} = 0$

$\Rightarrow 2(x^3 - 8) = 0 \Rightarrow (x)^3 - (2)^3 = 0$

Put in $(ii) f''(x) = 2 + \frac{32}{(2)^2} > 0$

- $f(x)$ *hax minimum (least) value at* $x = 2$ *and*

put in $(i) \Rightarrow y = \frac{4}{4} = 1$

Hence

2dm, 2dm and 1 dm is the dimension of box

Question No.8 find the dimensions of a rectangular garden having perimeter 80m.if its area is to b maximum.

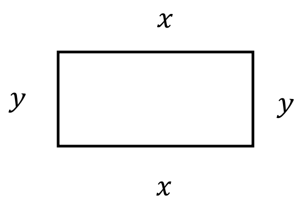
Solution: ***Let x and be required dimensions.***

$$perimeter = x + y + x + y$$

$80 = 2x + 2y \Rightarrow x + y = 40$

$\Rightarrow \boldsymbol{y = 40 - x \rightarrow (i)}$

area of rectangular $= f(x) = xy$



$$f(x) = xy = x(40 - x) = 40x - x^2$$

$f'(x) = 40 - 2x \Rightarrow f''(x) = -2 \rightarrow (ii)$

put $f'(x) = 0$ *so* $40 - 2x = 0 \Rightarrow 2x = 40$

$\Rightarrow x - 20$ *put in* $(ii) \Rightarrow f''(20) = -< 0$

- $f(x)$ *hax maximum value at* $x = 2$

and put in (i) $\; y = 40 - 20 \Rightarrow y = 20$

So dimensions of rectangular garden are 20m and 20m

Question No.9 An open tank of square base of side x and vertical sides is to be constructed to contain a given quantity of water. Find the depth in terms of x if the tank with lead will be least.

Solution:

Let length, width and height of tank with square base are $x, x, and\ y$ respectively.

$$\because volume = x \times x \times y = x^2y$$
$$y = \frac{v}{x^2} \rightarrow (i)$$
$$\because surface\ area =$$
$$area\ of\ square + area\ of\ four\ verticle\ walls$$
$$\Rightarrow f(x) = x^2 + 4xy$$
$$\Rightarrow f(x) = x^2 + 4x\left(\frac{v}{x^2}\right) = x^2 + \frac{4v}{x}$$
$$f(x) = x^2 + 4vx^{-1}$$
$$\Rightarrow f'(x) = 2x + 4v(-1)(x^{-2}) = 2x - \frac{4v}{x^2}$$
$$\Rightarrow f''(x) = 2 + 4v(-1)(-2)x^{-3} = 2 + \frac{8v}{x^3} \rightarrow (ii)$$
$$Put\ f'(x) = 0 \Rightarrow 2 + 4v(-1)(-2)(x^{-3}) = 2 + \frac{8v}{x^3}$$
$$\rightarrow (ii)$$
$$Put\ f'(x) = 0 \Rightarrow 2x - \frac{4v}{x^2} = 0$$
$$\Rightarrow \frac{2x^3 - 4v}{x^2} = 0 \Rightarrow 2^3 = 4v$$
$$\Rightarrow x^3 = \frac{4v}{2} = 0 \Rightarrow x = (2v)^{\frac{1}{3}} put\ in\ (ii)$$
$$f''(2v)^{\frac{1}{3}} = 2 + 8v\left[(2v)^{\frac{1}{3}}\right]^{-3} = 2 + 8v(2v)^{-1}$$
$$= 2 + \frac{8v}{2v} = 2 + 4 = 6 > 0$$
$$\Rightarrow f(x) has\ minimum\ value\ at\ x = (2v)^{\frac{1}{3}}$$
$$and\ put\ in\ (i) y = \frac{\frac{x^3}{2}}{x^2} = \frac{x}{2}$$
$$y = \frac{x}{2}\ which\ is\ required\ height\ (depth) interms\ of\ x.$$

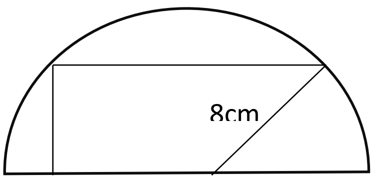
Question No.10 find the dimensions of the rectangular of maximum area which fits inside the semi –circle of radius 8cm as shown in figure.

Solution:

Let O be centre of semi – circle

let ABCD be rectangular with fits inside the semi – circle

Of radius 8cm as Shown in figure.



$$let\ |AC| = |BD| = y$$
$$ALSO\ let\ |AC| = |BD| = x$$
$$so\ lenghtof\ rectangual$$
$$= 2x\ and\ width\ of\ rectangual = y$$
$$\because radius\ 8cm(given) so\ |OC| = 8cm$$
$$in\ \Delta OAC\ \ x^2 + y^2 = (8)^2\ \ using\ pathagourm\ theorem$$
$$\Rightarrow y^2 = 64 - x^2 \Rightarrow y = \pm\sqrt{64 - x^2}$$
$$y = \sqrt{64 - x62} \rightarrow (i)$$
$$y = -\sqrt{64 - x^2}\ rejected\ being\ length$$
$$\because Area\ of\ rectangular = 2xy \Rightarrow A = f(x) = 2xy$$
$$\Rightarrow A^2 = f(x) = 4x^2(64 - x^2) = 256x^2 - 4x^4$$
$$\Rightarrow f'(x) = 519x2x - 16x^3$$
$$f'' = 512 - 48x^2 \rightarrow (ii)$$
$$put\ f'(x) = 0 \Rightarrow 512x - 16x^3 = 0$$
$$\Rightarrow 16x(32 - x^2) = 0$$
$$\Rightarrow 16x = 0 \quad x^2 = 32 \Rightarrow x = \pm\sqrt{32}$$
$$\Rightarrow x = 0, \quad x = \sqrt{32}\ ,$$
$$x = -\sqrt{32}\ rejected\ beinglength$$

So put $x = put\ x = \sqrt{32}\ in\ (ii) \quad \Rightarrow f''(x)(\sqrt{32})$

$$= 512 - 48\left(\sqrt{32}\right)^2$$
$$\Rightarrow f''\left(\sqrt{32}\right) = 512 - 48(32) = -1024 < 0$$
$$\Rightarrow f(x) has\ \ maximum\ value\ at\ x = \sqrt{32}$$
$$and\ put\ in(i) y = \sqrt{64 - \left(\sqrt{32}\right)^2} = \sqrt{64 - 32} = \sqrt{32}$$
$$y = \sqrt{32}$$

Thus length of rectangular $= 2x = 2(4\sqrt{2}) = 8\sqrt{2}$

Width of rectangular $= y = 4\sqrt{2}$

Hence dimensions are $8\sqrt{2}\ and\ 4\sqrt{2}$

Question No.11 find the point on the curve $y = x^2 - 1$ that is closed to the point $(3, -1)$

Solution:

$$\because y = x^2 - 1 \rightarrow (i)$$

Let $p(x, y)$ be required point.

$$Let\ d = distance\ b\backslash w(x, y) and\ (3, -1)$$
$$\Rightarrow d = \sqrt{(x - 3)^2 + (y + 1)^2}$$
$$\Rightarrow d^2 = (x - 3)^2 + (y + 1)^2 \qquad take\ d^2 = f(x)$$
$$\Rightarrow f(x) = (x - 3)^2 + (y + 1)^2$$
$$f(x) = x^2 + 9 - 6x + (x^2 - 1)^2 + 1 + 2(x^2 - 1)$$
$$\because y = x^2 - 1$$
$$= x^2 + 9 - 6x + x^4 + 1 - 2x^2 + 1 + 2x^2 - 2$$
$$f(x) = x^4 + x^2 - 6x + 9$$
$$\Rightarrow f'(x) = 4x^3 + 2x - 6$$
$$\Rightarrow f''(x) = 12x^2 + 2 \rightarrow (ii)$$

Put $f'(x) = 0 \ \Rightarrow 4x^3 + 2x - 6 = 0$

$$2x^3 + x - 3 = 0 \quad put\ x = 1$$
$$\Rightarrow (x - 1)(2x^2 + 2x + 3) = 0$$
$$\Rightarrow x = 1, 2x^2 + 2x + 3 = 0 \quad (rejected\ being\ complex)$$

$so\ put\ x = 1\ in\ (ii)$

1	2	0	1	-3
	↓	2	2	3
	2	2	3	0

$2x^2 + 2x + 3\ and\ x - 1\ are\ factors\ of\ 2x^3 + x - 3 = 0$

$$f''(x) = 12(1)^2 + 2 = 14 > 0$$

$\Rightarrow f(x) has\ minimum\ value\ at\ x = 1\ and\ put\ in\ (i)$

$$\Rightarrow y = (1)^2 - 1 = 0$$

So $(1,0) is\ the\ required\ point\ closest\ to\ (3, -1)$

Question No.12

Find the point on the curve $y = x^2 - 1$ so $(1, 0) is\ the\ required\ point\ closet\ to\ (18, -1)$

Solution:

$$\because y = x^2 - 1 \rightarrow (i)$$

Let $p)x, y)$ be the required point.

$$\because d = distance\ b\backslash w\ p(x, y) amd\ (18,1)$$
$$\Rightarrow d = \sqrt{(x - 18)^2 + (y - 1)^2}$$
$$\Rightarrow d^2 = \sqrt{(x - 18)^2 + (y - 1)^2} \qquad take\ d^2 = f(x)$$
$$\Rightarrow f(x) = (x - 18)^2 + (y - 1)^2$$
$$f(x) = (x - 18)^2 + (x^2 + 1 - 1) \because y = x^2 + 1$$

$$= x^2 + 324 - 36x + x^4$$
$$f(x) = x^4 + x^2 - 36x + x^4$$
$$f'(x) = 4x^3 + 2x - 36$$
$$f''(x) = 12x\text{^}2 + 2 \to (ii)$$
$$Put\ f'(x) = 0 \Rightarrow 4x^3 + 2x - 36 = 0$$
$$\Rightarrow 2x^3 + x - 18 = 0 \qquad for\ x = 2$$
$$\Rightarrow (x-2)(2x^2 + 4x + 9) = 0$$
$$x = 2,\ \ 2x^2 + 4x + 9 = 0\ \ rejected\ being\ complex$$
$$\text{Put} x = 2\ in\ (ii)$$

	2	0	1	-18
2	↓	4	8	18
	2	2	9	0

$$2x^2 + 4x + 9\ and\ x - 2\ are\ factors\ of$$
$$2x^3 + x - 18 = 0$$
$$put\ x = 2\ in\ (ii)$$
$$\Rightarrow f''(x) = 12(2)^2 + 2 > 0$$
$$\Rightarrow f(x) has\ minimum\ value\ at\ x = 2\ put\ in\ (i)$$
$$\Rightarrow y = (2)^2 + 1 = 5$$

So $(2,5)$ *is the required point which is closest to* $(18,1)$