Chapter 8

MATHEMATICAL INDUCTION AND BINOMIAL THEOREM

PRINCIPLE OF MATHEMATICAL INDUCTION (Lahore Board 2009)

The principle of mathematical induction is stated as follows:

$S(1)$ is true i.e., $S(n)$ is true for $n = 1$ and $S(k+1)$ is true whenever $S(k)$ is true for any positive integer k, then $S(n)$ is true for all positive integers.

PROCEDURE

Condition 1: Substituting $n = 1$, show that the statement is true for $n = 1$.

Condition 2: Assuming that the statement is true for positive integer k, then show that it is true for the next higher integer.

EXERCISE 8.1

Q.1 Use mathematical induction to prove that the following formula for every positive integer n.

$$1 + 5 + 9 + \ldots\ldots + (4n-3) = n(2n-1)$$

Solution:

Let $S(n)$ be the given statement, that is

$S(n): 1 + 5 + 9 + \ldots\ldots + (4n-3) = n(2n-1)$

Condition 1

When $n = 1$, $S(1)$ becomes

$S(1):\ 4(1) - 3 = 1(2-1)$

$1 = 1$

L.H.S. = R.H.S. Thus $C-1$ is satisfied.

Condition 2

Let us suppose that $S(n)$ is true for any $n = K \in N$, that is $S(k)$;

$1 + 5 + 9 + ... + (4k - 3) = k(2k - 1)$ .... (1)

Adding $[4(k+1) - 3]$ an both sides of (1),

$$1 + 5 + 9 + ... + (4k - 3) + [4(k+1) - 3] = k(2k - 1) + [4(k+1) - 3]$$

$$= 2k^2 - k + 4k + 4 - 3$$

$$= 2k^2 + 3k + 1$$

$$= 2k^2 + 2k + k + 1$$

$$= 2k(k+1) + 1(k+1)$$

$$= (k+1)(2k+1)$$

$$= (k+1)(2k+2-1)$$

$$= (\overline{k+1})[2(\overline{k+1}) - 1]$$

$\Rightarrow$ $n = k + 1$

Thus $S(k+1)$ is true if $S(k)$ is true. Therefore C–2 is satisfied.

Since both conditions are satisfied. Therefore $S(n)$ is true for any positive integer.

Q.2 Use mathematical induction to prove that the following formula for every positive integer n.

$1 + 3 + 5 + ... + (2n - 1) = n^2$ (Lahore Board 2005, 2008)

Solution:

Let $S(n)$ be the given statement, that is

$S(n)$: $1 + 3 + 5 + ... + (2n - 1) = n^2$

Condition 1

When $n = 1$

$2 - 1 = 1$

$1 = 1$

L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that $S(n)$ is true for any $n = K \in N$, that is

$S(k)$: $1 + 3 + 5 + ... + (2k - 1) = k^2$ .... (1)

Adding $[2(k+1) - 1]$ on both sides

$$1 + 3 + 5 + \ldots\ldots + (2k - 1) + [2(k + 1) - 1] = k^2 + [2(k + 1) - 1]$$

$$= k^2 + 2k + 2 - 1$$

$$= k^2 + 2k + 1$$

$$= (k + 1)^2$$

Thus $S(k + 1)$ is true when $S(k)$ is true. Therefore condition 2 is satisfied.

Since both conditions are satisfied, so $S(n)$ is true for every positive integer.

Q.3 Use mathematical induction to prove that the following formula for every positive integer n.

$$\mathbf{1 + 4 + 7 + \ldots\ldots + (3n - 2) = \frac{n(3n - 1)}{2}}$$

Solution:

Let $S(n)$ be the given statement, that is

$$S(n):\ 1 + 4 + 7 + \ldots\ldots + (3n - 2) = \frac{n(3n - 1)}{2}$$

Condition 1

When $n = 1$, $S(1)$ becomes

$$1 = \frac{1(3 - 1)}{2}$$

$$1 = \frac{2}{2}$$

$$1 = 1$$

L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that $S(n)$ is true for any $n = K \in N$, that is

$$S(k):\ 1 + 4 + 7 + \ldots\ldots + (3k - 2) = \frac{k(3k - 1)}{2} \qquad \ldots\ldots\ldots (1)$$

Adding $[3(k + 1) - 2]$ on both sides of (1)

$$1 + 4 + 7 + \ldots\ldots + (3k - 2) + [3(k + 1) - 2] = \frac{k(3k - 1)}{2} + [3(k + 1) - 2]$$

$$= \frac{k(3k - 1)}{2} + 3k + 1$$

$$= \frac{3k^2 - k + 6k + 2}{2}$$

$$= \frac{3k^2 + 5k + 2}{2}$$

$$= \frac{3k^2 + 3k + 2k + 2}{2}$$

$$= \frac{3k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(3k+2)}{2}$$

$$= \frac{(k+1)(3k+3-1)}{2}$$

$$= \frac{(k+1)(3(k+1)-1)}{2}$$

$\Rightarrow$ $n = k + 1$

Thus condition (2) is satisfied.

Hence S (n) is true for every +ve integer.

Q.4 Use mathematical induction to prove that the following formula for every positive integer n.

$$1 + 2 + 4 + ... + 2^{n-1} = 2^n - 1$$

Solution:

Let S (n) be the given statement that is

S (n): $1 + 2 + 4 + ... + 2^{n-1} = 2^n - 1$

Condition 1

When $n = 1$, S (1) becomes

$2^{1-1} = 2^1 - 1$

$2^0 = 2 - 1$

$1 = 1$

L.H.S. = R.H.S. therefore condition 1 is satisfied.

Condition 2

Let us assume that S (n) is true for any $n = k \in N$, that is

S (k): $1 + 2 + 4 + ... + 2^{k-1} = 2^k - 1$ .... (1)

Adding 2^{k+1-1} on both sides

$$1 + 2 + 4 + \ldots\ldots + 2^{k-1} + 2^{k+1-1} = 2^k - 1 + 2^{k+1-1}$$
$$= 2^k - 1 + 2^k$$
$$= 2 \,.\, 2^k - 1 \quad (2^k + 2^k = 2k^k)$$
$$= 2^{k+1} - 1$$

Therefore condition 2 is satisfied. Both conditions are satisfied. Hence S (n) is true for every +ve integer.

Q.5 Use mathematical induction to prove that the following formula for every positive integer n. (Lahore Board 2010)

$$\mathbf{1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots + \frac{1}{2^{n-1}} = 2\left[1 - \frac{1}{2^n}\right]}$$

Solution:

Let S (n) be the given statement, that is

$$S(n): \quad 1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots + \frac{1}{2^{n-1}} = 2\left[1 - \frac{1}{2^n}\right]$$

Condition 1

When n = 1; S (n) becomes

$$S(1): \quad 1 = 2\left[1 - \frac{1}{2^1}\right]$$

$$1 = 2 \times \frac{1}{2} = 1$$

L.H.S. = R.H.S. therefore condition 1 is satisfied.

Condition 2

Let us suppose that S (n) is true for any $n = k \in N$, that is

$$1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots + \frac{1}{2^{k-1}} = 2\left[1 - \frac{1}{2^k}\right] \qquad \ldots\ldots\ldots (1)$$

Adding $\frac{1}{2^{k+1-1}}$ on both sides of (1)

$$1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots + \frac{1}{2^{k-1}} + \frac{1}{2^{k+1-1}} = 2\left[1 - \frac{1}{2^k}\right] + \frac{1}{2^{k+1-1}}$$

$$= 2\left[\frac{2^k - 1}{2^k}\right] + \frac{1}{2^k}$$

$$= \frac{22^k - 2 + 1}{2^k} = \frac{2^{k+1} - 1}{2^k}$$

$$= \frac{2\,[2^{k+1} - 1]}{22^k} \quad \text{(Multiplying \& dividing by 2)}$$

$$= \frac{2\,[2^{k+1} - 1]}{2^{k+1}}$$

$$= 2\left[\frac{2^{k+1}}{2^{k+1}} - \frac{1}{2^{k+1}}\right]$$

$$= 2\left[1 - \frac{1}{2^{k+1}}\right]$$

Thus S (k + 1) is true of S (k) is true. Therefore condition 2 is satisfied. Hence S (n) is true for every positive integer.

Q.6 Use mathematical induction to prove that the following formula for every positive integer n.

$2 + 4 + 6 \ldots\ldots + 2n = n(n+1)$

Solution:

Let S (n) be the given statement given, that is

S (n): $2 + 4 + 6 \ldots\ldots + 2n = n(n+1)$

Condition 1

When $n = 1$; S (n) becomes

S (1), $2 = 1(1+1)$

$2 = 2$

L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that S (n) is true for $n = k \in N$ that is

S (k): $2 + 4 + 6 + \ldots\ldots + 2k = k(k+1)$ .... (1)

Adding $2(k+1)$ on both sides

$$2 + 4 + 6 + \ldots\ldots + 2k + 2(k+1) = k(k+1) + 2(k+1)$$

$$= k^2 + k + 2k + 2$$

$$= k(k+1) + 2(k+1)$$

$$= (k+1)(k+2)$$

$$= (\overline{k+1})(\overline{k+1}+1)$$

Thus S (k + 1) is true of S (k) is true. Therefore condition 2 is satisfied.

Since both conditions are satisfied. Therefore S (n) is true for every +ve integer.

Q.7 Use mathematical induction to prove that the following formula for every positive integer n. (Lahore Board 2006)

$$\mathbf{2 + 6 + 18 + \ldots\ldots + 2 \times 3^{n-1} = 3^n - 1}$$

Solution:

Let S (n) be the given statement, that is

S (n): $2 + 6 + 18 + \ldots\ldots. + 2 \times 3^{n-1} = 3^n - 1$

Condition 1

When n = 1; S (n) becomes

S (1): $2 = 3 - 1$

$2 = 2$

⇒ L.H.S. = R.H.S. therefore condition 1 is satisfied.

Condition 2

Let us suppose that S (n) is true for $n = k \in N$, that is

S (k) ; $2 + 6 + 8 + \ldots\ldots. + 2 \times 3^{k-1} = 3^k - 1$ ……… (1)

Adding $2 \times 3^{k+1-1}$ on both sides

$$2 + 6 + 8 + \ldots\ldots.. + 2 \times 3^{k-1} + 2 \times 3^{k+1-1} = 3^k - 1 + 2 \times 3^{k+1-1}$$

$$= 3^k + 2 \times 3^k - 1$$

$$= 3^k(1+2) - 1$$

$$= 3^{\overline{k+1}} - 1$$

Thus S (k + 1) is true when S (k) is true.

Therefore condition 2 is satisfied. Since both conditions are satisfied. Therefore S (n) is true for every positive integer.

Q.8 Use mathematical induction to prove that the following formula for every positive integer n. (Gujranwala Board 2003)

$$\mathbf{1 \times 3 + 2 \times 5 + 3 \times 7 + \ldots. + n \times (2n+1) = \frac{n(n+1)(4n+5)}{6}}$$

Solution:

Let S (n) be the given statement, that is

$$S(n):\quad 1\times 3 + 2\times 5 + 3\times 7 + \ldots\ldots + n(2n-1) = \frac{n(n+1)(4n+5)}{6}$$

Condition 1

When n = 1, S (n) becomes

$$S(1):\quad 1\times 3 = \frac{1(2)(9)}{6}$$

$$3 = 3$$

$\Rightarrow$ L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let us assume that S (n) is true for $n = k \in N$, i.e.

$$S(k):\quad 1\times 3 + 2\times 5 + 3\times 7 + \ldots\ldots + k(2k+1) = \frac{k(k+1)(4k+5)}{6} \quad \ldots\ldots\ldots (1)$$

Adding $(k+1)(2(k+1)+1)$ on both sides of (1)

$$1\times 3 + 2\times 5 + 3\times 7 + \ldots\ldots\ldots + k(2k+1) + (k+1)\left[2(k+1)+1\right]$$

$$= \frac{k(k+1)(4k+5)}{6} + (k+1)(2k+3)$$

$$= \frac{k(k+1)(4k+5) + 6(k+1)(2k+3)}{6}$$

$$= \frac{(k+1)\left[k(4k+5) + 6(2k+3)\right]}{6}$$

$$= \frac{(k+1)\left[4k^2 + 5k + 12k + 18\right]}{6}$$

$$= \frac{(k+1)(4k^2 + 17k + 18)}{6}$$

$$= \frac{(k+1)(4k^2 + 8k + 9k + 18)}{6}$$

$$= \frac{(k+1)\left[4k(k+2) + 9(k+2)\right]}{6}$$

$$= \frac{(k+1)(k+2)(4k+9)}{6}$$

$$= \frac{(k+1)(k+2)(4k+4+5)}{6}$$

$$= \frac{(k+1)(k+2)(4(k+1)+5)}{6}$$

$$= \frac{(\overline{k+1})(\overline{k+1}+1)[4(\overline{k+1})+5]}{6}$$

Thus S (k + 1) is true when S (k) is true. Condition 2 is satisfied. Since both conditions are satisfied. Therefore S (n) is true for every +ve integer.

Q.9 Use mathematical induction to prove that the following formula for every positive integer n.

$$\mathbf{1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots\ldots n(n+1) = \frac{n(n+1)(n+2)}{3}}$$

Solution:

Let S (n) be the given statement, that is

$$S(n):\ 1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots\ldots n(n+1) = \frac{n(n+1)(n+2)}{3}$$

Condition 1

When n = 1; S (n) becomes

$$S(1):\ 1 \times 2 = \frac{1(2)(3)}{3}$$

$$2 = 2$$

$\Rightarrow$ L.H.S. = R.H.S.

Hence condition 1 is satisfied.

Condition 2

Let us assume that S (n) is true for any $n = k \in N$, i.e.

$$S(k):\ 1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots.. + k(k+1) = \frac{k(k+1)(k+2)}{3} \quad \ldots\ldots\ldots (1)$$

Adding $(k+1) \times (k+1+1)$ on both sides

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots.. + k(k+1) + (k+1)(k+1+1)$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$

$$= \frac{(k+1)(k+2)[k+3]}{3} \quad \text{(taking common)}$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

$$= \frac{(\overline{k+1})(\overline{k+1}+1)(\overline{k+1}+2)}{3}$$

Thus $S(k+1)$ is true if $S(k)$ is true.

Therefore condition 2 is satisfied. Since both conditions are satisfied. Therefore, $S(n)$ is true for every +ve integer.

Q.10 Use mathematical induction to prove that the following formula for every positive integer n.

$$\mathbf{1 \times 2 + 3 \times 4 + 5 \times 6 + \ldots\ldots + (2n-1) \times 2n = \frac{n(n+1)(4n-1)}{3}}$$

Solution:

Let $S(n)$ be the given statement, that is

$$S(n):\ 1 \times 2 + 3 \times 4 + 5 \times 6 + \ldots\ldots + (2n-1)(2n) = \frac{n(n+1)(4n-1)}{3}$$

Condition 1

When $n = 1$, $S(n)$ becomes

$$S(1):\ 1 \times 2 = \frac{1(1+1)(4-1)}{3}$$

$$2 = \frac{2 \times 3}{3}$$

L.H.S. = R.H.S. therefore condition 1 is satisfied.

Condition 2

Let us suppose that $S(n)$ is true for $n = k \in N$ that is

$$S(k);\ 1 \times 2 + 3 \times 4 + 5 \times 6 + \ldots\ldots + (2k-1) \times 2k = \frac{k(k+1)(4k-1)}{3} \quad \ldots\ldots\ldots (1)$$

Adding $[2(k+1)-1] \times 2(k+1)$ on both sides of (1)

$$1 \times 2 + 3 \times 4 + 5 \times 6 + \ldots\ldots + (2k-1) \times 2k + [2(k+1)-1]\, 2(k+1)$$

$$= \frac{k(k+1)(4k-1)}{3} + [2(k+1)-1]\,2(k+1)$$

$$= \frac{k(k+1)(4k-1)}{3} + (2k+1) \times 2(k+1)$$

$$= \frac{k(k+1)(4k-1) + 6(2k+1)(k+1)}{3}$$

$$= \frac{(k+1)[4k^2 - k + 12k + 6]}{3}$$

$$= \frac{(k+1)(4k^2 + 11k + 6)}{3}$$

$$= \frac{(k+1)(4k^2 + 8k + 3k + 6)}{6}$$

$$= \frac{(k+1)(4k+3)(k+2)}{6}$$

$$= \frac{(k+1)(k+1+1)(4k+4-1)}{6}$$

$$= \frac{(\overline{k+1})(\overline{k+1}+1)(4\,\overline{k+1}-1)}{6}$$

Thus $S(k+1)$ is true if $S(k)$ is true.

Therefore condition 2 is satisfied. Since both conditions are satisfied.

Therefore $S(n)$ is true for every +ve integer.

Q.11 Use mathematical induction to prove that the following formula for every positive integer n.

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$$

Solution:

Let $S(n)$ be the given statement, that is

$$S(n): \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

Condition 1

When $n = 1$; $S(n)$ becomes

$$S(1); \frac{1}{1 \times 2} = \frac{1}{1+1}$$

$$\frac{1}{2} = \frac{1}{2}$$

$\Rightarrow$ L.H.S. = R.H.S. therefore condition 1 is satisfied.

Condition 2

Let us suppose that S (n) is true for $n = k \in N$ that is

$$S(k); \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots\ldots + \frac{1}{k(k+1)} = \frac{k}{k+1} \qquad \ldots\ldots\ldots (1)$$

Adding $\frac{1}{(k+1)(k+1+1)}$ on both sides

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+1+1)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{(k+1)}{k+1+1}$$

Thus S (k + 1) is true when S (k) is true.

Therefore condition 2 is satisfied. Since both conditions are satisfied.

Therefore S (n) is true for every +ve integer.

Q.12 Use mathematical induction to prove that the following formula for every positive integer n.

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \ldots\ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

Solution:

Let S (n) be the given statement, that is

$$S(n): \frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \ldots\ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

Condition 1

When $n = 1$, $S(n)$ becomes

$$S(1): \frac{1}{(1)(3)} = \frac{1}{2+1}$$

$$\frac{1}{3} = \frac{1}{3}$$

$\Rightarrow$ L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that $S(n)$ is true for $n = k \in N$ that is

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \dots\dots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1} \qquad \dots\dots\dots (1)$$

Adding $\dfrac{1}{[2(k+1)-1][2(k+1)+1]}$ on both sides of (1)

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \dots\dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{[2(k+1)-1][2(k+1)+1]}$$

$$= \frac{k}{(2k+1)} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+2k+k+1}{(2k+1)(2k+3)}$$

$$= \frac{2k(k+1)+1(k+1)}{(2k+1)(2k+3)}$$

$$= \frac{(k+1)(2k+1)}{(2k+1)(2k+3)}$$

$$= \frac{k+1}{2k+2+1} = \frac{(\overline{k+1})}{2(\overline{k+1})+1}$$

Thus $S(k+1)$ is true, when $S(k)$ is true. Therefore condition 2 is satisfied.

Hence $S(n)$ is true for every +ve integer.

Q.13 Use mathematical induction to prove that the following formula for every positive integer n.

$$\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\dots\dots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{2(3n+2)}$$

Solution:

Let S (n) be the given statement, that is

$$S(n):\ \frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\dots\dots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{2(3n+2)}$$

Condition 1

When n = 1, S (n) becomes

$$S(1):\ \frac{1}{2\times 5}=\frac{1}{2(3+2)}$$

$$\frac{1}{10}=\frac{1}{10}$$

$\Rightarrow$ L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that S (n) is true for n = k $\in$ N, i.e.

$$S(k);\ \frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\dots\dots+\frac{1}{(3k-1)(2k+2)}=\frac{k}{2(3k+2)} \qquad \dots\dots\dots (1)$$

Adding $\frac{1}{[3(k+1)-1][3(k+1)+2]}$ on both sides

$$\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\dots\dots+\frac{1}{(3k-1)(3k+2)}+\frac{1}{[3(k+1)-1][3(k+1)+2]}$$

$$=\frac{k}{2(3k+2)}+\frac{1}{[3(k+1)-1][3(k+1)+2]}$$

$$=\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}$$

$$=\frac{k(3k+5)+2}{2(3k+2)(3k+5)}$$

$$= \frac{3k^2 + 5k + 2}{2(3k+2)(3k+5)}$$

$$= \frac{3k^2 + 3k + 2k + 2}{2(3k+2)(3k+5)}$$

$$= \frac{3k(k+1)\,2(k+1)}{2(3k+2)(3k+5)}$$

$$= \frac{(k+1)(3k+2)}{2(3k+2)(3k+5)}$$

$$= \frac{k+1}{2(3k+5)} = \frac{k+1}{2(3k+3+2)}$$

$$= \frac{(\overline{k+1})}{2[3(\overline{k+1})+2]}$$

Thus S (k + 1) is true, when S (k) is true.

Therefore condition 2 is satisfied. Thus S (n) is true for every +ve integer.

Q.14 Use mathematical induction to prove that the following formula for every positive integer n.

$$\mathbf{r + r^2 + r^3 + \dots\dots\dots + r^n = \frac{r(1-r^n)}{1-r}}$$

Solution:

Let S (n) be the given statement, that is

$$S(n):\ r + r^2 + r^3 + \dots\dots + r^n = \frac{r(1-r^n)}{1-r}$$

Condition 1

When n = 1, S (n) becomes

$$S(1):\quad r = \frac{r(1-r)}{(1-r)}$$

$r = r$

$\Rightarrow$ L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that S (n) is true for any $n = k \in N$, that is

$$S(k):\ r + r^2 + r^3 + \dots\dots\dots + r^k = \frac{r(1-r^k)}{1-r} \qquad \dots\dots\dots (1)$$

Adding both sides by r^{k+1}

$$r + r^2 + r^3 + \ldots\ldots r^k + r^{k+1} = \frac{r(1 - r^k)}{(1 - r)} + r^{k+1}$$

$$= \frac{r(1 - r^k) + (1 - r)\, r^{k+1}}{(1 - r)}$$

$$= \frac{r - r^{k+1} + r^{k+1} - r^{k+1+1}}{1 - r}$$

$$= \frac{r - r^{k+1+1}}{1 - r}$$

$$= \frac{r\left(1 - r^{k+1}\right)}{1 - r}$$

Thus $S(k + 1)$ is true when $S(k)$ is true.

Therefore condition 2 is satisfied.

Hence $S(n)$ is true for every +ve integer.

Q.15 Use mathematical induction to prove that the following formula for every positive integer n.

$$\mathbf{a + (a + d) + (a + 2d) + \ldots\ldots\ldots [a + (n - 1)\, d] = \frac{n}{2}[2a + (n - 1)\, d]}$$

Solution:

Let $S(n)$ be the given statement that is

$$S(n):\ a + (a + d) + (a + 2d) + \ldots\ldots\ldots [a + (n - 1)\, d] = \frac{n}{2}[2a + (n - 1)\, d]$$

Condition 1

When $n = 1$, $S(n)$ becomes

$$S(1):\ a = \frac{1}{2}[2a + (1 - 1)\, d] = \frac{1}{2}(2a)$$

$a = a$

$\Rightarrow$ L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that $S(n)$ is true for any $n = k \in N$, i. e.

$$S(k):\ a + (a + d) + (a + 2d) + \ldots\ldots\ldots [a + (k - 1)\, d] = \frac{k}{2}[2a + (k - 1)\, d] \ \ldots\ldots\ldots (1)$$

Adding both sides by $\left[a + (k + 1 - 1)\, d\right]$

$$a + (a + d) + (a + 2d) + \ldots\ldots + \left[a + (k - 1)\, d\right] + \left[a + (k + 1 - 1)\, d\right]$$

$$= \frac{k}{2}\left[2a + (k - 1)\, d\right] + \left[a + (k + 1 - 1)\, d\right]$$

$$= \frac{k}{2}\left[2a + (k - 1)\, d\right] + \left[a + kd\right]$$

$$= \frac{k\,[2a + kd - d] + 2\,[a + kd]}{2}$$

$$= \frac{2ak + k^2 d - kd + 2a + 2kd}{2}$$

$$= \frac{2ak + 2a + k^2 d + kd}{2}$$

$$= \frac{2a\,(k + 1) + kd\,(k + 1)}{2}$$

$$= \frac{(k + 1)\,(2a + kd)}{2}$$

$$= \frac{(\overline{k + 1})}{2}\left[2a + (\overline{k + 1} - 1)\, d\right]$$

Thus $S\,(k + 1)$ is true when $S\,(k)$ is true.

Therefore condition 2 is satisfied.

Hence $S\,(n)$ is true for every +ve integer.

Q.16 Use mathematical induction to prove that the following formula for every positive integer n.

$$\mathbf{1\,.\,1! + 2\,.\,2! + 3\,.\,3! + \ldots\ldots\, n\,.\,n! = (n + 1)! - 1.}$$

Solution:

Let $S\,(n)$ be the given statement, that is

$S\,(n):\ 1\,.\,1! + 2\,.\,2! + 3\,.\,3! + \ldots\ldots + n\,n! = (n + 1)! - 1$

Condition 1

When $n = 1$, $S\,(n)$ becomes

$S\,(1):\ 1\,.\,1! = (1 + 1)! - 1$

$1 = 2! - 1$

$1 = 2 - 1$

$1 = 1$

$\Rightarrow$ L.H.S. = R.H.S.

Thus condition 1 is satisfied.

Condition 2

Let us assume that $S(n)$ is true for $n = k \in N$, i.e.

$$S(k):\ 1 . 1! + 2 . 2! + 3 . 3! + \ldots\ldots k . k! = (k+1)! - 1 \qquad \ldots\ldots\ldots (1)$$

Adding $(k+1)(k+1)!$ on both sides of (1)

$$1 . 1! + 2 . 2! + 3 . 3! + \ldots\ldots + k . k! + (k+1)(k+1)!$$

$$= (k+1)! - 1 + (k+1)(k+1)!$$

$$= (k+1)! + (k+1)(k+1)! - 1$$

$$= (k+1)!\,[1 + k + 1] - 1$$

$$= (k+1)!\,(k+2) - 1$$

$$= (k+2)(k+1)! - 1$$

$$= (k+2)! - 1$$

$$= \left(\overline{k+1} + 1\right) - 1$$

Thus $S(k+1)$ is true, when $S(k)$ is true.

Therefore condition 2 is satisfied. Since both conditions are satisfied.

Therefore, $S(n)$ is true for every +ve integer.

Q.17 Use mathematical induction to prove that the following formula for every positive integer n.

$a_n = a_1 + (n-1)d$ when $a_1, a_1 + d, a_1 + 2d, \ldots\ldots$ forms A.P.

Solution:

Let $S(n)$ be the given statement, that is

$$S(n):\quad a_n = a_1 + (n-1)d$$

Condition 1

When $n = 1$, $S(n)$ becomes

$$S(1):\quad a_1 = a_1 + (1-1)d$$

$$a_1 = a_1$$

$\Rightarrow$ L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that S (n) is true for $n = k \in N$ that is

$S(k): \quad a_k = a_1 + (k - 1) d \qquad \ldots\ldots\ldots (1)$

Adding "d" on both sides

$a_{k+d} = a_1 + (k - 1) d + d$

$a_{k+1} = a_1 + d [k - 1 + 1]$

$a_{\overline{k+1}} = a_1 + \left(\overline{k+1} - 1\right) d$

Thus S (k + 1) is true when S (k) is true.

Therefore condition 2 is satisfied.

Since both conditions are satisfied. Therefore S (n) is true for every +ve integer.

Q.18 Use mathematical induction to prove that the following formula for every positive integer n.

$a_n = a_1 r^{n-1}$ when $a_1, a_1 r, a_1 r^2, \ldots\ldots$ from a G. P

Solution:

Let S (n) be the given statement, that is

$S(n): \quad a_n = a_1 r^{n-1}$

Condition 1

When $n = 1$, then S (n) becomes

$S(1): \quad a_1 = a_1 r^{1-1}$

$a_1 = a_1$

$\Rightarrow$ L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that S (n) is true for $n = k \in N$, i.e.

$S(k): \quad a_k = a_1 r^{k-1} \qquad \ldots\ldots\ldots (1)$

Multiplying both sides by r

$a_k \times r = a_1 r^{k-1} \times r$

$a_{k+1} = a_1 r^{k-1+1}$

$a_{\overline{k+1}} = a_1 r^{\overline{k+1} - 1}$

Thus S (k + 1) is true when S (k) is true.

Therefore condition 2 is satisfied.

Since both conditions are true.

Therefore S (n) is true for every +ve integer.

Q.19 Use mathematical induction to prove that the following formula for every positive integer n.

$$1^2 + 3^2 + 5^2 + \ldots\ldots + (2n - 1)^2 = \frac{n(4n^2 - 1)}{3}$$

Solution:

Let S (n) be the given statement, that is

$$S(n):\quad 1^2 + 3^2 + 5^2 + \ldots\ldots + (2n - 1)^2 = \frac{n(4n^2 - 1)}{3}$$

Condition 1

When n = 1, S (n) becomes

$$S(1):\quad 1^2 = \frac{1(4 - 1)}{3}$$

$$1 = 1$$

$\Rightarrow$ L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that S (n) is true for n = k ∈ N, i.e.

$$S(k):\quad 1^2 + 3^2 + 5^2 + \ldots\ldots + (2k - 1)^2 = \frac{k(4k^2 - 1)}{3} \qquad \ldots\ldots\ldots (1)$$

Adding $[2(k + 1) - 1]^2$ on both sides

$$1^2 + 3^2 + 5^2 + \ldots\ldots + (2k - 1)^2 + [2(k + 1) - 1]^2 = \frac{k(4k^2 - 1)}{3} + (2k + 1)^2$$

$$= \frac{k(2k - 1)(2k + 1)}{3} + (2k + 1)^2$$

$$= \frac{k(2k - 1)(2k + 1) + 3(2k + 1)^2}{3}$$

$$= \frac{(2k + 1)[k(2k - 1) + 3(2k + 1)]}{3}$$

$$= \frac{(2k+1)(2k-1)(k+3)}{3}$$

$$= \frac{(2k+1)(2k^2+6k-k-3)}{3}$$

$$= \frac{(2k+1)(2k+3)(k+1)}{3}$$

$$= \frac{(k+1)\left[(2k+1)(2k+3)\right]}{3}$$

$$= \frac{(k+1)\left[4k^2+6k+2k+3\right]}{3}$$

$$= \frac{(k+1)\left[4k^2+8k+4-1\right]}{3}$$

$$= \frac{(k+1)\left[4(k^2+2k+1)-1\right]}{3}$$

$$= \frac{(\overline{k+1})\left[4(\overline{k+1})^2-1\right]}{3}$$

Thus S (k + 1) is true, when S (k) is true.

Therefore condition 2 is satisfied.

Since both the conditions are satisfied.

Therefore S (n) is true for every +ve integer.

Q.20 Use mathematical induction to prove that the following formula for every positive integer n. (Lahore Board 2010)

$$\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \ldots + \binom{n+2}{3} = \binom{n+3}{4}$$

Solution:

Let S (n) be the given statement, therefore

$$S(n): \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \ldots + \binom{n+2}{3} = \binom{n+3}{4}$$

Condition 1

When n = 1, S (n) becomes

$$\binom{3}{3} = \binom{1+3}{4}$$

1 = 1

$\Rightarrow$ L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that S (n) is true for $n = k \in N$, i.e.

$$S(k): \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + ... + \binom{k+2}{3} = \binom{k+3}{4} \qquad .... (1)$$

Adding $\binom{k+2+1}{3}$ on both sides of (1) gives

$$\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + ... + \binom{k+2}{3} \binom{k+2+1}{3} = \binom{k+3}{4} + \binom{k+2+1}{4}$$

$$= \binom{k+3+1}{4} \qquad \left(\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r\right)$$

$$= \binom{\overline{k+1}+3}{4}$$

Thus S (k + 1) is true when S (k) is true.

Therefore condition 2 is satisfied. Since both conditions are satisfied.

Therefore S (n) is true for every +ve integer.

Q.21 Use mathematical induction to prove that the following formula for every positive integer n. (Lahore Board 2011)

(i) $n^2 + n$ **is divisible by 2**

(ii) $5^n - 2^n$ **is divisible by 3**

(iii) $5^n - 1$ **is divisible by 4**

(iv) $8 \times 10^n - 2$ **is divisible by 6**

(v) $n^3 - n$ **is divisible by 6**

Solution:

(i) $n^2 + n$ **is divisible by 2**

Let S (n) be the given statement > that is

S (n); $n^2 + n$ is divisible by 2

Condition 1

When $n = 1$, S (n) becomes

$$S(1): 1^2 + 1 = 1 + 1 = 2$$

Clearly 2 is divisible by 2

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that $S(n)$ is true for $n = k \in N$, i.e.

$S(k)$: $k^2 + k$ is divisible by 2 .... (1)

We want to prove that $S(k+1)$ is also divisible by 2

For $n = k + 1$ $S(n)$ becomes

$$S(k+1) = (k+1)^2 + (k+1)$$
$$= k^2 + 1 + 2k + k + 1$$
$$= k^2 + k + 2k + 2$$
$$= (k^2 + k) + 2(k+1)$$

$(k^2 + k)$ is divisible by 2 by expression (1) and $2(k+1)$ is also divisible by 2. Thus $S(k+1)$ is divisible by 2.

Therefore condition 2 is satisfied. Since both conditions are satisfied.

Therefore $S(n)$ is divisible by 2 for all +ve integers.

(ii) $5^n - 2^n$ is divisible by 3 (Gujranawala Board, 2006)

Let $S(n)$ be the given statement, i.e.

$S(n)$: $5^n - 2^n$ is divisible by 3

Condition 1

When $n = 1$; $S(n)$ becomes

$S(1)$; $5 - 2 = 3$, clearly 3 is divisible by 3

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that $S(n)$ is true for $n = k \in N$, that is

$S(k)$; $5^k - 2^k$ is divisible by 3 (1)

We want to prove that $S(k+1)$ is also divisible by 3

For $n = k + 1$ $S(n)$ becomes

$$S(k+1);\ 5^{k+1} - 2^{k+1} = 5^k \cdot 5 - 2^k \cdot 2$$
$$= (3+2)\,5^k - 2\,2^k$$
$$= 3 \cdot 5^k + 2 \cdot 5^k - 2\,2^k$$
$$= 3 \cdot 5^k + 2(5^k - 2^k)$$

The first term is clearly divisible by 3. The 2nd term also divisible by 3 by (1). Thus the whole term is divisible by 3. Therefore condition 2 is satisfied. Thus $S(n)$ is divisible by every +ve integer.

(iii) $5^n - 1$ is divisible by 4

Let S (n) be the given statement, i.e.

$S(n) = 5^n - 1$ is divisible by 4

Condition 1

When $n = 1$, then S (n) becomes

$S(1) = 5 - 1 = 4$

Clearly 4 is divisible by 4. Therefore condition 1 is satisfied.

Condition 2

Let us suppose that S (n) is true for $n = k \in N$, i.e

$S(k) = 5^k - 1$ is divisible by 4 .... (1)

We want to prove that S (k + 1) is also divisible by 4

For $n = k + 1$ S (n) becomes

$$S(k+1);\ 5^{k+1} - 1 = 5^k . 5 - 1$$
$$= (4 + 1)\, 5^k - 1$$
$$= 4 . 5^k + (5^k - 1)$$

The first term is clearly divisible by 4. The second term is also divisible by 4 by (1).

Thus S (k + 1) is divisible by 4. Therefore condition 2 is satisfied.

Thus S (n) is divisible for all +ve integral values of n.

(iv) $8 \times 10^n - 2$ is divisible by 6 (Lahore Board 2010)

Let S (n) be the given statement, that is

$S(n);\ 8 \times 10^n - 2$ is divisible by 6

Condition 1

When $n = 1$; S (n) becomes

$S(1);\ 8 \times 10 - 2 = 80 - 2 = 78$ clearly divisibly by 6

Condition 2

Let us suppose that S (n) is true for $n = k \in N$ that is

$S(k)$; $8 \times 10^k - 2$ is divisible by 6 .... (1)

We want to prove that $S(k+1)$ is also divisible by 6.

For $n = k + 1$ $S(n)$ becomes

$$S(k+1);\ 8 \times 10^{k+1} - 2 = 8 \times 10^k . 10 - 2$$
$$= 8 \times 10^k . 10 - 2$$
$$= 80 \times 10^k - 2$$
$$= (72 + 8)\, 10^k - 2$$
$$= 72 \times 10^k + (8 \times 10^k - 2)$$

The first term is clearly divisible by 6. The second term is also divisible by 6 by (1).

Thus $S(k+1)$ is divisible by 6. Therefore condition 2 is satisfied.

Hence $S(n)$ is divisible by 6 for all +ve integers.

(v) **$n^3 - n$ is divisible by 6**

Let $S(n)$ be the given statement, that is

$S(n)$; $n^3 - n$ is divisible by 6

Condition 1

When $n = 1$, $S(n)$ becomes

$S(1) = 1 - 1 = 0$ which is divisible by 6. Therefore condition 1 is satisfied.

Condition 2

Let us suppose that $S(n)$ is true for $n = k \in N$, i.e.

$S(k) = k^3 - k$ is divisible by 6 .... (1)

We want to prove that $S(k+1)$ is divisible by 6

For $n = k + 1$ $S(n)$ becomes

$$S(k+1) = (k+1)^3 - (k+1)$$
$$= (k+1)\left[(k+1)^2 - 1\right]$$
$$= (k+1)\left[k^2 + 2k + 1 - 1\right]$$
$$= (k+1)(k^2 + 2k)$$
$$= (k+1)\, k\, (k+2)$$
$$= k\,(k+1)(k+2)$$

Since the product of three consecutive terms is divisible by 6. Thus $S(k+1)$ is divisible by 6. Therefore condition 2 is satisfied. Hence $S(n)$ is divisible by 6 for all +ve integral values of n.

Q.22 Use mathematical induction to prove that the following formula for every positive integer n. **(Lahore Board 2005)**

$$\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\ldots\frac{1}{3^n} = \frac{1}{2}\left[1-\frac{1}{3^n}\right]$$

Solution:

Let S (n) be the given statement, i.e.

$$S(n): \frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\ldots\frac{1}{3^n} = \frac{1}{2}\left[1-\frac{1}{3^n}\right]$$

Condition 1

When n = 1, S (n) becomes

$$S(1): \frac{1}{3} = \frac{1}{2}\left[1-\frac{1}{3}\right]$$

$$\frac{1}{3} = \frac{1}{2}\left(\frac{2}{3}\right)$$

$$\frac{1}{3} = \frac{1}{3}$$

$\Rightarrow$ L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that S (n) is true for n = k $\in$ N, i.e.

$$S(k): \frac{1}{3}+\frac{1}{3^2}+\ldots\ldots\frac{1}{3^k} = \frac{1}{2}\left[1-\frac{1}{3^k}\right] \quad \ldots\ldots\ldots (1)$$

Adding $\frac{1}{3^{k+1}}$ on both sides of (1)

$$\frac{1}{3}+\frac{1}{3^2}\ldots\ldots\frac{1}{3^k}+\frac{1}{3^{k+1}} = \frac{1}{2}\left[1-\frac{1}{3^k}\right]+\frac{1}{3^{k+1}}$$

$$= \frac{1}{2}\left[\frac{3^k-1}{3^k}\right]+\frac{1}{3^{k+1}}$$

$$= \frac{3\,[3^k-1]+2\,[1]}{2\,.\,3^{k+1}}$$

$$= \frac{3^{k+1}-3+2}{2\,.\,3^{k+1}}$$

$$= \frac{1}{2}\left[\frac{3^{k+1}-1}{3^{k+1}}\right]$$

$$= \frac{1}{2}\left[1-\frac{1}{3^{k+1}}\right]$$

Thus S (k + 1) is true, when S (k) is true.

Therefore condition 2 is satisfied.

Since both conditions are satisfied. Therefore S (n) is true for every +ve integral values of n.

Q.23 Use mathematical induction to prove that the following formula for every positive integer n.

$$\mathbf{1^2 - 2^2 + 3^2 - 4^2 + \dots\dots + (-1)^{n-1} \cdot n^2 = \frac{(-1)^{n-1} \cdot n(n+1)}{2}}$$

Solution:

Let S (n) be the given statement, that is

$$S(n):\ 1^2 - 2^2 + 3^2 - 4^2 + \dots\dots + (-1)^{n-1} \cdot n^2 = \frac{(-1)^{n-1} \cdot n(n+1)}{2}$$

Condition 1

When n = 1, S (n) becomes

$$S(1):\ 1^2 = \frac{(-1)^{1-1}\, 1(1+1)}{2}$$

$$1 = 1$$

Since L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that S (n) be true for n = k ∈ N, i.e.

$$S(k);\ 1^2 - 2^2 + 3^2 - 4^2 + \dots\dots + (-1)^{k-1} \cdot k^2 = \frac{(-1)^{k-1} \cdot k(k+1)}{2} \qquad \dots\dots\dots (1)$$

Adding $(-1)^{k+1-1}(k+1)^2$ on both sides of equation (1)

$$1^2 - 2^2 + 3^2 - 4^2 + \dots\dots + (-1)^{k-1} k^2 + (-1)^{k+1-1}(k+1)^2$$

$$= \frac{(-1)^{k-1}\, k(k+1)}{2} + (-1)^k (k+1)^2$$

$$= \frac{(-1)^k (-1)^{-1}\, k(k+1) + 2(-1)^k (k+1)^2}{2}$$

$$= \frac{(-1)^k (k+1)[-k+2k+2]}{2}$$

$$= \frac{(-1)^k (k+1)(k+2)}{2}$$

$$= \frac{(-1)^{\overline{k+1}-1} \left(\overline{k+1}\right) \left(\overline{k+1}+1\right)}{2}$$

Thus $S(k+1)$ is true if $S(k)$ is true.

Therefore condition 2 is satisfied.

Hence $S(n)$ is true for all positive integer n.

Q.24 Use mathematical induction to prove that the following formula for every positive integer n.

$$\mathbf{1^3 + 3^3 + 5^3 + \ldots\ldots + (2n-1)^3 = n^2[2n^2 - 1]}$$

Solution:

Let $S(n)$ be the given statement, that is

$S(n)$: $1^3 + 3^3 + 5^3 + \ldots\ldots + (2n-1)^3 = n^2[2n^2 - 1]$

Condition 1

When $n = 1$, $S(n)$ becomes

$S(1)$: $1^3 = 1[2-1]$

$1 = 1$

$\Rightarrow$ L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that $S(n)$ is true for $n = k \in N$ that is

$$1^3 + 3^3 + 5^3 + \ldots + (2k-1)^3 = k^2[2k^2 - 1] \qquad \ldots\ldots\ldots (1)$$

Adding $[2(k+1)-1]^3$ on both sides

$$1^3 + 3^3 + 5^3 + \ldots\ldots\ldots + (2k-1)^3 + [2(k+1)-1]^3$$

$$= k^2[2k^2 - 1] + [2k+1]^3$$

$$= 2k^4 - k^2 + 8k^3 + 1 + 12k^2 + 6k$$

$$= 2k^4 + 8k^3 + 11k^2 + 6k + 1$$

$$= 2k^4 + 2k^3 + 6k^3 + 6k^2 + 5k^2 + 5k + k + 1$$

$$= 2k^3 (k + 1) + 6k^2 (k + 1) + 5k (k + 1) + 1 (k + 1)$$

$$= (k + 1) \left[2k^3 + 6k^2 + 5k + 1\right]$$

$$= (k + 1) \left[2k^3 + 2k^2 + 4k^2 + 4k + k + 1\right]$$

$$= (k + 1) \left[2k^3 (k + 1) + 4k (k + 1) + (k + 1)\right]$$

$$= (k + 1) (k + 1) (2k^2 + 4k + 1)$$

$$= (k + 1)^2 (2k^2 + 4k + 2 - 1)$$

$$= (k + 1)^2 \left[2 (k^2 + 2k + 1) - 1\right]$$

$$= \left(\overline{k + 1}\right)^2 \left[2 \left(\overline{k + 1}\right)^2 - 1\right]$$

Thus $S (k + 1)$ is true when $S (k)$ is true.

Therefore condition 2 is satisfied.

Hence $S (n)$ is true for every positive integer.

Q.25 Use mathematical induction to prove that the following formula for every positive integer n.

$(x + 1)$ is a factor of $x^{2n} - 1$

Solution:

Let $S (n)$ be the given statement, that is

$S (n) = x + 1$ is a factor of $x^{2n} - 1$

Condition 1

When $n = 1$, $S (n)$ becomes

$S (1) = x^2 - 1 = (x + 1) (x - 1)$

Clearly $x + 1$ is a factor of $x^2 - 1$.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that $S (n)$ is true for $n = k \in N$, i.e.

$S (k)$; $x + 1$ is a factor of $x^{2k} - 1$ .... (1)

We want to prove that $S (k + 1)$ has a factor $x + 1$

$S (k + 1)$; $x^{2 (k + 1)} - 1 = x^{2k + 2} - 1 = x^{2k} x^2 - 1$

Adding & subtracting $x^2 . 1$

$$= x^{2k} x^2 - x^2 - 1 + x^2 . 1 - 1 . 1$$

$$= x^2 (x^{2k} - 1) + 1 (x^2 - 1)$$

$$= x^2 (x^{2k} - 1) + 1 (x - 1) (x + 1)$$

The first term has a factor $x + 1$ by assumption 1 & 2nd term clearly has a factor $x + 1$. Thus the whole term has a factor $x + 1$. Thus $S (k + 1)$ has a factor $x + 1$. Therefore condition 2 is satisfied. Thus $S (n)$ is true for all +ve integral values of n.

Q.26 Use mathematical induction to prove that the following formula for every positive integer n.

$(x - y)$ is a factor of $x^n - y^n$

Solution:

Let $S(n)$ be the given statement, that is

$S(n) = x - y$ is a factor of $x^n - y^n$

Condition 1

When $n = 1$, $S(n)$ becomes

$S(1) = x - y$ is a factor of $x^n - y^n$.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that $S(n)$ is true for $n = k \in N$ i.e.

$S(k)$; $x - y$ is a factor of $x^k - y^k$ .... (1)

We want to prove that $S(k+1)$ has a factor $x - y$.

$S(k+1)$; $x^{k+1} - y^{k+1}$

$x^k x - y^k y$

Subtracting & adding $x y^k = x^k x - x y^k + x y^k - y^k y = x(x^k - y^k) + y^k (x - y)$

The first term has a factor $x - y$ by substitution (1).

The 2nd term clearly has a factor $x - y$. Thus $S(k+1)$ has a factor $x - y$. Therefore condition 2 is satisfied. Hence $S(n)$ is true for all +ve integers.

Q.27 Use mathematical induction to prove that the following formula for every positive integer n.

$x + y$ is a factor of $x^{2n-1} + y^{2n-1}$.

Solution:

Let $S(n)$ be the given statement, i.e.

$S(n)$; $x + y$ is a factor of $x^{2n-1} + y^{2n-1}$

Condition 1

When $n = 1$, $S(n)$ becomes

$S(1)$ $x + y$ is a factor of $x^{2-1} + y^{2-1} = x + y$ (True)

Thus condition 1 is satisfied.

Let us suppose that $S(n)$ is true for $n = k \in N$ i.e.

$S(k)$; $x + y$ is a factor of $x^{2k-1} + y^{2k-1}$ .... (1)

We want to prove that $S(k+1)$ has a factor $x + y$

$$S(k+1);\ x^{2(k+1)-1} + y^{2(k+1)-1} = x^{2k+2-1} + y^{2k+2-1}$$
$$= x^{2k-1}x^2 + y^{2k-1}y^2$$

Adding $x^2 y^{2k-1}$ & subtracting

$$= x^{2k-1}x^2 + x^2y^{2k-1} - x^2y^{2k-1} + y^{2k-1}y^2$$
$$= x^2(x^{2k-1} + y^{2k-1}) - y^{2k-1}(x^2 - y^2)$$

The first term has a factor $x + y$ by (1). The second term clearly has a factor $x + y$. Thus $S(k+1)$ has a factor $x + y$. Therefore condition 2 is satisfied. Therefore $S(n)$ is true for all +ve integer n.

Q.28 Use mathematical induction to show that

$1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1$ for all non–negative integers n.

Solution:

Let $S(n)$ be the statement, i.e.

$S(n)$: $1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1$

Condition 1

When $n = 0$, $S(n)$ becomes

$S(0)$: $1 = 2^0 - 1 = 2 - 1 = 1$

$\Rightarrow$ L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let us suppose that $S(n)$ is true for $n = k \in N$, i.e.

$S(k)$: $1 + 2 + 2^2 + ... + 2^k = 2^{k+1} - 1$ .... (1)

Adding 2^{k+1} on both sides

$$1 + 2 + 2^2 + ... + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$
$$= 2\,2^{k+1} - 1$$
$$= 2^{k+1+1} - 1$$

Therefore condition 2 is satisfied.

Thus $S(k+1)$ is true when $S(k)$ is true.

Hence $S(n)$ is true for all non – negative integer n.

Q.29 If A and B are two matrices and $A B = B A$, then show by mathematical induction that $AB^n = B^nA$ for any positive integers.

Solution:

Let $S(n)$ the given statement, i.e.

$S(n)$: $A . B^n = B^n A$ for any +ve integer.

Condition 1

When $n = 1$; $S(n)$ becomes

$S(1)$: $A B = B A$

$AB = AB$ (Given)

L.H.S. = R.H.S.

Therefore condition 1 is satisfied.

Condition 2

Let $S(n)$ be true for $n = k \in N$, that is $S(k)$: $A B^k = B^k . A$ .... (1)

We want to prove that $S(k + 1)$ is also true for that post multiply by B, we have

$A B^k B = B^k A B$

$A B^{k+1} = B^k BA$ (Given $AB = BA$)

$AB^{\overline{k+1}} = B^{\overline{k+1}} A$

Thus $S(k + 1)$ is true, when $S(k)$ is true. Therefore condition 2 is satisfied.

Hence $S(n)$ is true for any +ve integer.

Q.30 Prove that by principle of mathematical induction that $n^2 - 1$ is divisible by 8 when n is an odd positive integer.

Solution:

Let $S(n)$ be the given statement i.e.

$S(n)$; $n^2 - 1$ is divisible by 8.

Condition 1

When $n = 1$, we have

$S(1) = 1^2 - 1 = 1 - 1 = 0$ that is divisible by 8.

Therefore condition 1 is satisfied.

$S(k)$; $k^2 - 1$ is divisible by 8. .... (1)

We want to prove that S(k + 1) is also true.

$S(k+1)\ ;\ (k+1)^2 - 1$

$S(k+2) = (k+1+1)^2 - 1 = (k+2)^2 - 1 = k^2 + 4k + 4 - 1$

$= k^2 + 4k + 4 - 1 = (k^2 - 1) + 4(k+1)$ .... (2)

Clearly $(k^2 - 1)$ is divisible by 8 by (1). As is odd +ve integer, so k + 1 is an even integer. Hence k + 1 = 2m (say) where $m \in z^+$. Therefore (2) become $(k^2 - 1) + 4(2m)$.

$= (k^2 - 1) + 8m$

Therefore now 8m is also clearly divisible by 8. Thus S(k + 2) is true when S(k) is true. Therefore condition (2) is satisfied. Hence S(n) is true.

Q.31 Use mathematical induction to prove that $l\text{n}x^n = nl\text{n}x$, for any integer $n \geq 0$, if x is +ve number.

Solution:

Let S(n) be the given statement that is S(n) ; $l\text{n}x^n = nl\text{n}x$

Condition 1:

When, n = 0 S(n) becomes

$S(0)\ ;\ l\text{n}x^0 = 0\ l\text{n}x$

$l\text{n}1 = 0\ .\ l\text{n}x$

$0 = 0 \Rightarrow$ L.H.S = R.H.S

Therefore condition 1 is satisfied for $n \geq 0$.

Condition 2:

Let the given statement S (n) be true for $n = k \in N$ i.e.

$S(k);\ \ln x^k = k \ln x$ .... (1)

Adding ln x on both sides

$\ln x^k + \ln x = k \ln x + \ln x$

$\ln x^k x = (k+1) \ln x$

$\ln x^{\overline{k+1}} = (\overline{k+1}) \ln x$

Thus S (k + 1) is true when S (k) is true.

Therefore condition 2 is satisfied.

Since both conditions are satisfied. Therefore S (n) is true for any integer $n \geq 0$.

Q.32 Use the principle of extended mathematical induction to prove that $n! > 2^n - 1$ for integral values of $n \geq 4$.

Solution:

Let S (n) be the given statement, that is

$S(n):\ n! > 2^n - 1$ for integral values of $n \geq 4$.

Condition 1

When $n = 4$, S (n) becomes

$S(4):\ 4! > 2^4 - 1$

$24 > 16 - 1$

$24 > 15$ which is true.

Therefore condition 1 is satisfied for $n \geq 4$.

Condition 2

Let us suppose that S (n) is true for $n = k$

$S(k):\ k! > 2^k - 1$ for integral values of $k \geq 4$ .... (1)

Now multiplying throughout by $k + 1$

$(k+1)\,k! > (k+1)(2^k - 1)$

$(k+1)! > (k-1+2)(2^k - 1)$

$> (k-1)(2^k - 1) + 2(2^k - 1)$

$> (k-1)(2^k - 1) + 2^{k+1} - 2$

$> 2^{k+1} - 1 + \left[(k-1)(2^k - 1) - 1\right]$

$\Rightarrow\ (\overline{k+1})! > 2^{k+1} - 1$ $\because$ As $\left[(k-1)(2^k - 1) - 1\right]$ is +ve so ignore it.

Thus $S(k+1)$ is true, when $S(k)$ is true.

Therefore condition 2 is satisfied. Hence S (n) is true for all the integral values of $n \geq 4$.

Q.33 $n^2 > n + 3$, for integral values of $n \geq 3$. (Gujranawala Board, 2003)

Solution:

Let S (n) be the given statement, that is

$S(n);\ n^2 > n + 3$ for integral values of $n \geq 3$.

Condition 1

When $n = 3$, S (n) becomes

$S(3);\ 3^2 > 3 + 3$

$9 > 6$ which is true. Therefore condition 1 is satisfied for $n \geq 3$.

Condition 2

Let us suppose that S (n) is true for n = k

S (k); $k^2 > k + 3$ for integral values of $k \geq 3$ .... (1)

Adding $2k + 1$, throughout

$$k^2 + 2k + 1 > k + 3 + 2k + 1$$

$$(k + 1)^2 > k + 1 + 3 + (2k)$$

$$(\overline{k + 1})^2 > \overline{k + 1} + 3 \qquad (\because \text{ As } 2k \text{ is +ve, so ignore it})$$

Thus S (k + 1) is true, when S (k) is true.

Therefore condition 2 is satisfied. Therefore S (n) is true for $n \geq 3$.

Q.34 $4^n > 3^n + 2^{n-1}$, for integral values of $n \geq 2$. (Lahore Board, 2003)

Solution:

Let S (n) be the given statement, that is

S (n): $4^n > 3^n + 2^{n-1}$, for integral values of $n \geq 2$.

Condition 1

When n = 2 then S (n) becomes

S (2): $4^2 > 3^2 + 2^{2-1}$

$16 > 9 + 2$

$16 > 11$ which is true, therefore condition 1 is true for $n \geq 2$.

Condition 2

Let us suppose that S (n) is true for n = k

S (k); $4^k > 3^k + 2^{k-1}$ for integral values of $k \geq 2$ .... (1)

Multiplying throughout by 4

$$44^k > 4.3^k + 4.2^{k-1}$$

$$4^{k+1} > (3 + 1)\, 3^k + (2 + 2)\, 2^{k-1}$$

$$> 3^{k+1} + 3^k + 2^{k+1-1} + 2^{k-1+1}$$

$$> 3^{k+1} + 2^{k+1-1} + (3^k + 2^{k-1+1})$$

$$4^{\overline{k+1}} > 3^{\overline{k+1}} + 2^{k+1-1} \qquad (\because \text{ As } 3^k + 2^k \text{ is +ve so ignore it})$$

Thus S (k + 1) is true, when S (k) is true. Therefore condition 2 is satisfied.

Hence S (n) is true for the integral values of $n \geq 2$.

Q.35 $3^n < n!$ for integral values of $n > 6$.

Solution:

Let S (n) be the given statement, that is

S (n): $3^n < n!$ for integral values of $n > 6$

Condition 1

When $n = 7$, S (n) becomes

S (7); $3^7 < 7!$

$2187 < 5040$ which is true. Condition is satisfied.

Condition 2

Let us suppose that S (n) is true for $n = k$

S (k); $3^k < k!$ for integral values of $k > 6$ .... (1)

Multiplying throughout by $(k + 1)$

$(k + 1)\, 3^k < (k + 1)\, k!$

$(k - 2 + 3)\, 3^k < (k + 1)!$

$(k - 2)\, 3^k + 3^{k+1} < (k + 1)!$

$3^{\overline{k+1}} < (\overline{k+1})!$ $(\because$ As $(k - 2)\, 3^k$ is +ve, so ignore it$)$

Thus S (k + 1) is true, when S (k) is true. Therefore condition 2 is satisfied.

Since both conditions are satisfied.

Therefore, S (n) is true for integral values of $n > 6$.

Q.36 $n! > n^2$ for integral values of $n \geq 4$.

Solution:

Let S (n) be the given statement, that is

S (n); $n! > n^2$ for integral values of $n \geq 4$

Condition 1

When $n = 4$, S (n) becomes

S (4); $4! > 4^2$

$24 > 16$ which is true. Therefore condition 1 is satisfied for $n \geq 4$.

Condition 2

Let us suppose that S (n) is true for $n = k$.

S (k); $k! > k^2$ for integral values of $k \geq 4$ .... (1)

Multiplying throughout by $k + 1$

$$(k+1)\,k! > (k+1)\,k^2$$

$$(k+1)! > k^3 + k^2$$

$$(k+1)! > k^2 + 2k + 1 + (k^3 - 2k - 1)$$

$$(\overline{k+1})! > (\overline{k+1})^2 \qquad (\because \text{ As } k^3 - 2k - 1 \text{ is +ve so ignore it})$$

Thus $S(k+1)$ is true, when $S(k)$ is true.

Therefore condition 2 is satisfied. Hence $S(n)$ is true for integral values of $n \geq 4$.

Q.37 $3 + 5 + 7 + \ldots\ldots + (2n + 5) = (n + 2)(n + 4)$ for integral values of $n \geq -1$.

Solution:

Let $S(n)$ be the given statement, that is

$S(n)$: $3 + 5 + 7 + \ldots\ldots + (2n + 5) = (n + 2)(n + 4)$, for integral values of $n \geq -1$

Condition 1

When $n = -1$, $S(n)$ becomes

$S(-1)$; $3 = (-1 + 2)(-1 + 4)$

$3 = 3$

$\Rightarrow$ L.H.S. = R.H.S. Therefore condition 1 is satisfied for $n \geq -1$.

Condition 2

Let us suppose that statement is true for $n = k$.

$$3 + 5 + 7 + \ldots\ldots + (2k + 5) = (k + 2)(k + 4) \qquad \ldots\ldots\ldots (1)$$

Adding $(2(k+1) + 5)$ on both sides

$$\begin{aligned} 3 + 5 + 7 + \ldots\ldots (2k + 5)(2k + 7) &= (k + 2)(k + 4) + (2k + 7) \\ &= k^2 + 4k + 2k + 8 + 2k + 7 \\ &= k^2 + 8k + 15 \\ &= k^2 + 5k + 3k + 15 \\ &= (k + 5)(k + 3) \\ &= (\overline{k+1} + 4)(\overline{k+1} + 2) \end{aligned}$$

Condition 2 is satisfied. Thus $S(k+1)$ is true when $S(k)$ is true.

Hence $S(n)$ is true for integral values of $n \geq -1$.

Q.38 $1 + nx \le (1 + x)^n$ for $n \ge 2$ & $x > -1$

Solution:

Let S (n) be the given statement, i.e.

S (n): $1 + nx \le (1 + x)^n$ for $n \ge 2$ & $x > -1$

Condition 1

When n = 2, S (n) becomes

S (2); $1 + 2x \le (1 + x)^2$

$1 + 2x \le 1 + x^2 + 2x$, which is true.

Therefore condition 1 is satisfied for $n \ge 2$ and $x > -1$.

Condition 2

Let us suppose that S (n) is true for n = k

S (k); $1 + kx \le (1 + x)^k$ for $k \ge 2$ and $x > -1$ .... (1)

Multiplying throughout by $(1 + x)$

$(1 + kx)(1 + x) \le (1 + x)^k (1 + x)$

$1 + x + kx + kx^2 \le (1 + x)^{k+1}$

$1 + (k + 1)x + kx^2 \le (1 + x)^{k+1}$

$1 + (\overline{k + 1})x \le (1 + x)^{\overline{k+1}}$ ($\because$ kx^2 is +ve, so ignore it)

Thus S (k + 1) is true when S (k) is true.

Therefore condition 2 is satisfied.

Therefore S (n) is true for all $n \ge 2$ and $x > -1$.

BINOMIAL THEOREM (Definition Lahore Board 2010)

An algebraic expression consisting of two terms such as $a + x$, $x - 2y$, $ax + b$ etc. is called a binomial or binomial expression.

We know that $(a + x)^2 = a^2 + 2ax + x^2$

The R.H.S. is called **binomial expansion** and 2 is called index.

FORMULA OF BINOMIAL THEOREM (Lahore Board 2009–11)

$$(a + x)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} x + \binom{n}{2} a^{n-2} x^2 + \ldots\ldots + \binom{n}{r-1} a^{n-r+1} x^{r-1} + \binom{n}{r} a^{n-r} x^r + \ldots\ldots + \binom{n}{n} x^n$$

it can briefly written as

$$(a + x)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} x^r$$

NOTE:

(i) The rule or formula of a binomial for expansion raised to any positive integral power n.

(ii) It is finite series.

(iii) Number of terms in the expansion of $(a + x)^n$ is $n + 1$.

(iv) $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots\ldots, \binom{n}{n}$ are called binomial coefficients.

(v) $\binom{n}{0}, \binom{n}{2}, \binom{n}{4}, \ldots\ldots \binom{n}{n}$ are called even binomial coefficients.

(vi) $\binom{n}{1}, \binom{n}{3}, \binom{n}{5}, \ldots\ldots \binom{n}{n-1}$ are called odd binomial coefficients.

SUM OF BIN BINOMIAL COEFFICIENTS

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots\ldots + \binom{n}{n} = 2^n$$

SUM OF EVEN BINOMIAL COEFFICIENTS

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots\ldots + \binom{n}{n} = 2^{n-1}$$

SUM OF ODD BINOMIAL COEFFICIENTS

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots\ldots + \binom{n}{n-1} = 2^{n-1}$$

REMARK

Sum of even binomial coefficients = Sum of odd binomial coefficients.

EXERCISE 8.2

Q.1 Using binomial theorem, expand the following:

(i) $(a + 2b)^5$ **(ii)** $\left(\frac{x}{2} - \frac{2}{x^2}\right)^6$

(iii) $\left(3a - \frac{x}{3a}\right)^4$ **(iv)** $\left(2a - \frac{x}{a}\right)^7$

(v) $\left(\frac{x}{2y} - \frac{2y}{x}\right)^8$ **(vi)** $\left[\sqrt{\frac{a}{x}} - \sqrt{\frac{x}{a}}\right]^6$

Solution:

(i) $(a + 2b)^5$

$$= \binom{5}{0} a^5 (2b)^0 + \binom{5}{1} a^4 (2b)^1 + \binom{5}{2} a^3 (2b)^2 + \binom{5}{3} a^2 (2b)^3 + \binom{5}{4} a^1 (2b)^4 + \binom{5}{5} a^0 (2b)^5$$

$$= a^5 + 5a^4 (2b) + 10a^3 (4b^2) + 10a^2 (8b^3) + 5a (16b^4) + 32b^5$$

$$= a^5 + 10a^4 b + 40 a^3 b^2 + 80 a^2 b^3 + 80 a b^4 + 32 b^5$$

(ii) $\left(\dfrac{x}{2} - \dfrac{2}{x^2}\right)^6$

$$= \binom{6}{0}\left(\frac{x}{2}\right)^6 - \binom{6}{1}\left(\frac{x}{2}\right)^{6-1}\left(\frac{2}{x^2}\right)^1 + \binom{6}{2}\left(\frac{x}{2}\right)^{6-2}\left(\frac{2}{x^2}\right)^2 - \binom{6}{3}\left(\frac{x}{2}\right)^{6-3}\left(\frac{2}{x^2}\right)^3$$

$$+ \binom{6}{4}\left(\frac{x}{2}\right)^{6-4}\left(\frac{2}{x^2}\right)^4 - \binom{6}{5}\left(\frac{x}{2}\right)^{6-5}\left(\frac{2}{x^2}\right)^5 + \binom{6}{6}\left(\frac{x}{2}\right)^{6-6}\left(\frac{2}{x^2}\right)^6$$

$$= \frac{x^6}{64} - 6\,\frac{x^5}{32} \times \frac{2}{x^2} + 15 \times \frac{x^4}{16} \times \frac{4}{x^4} - 20 \times \frac{x^3}{8} \times \frac{8}{x^6} + 15 \times \frac{x^2}{4} \times \frac{16}{x^8} - 6 \times \frac{x}{2} \times \frac{32}{x^{10}} + \frac{64}{x^{12}}$$

$$= \frac{x^6}{64} - \frac{3}{8}x^3 + \frac{15}{4} - \frac{20}{x^3} + \frac{60}{x^6} - \frac{96}{x^9} + \frac{64}{x^{12}}$$

(iii) $\left(3a - \dfrac{x}{3a}\right)^4$

$$\binom{4}{0}(3a)^4\left(\frac{x}{3a}\right)^0 - \binom{4}{1}(3a)^{4-1}\left(\frac{x}{3a}\right)^1 + \binom{4}{2}(3a)^{4-2}\left(\frac{x}{3a}\right)^2 - \binom{4}{3}(3a)^{4-3}\left(\frac{x}{3a}\right)^3$$

$$+ \binom{4}{4}(3a)^{4-4}\left(\frac{x}{3a}\right)^4$$

$$= 81a^4 - 4 \times 27\,a^3 \times \frac{x}{3a} + 6 \times 9a^2 \times \frac{x^2}{9a^2} \times \frac{x^2}{9a^2} - 4 \times 3a \times \frac{x^3}{27a^3} + \frac{x^4}{81a^4}$$

$$= 81a^4 - \frac{108}{3}a^2x + 6x^2 - \frac{4x^3}{9a^2} + \frac{x^4}{81a^4}$$

$$= 81a^4 - 36a^2x + 6x^2 - \frac{4x^3}{9a^2} + \frac{x^4}{81a^4}$$

(iv) $\left(2a - \dfrac{x}{a}\right)^7$

$$= \binom{7}{0}(2a)^7 - \binom{7}{1}(2a)^6\left(\frac{x}{a}\right)^1 + \binom{7}{2}(2a)^5\left(\frac{x}{a}\right)^2 - \binom{7}{3}(2a)^4\left(\frac{x}{a}\right)^3$$

$$+ \binom{7}{4}(2a)^3\left(\frac{x}{a}\right)^4 - \binom{7}{5}(2a)^2\left(\frac{x}{a}\right)^5 + \binom{7}{6}(2a)^1\left(\frac{x}{a}\right)^6 - \binom{7}{7}(2a)^0\left(\frac{x}{a}\right)^7$$

$$= 128a^7 - 7\,(64a^6)\left(\frac{x}{a}\right) + 21 \times 32a^5\,\frac{x^2}{a^2} - 35\,(16a^4)\left(\frac{x^3}{a^3}\right)$$

$$+ 35\,(8a^3)\left(\frac{x^4}{a^4}\right) - 21\,(4a^2)\left(\frac{x^5}{a^5}\right) + 7\,(2a)\left(\frac{x^6}{a^6}\right) - \frac{x^7}{a^7}$$

$$= 128a^7 - 448\,x\,a^5 + 672\,a^3x^2 - 560\,a\,x^3 + \frac{280\,x^4}{a} - \frac{84\,x^5}{a^3} + \frac{14x^6}{a^5} - \frac{x^7}{a^7}$$

(v) $\left(\dfrac{x}{2y}-\dfrac{2y}{x}\right)^8$

$$= \binom{8}{0}\left(\frac{x}{2y}\right)^8 - \binom{8}{1}\left(\frac{x}{2y}\right)^7\left(+\frac{2y}{x}\right) + \binom{8}{2}\left(\frac{x}{2y}\right)^6\left(\frac{2y}{x}\right)^2 - \binom{8}{3}\left(\frac{x}{2y}\right)^5\left(\frac{2y}{x}\right)^3$$

$$+ \binom{8}{4}\left(\frac{x}{2y}\right)^4\left(\frac{2y}{x}\right)^4 - \binom{8}{5}\left(\frac{x}{2y}\right)^3\left(\frac{2y}{x}\right)^5 + \binom{8}{6}\left(\frac{x}{2y}\right)^2\left(\frac{2y}{x}\right)^6$$

$$- \binom{8}{7}\left(\frac{x}{2y}\right)^1\left(\frac{2y}{x}\right)^7 + \binom{8}{8}\left(\frac{x}{2y}\right)^0\left(\frac{2y}{x}\right)^8$$

$$= \frac{x^8}{256\,y^8} - 8\,\frac{x^7}{128\,y^7} \times \frac{2y}{x} + 28\left(\frac{x^6}{64y^6}\right)\left(\frac{4y^2}{x^2}\right) - 56\left(\frac{x^5}{32y^5}\right)\left(\frac{8y^3}{x^3}\right)$$

$$+ 70\left(\frac{x^4}{16y^4}\right)\left(\frac{16y^4}{x^4}\right) - 56\left(\frac{x^3}{8y^3}\right)\left(\frac{32y^5}{x^5}\right) + 28\,\frac{x^2}{4y^2} \times \frac{64y^6}{x^6} - 8 \times \frac{x}{2y} \times \frac{128y^7}{x^7} + \frac{256\,y^8}{x^8}$$

$$= \frac{x^8}{256y^8} - \frac{x^6}{8y^6} + \frac{7x^4}{16y^4} - 14\,\frac{x^2}{y^2} + 70 - \frac{224y^2}{x^2} + \frac{448y^4}{x^4} - \frac{512y^6}{x^6} + \frac{256y^8}{x^8}$$

(vi) $\left[\sqrt{\dfrac{a}{x}} - \sqrt{\dfrac{x}{a}}\right]^6$

$$= \binom{6}{0}\left(\sqrt{\frac{a}{x}}\right)^6 - \binom{6}{1}\left(\sqrt{\frac{a}{x}}\right)^{6-1}\left(\sqrt{\frac{x}{a}}\right)^1$$

$$+ \binom{6}{2}\left(\sqrt{\frac{a}{x}}\right)^{6-2}\left(\sqrt{\frac{x}{a}}\right)^2 - \binom{6}{3}\left(\sqrt{\frac{a}{x}}\right)^{6-3}\left(\sqrt{\frac{x}{a}}\right)^3$$

$$+ \binom{6}{4}\left(\sqrt{\frac{a}{x}}\right)^{6-4}\left(\sqrt{\frac{x}{a}}\right)^4 - \binom{6}{5}\left(\sqrt{\frac{a}{x}}\right)^{6-5}\left(-\sqrt{\frac{x}{a}}\right)^5 + \binom{6}{6}\left(\sqrt{\frac{a}{x}}\right)^0\left(\sqrt{\frac{x}{a}}\right)^6$$

$$= \left(\sqrt{\frac{a}{x}}\right)^6 - 6\left(\sqrt{\frac{a}{x}}\right)^5\left(\sqrt{\frac{a}{x}}\right)^{-1} + 15\left(\sqrt{\frac{a}{x}}\right)^4\left(\sqrt{\frac{a}{x}}\right)^{-2}$$

$$- 20\left(\sqrt{\frac{a}{x}}\right)^3\left(\sqrt{\frac{a}{x}}\right)^{-3} + 15\left(\sqrt{\frac{a}{x}}\right)^2\left(\sqrt{\frac{a}{x}}\right)^{-4} - 6\left(\sqrt{\frac{a}{x}}\right)\left(\sqrt{\frac{a}{x}}\right)^{-5} + \left(\sqrt{\frac{x}{a}}\right)^6$$

$$= \left(\sqrt{\frac{a}{x}}\right)^6 - 6\left(\sqrt{\frac{a}{x}}\right)^4 + 15\left(\sqrt{\frac{a}{x}}\right)^2 - 20\left(\sqrt{\frac{a}{x}}\right)^0$$

$$+ 15\left(\sqrt{\frac{a}{x}}\right)^{-2} - 6\left(\sqrt{\frac{a}{x}}\right)^{-4} + \left(\sqrt{\frac{x}{a}}\right)^6$$

$$= \left(\frac{a}{x}\right)^{6/2} - 6\left(\frac{a}{x}\right)^{4/2} + 15\left(\frac{a}{x}\right) - 20 + 15\left(\frac{a}{x}\right)^{-2/2} - 6\left(\frac{a}{x}\right)^{-4/2} + \left(\frac{x}{a}\right)^{6/2}$$

$$= \frac{a^3}{x^3} - 6\,\frac{a^2}{x^2} + 15\,\frac{a}{x} - 20 + 15\left(\frac{a}{x}\right)^{-1} - 6\left(\frac{a}{x}\right)^{-2} + \left(\frac{x}{a}\right)^3$$

$$= \frac{a^3}{x^3} - 6\,\frac{a^2}{x^2} + 15\,\frac{a}{x} - 20 + 15\,\frac{x}{a} - 6\,\frac{x^2}{a^2} + \frac{x^3}{a^3}$$

Q.2 Calculate the following by means of binomial theorem.

(i) $(0.97)^3$ **(Lahore Board 2010)**

(ii) $(2.02)^4$ **(Lahore Board 2011)**

(iii) $(9.98)^4$

(iv) $(2.9)^5$

Solution:

(i) $(0.97)^3 = (1 - 0.03)^3$

$$= \binom{3}{0}(1)^3 - \binom{3}{1}(1)^2(.03)^1 + \binom{3}{2}(1)^1(.03)^2 - \binom{3}{3}(1)^0(.03)^3$$

$$= 1 - 0.09 + 0.0027 - 0.000027$$

$$= 0.9127$$

(ii) $(2.02)^4 = (2 + 0.02)^4$

$$= \binom{4}{0}(2)^4 + \binom{4}{1}(2)^3(.02)^1 + \binom{4}{2}(2)^2(.02)^2 + \binom{4}{3}(2)^1(.02)^3 + \binom{4}{4}(2)^0(.02)^4$$

$$= 16 + 4(8)(.02) + 6(4)(0.0004) + 4(2)(0.000008) + 0.00000016$$

$$= 16.64 + 0.0096 + 0.000064$$

$$= 16.64$$

(iii) $(9.98)^4 = (10 - 0.02)^4$

$$= \binom{4}{0}(10)^4(.02)^0 - \binom{4}{1}(10)^3(.02)^1 + \binom{4}{2}(10)^2(.02)^2 - \binom{4}{3}(10)^1(.02)^3 + \binom{4}{4}(10)^0(.02)^4$$

$$= 10000 - 80 + 600(0.0004) - 40(0.000008) + 0.00000016$$

$$= 9920.24$$

(iv) $(2.9)^5 = (3 - 0.1)^5$

$$= \binom{5}{0}(3)^5 - \binom{5}{1}(3)^4(.01) + \binom{5}{2}(3)^3(.01)^2 - \binom{5}{3}(3)^2(.01)^3 + \binom{5}{4}(3)^1(.01)^4 - \binom{5}{5}(3)^0(.01)^5$$

$$= 243 - 4.05 + 10(27)(0.0001) - 10(9)(0.000001) + 15(0.00000001) - 0.0000000001$$

$$= 24.3 + 5 \times 81 - 0.01 + 10 \times 8 \times 0.0001$$

$$= 205.2$$

Q.3 Expand and simplify the following:

(i) $(a+\sqrt{2}\,x)^4 + (a-\sqrt{2}\,x)^4$

(ii) $(2+\sqrt{3})^5 + (2-\sqrt{3})^5$

(iii) $(2+i)^5 - (2-i)^5$

(iv) $(x+\sqrt{x^2-1})^3 + (x-\sqrt{x^2-1})^3$

Solution:

(i) $(a+\sqrt{2}\,x)^4 + (a-\sqrt{2}\,x)^4$

$$= \binom{4}{0}a^4 + \binom{4}{1}(a)^3(\sqrt{2}\,x)^1 + \binom{4}{2}(a)^2(\sqrt{2}\,x)^2 + \binom{4}{3}a(\sqrt{2}\,x)^3 + \binom{4}{4}a^0(\sqrt{2}\,x)^4$$

$$(a+\sqrt{2}\,x)^4 = a^4 + 4a^3\sqrt{2}\,x + 6a^2(\sqrt{2}\,x)^2 + 4a(\sqrt{2}\,x)^3 + (\sqrt{2}\,x)^4 \quad \text{......... (i)}$$

$$(a-\sqrt{2}\,x)^4 = a^4 - 4a^3\sqrt{2}\,x + 6a^2(\sqrt{2}\,x)^2 - 4a(\sqrt{2}\,x)^3 + (\sqrt{2}\,x)^4 \quad \text{......... (ii)}$$

By adding (i) and (ii)

$$(a+\sqrt{2}\,x)^4 + (a-\sqrt{2}\,x)^4 = 2a^4 + 12a^2(\sqrt{2}\,x)^2 + 2(\sqrt{2}\,x)^4$$

$$= 2a^4 + 12a^2(2x^2) + 2(4x^4)$$

$$= 2a^4 + 24a^2x^2 + 8x^4$$

(ii) $(2+\sqrt{3})^5 + (2-\sqrt{3})^5$

$$(2+\sqrt{3})^5 = \binom{5}{0}(2)^5 + \binom{5}{1}(2)^4(\sqrt{3})^1 + \binom{5}{2}(2)^3(\sqrt{3})^2 + \binom{5}{3}(2)^2(\sqrt{3})^3$$

$$+ \binom{5}{4}(2)(\sqrt{3})^4 + \binom{5}{5}(2)^0(\sqrt{3})^5$$

$$= 32 + 5\times 16\sqrt{3} + 10\times 8(\sqrt{3})^2 + 10\times 4(\sqrt{3})^3 + 5(2)(\sqrt{3})^4 + (\sqrt{3})^5$$

$$= 32 + 80\sqrt{3} + 80(\sqrt{3})^2 + 40(\sqrt{3})^3 + 10(\sqrt{3})^4 + (\sqrt{3})^5$$

$$(2-\sqrt{3})^5 = 32 - 80\sqrt{3} + 80(\sqrt{3})^2 - 40(\sqrt{3})^3 + 10(\sqrt{3})^4 - (\sqrt{3})^5$$

Adding

$$(2+\sqrt{3})^5 + (2-\sqrt{3})^5 = 64 + 480 + 180 = 724$$

(iii) $(2+i)^5 - (2-i)^5$

$$(2+i)^5 = \binom{5}{0}(2)^5 + \binom{5}{1}(2)^4(i) + \binom{5}{2}(2)^3(i)^2 + \binom{5}{3}(2)^2(i)^3$$

$$+ \binom{5}{4}(2)^1(i)^4 + \binom{5}{5}(2)^0(i)^5$$

$$= 32 + 5 \times 16\,(i) + 10 \times 8 \times (i)^2 + 10 \times 4\,(i)^3 + 5 \times 2\,(i)^4 + i^5$$

$$= 32 + 80i + 80i^2 + 40i^3 + 10i^4 + i^5$$

$$(2 - i)^5 = 32 - 80i + 80i^2 - 40i^3 + 10i^4 - i^5$$

$$- \quad + \quad - \quad + \quad - \quad +$$

Subtracting

$$(2 + i)^5 - (2 - i)^5 = 160i + 80i^3 + 2i^5$$

$$= 160i - 80i + 2i = 82i$$

(iv) $\left(x + \sqrt{x^2 - 1} \right)^3 + \left(x - \sqrt{x^2 - 1} \right)^3$

First we take $\left(x + \sqrt{x^2 - 1}\right)^3$

$$= x^3 + \binom{3}{1}(x)^{3-1}\left(\sqrt{x^2-1}\right) + \binom{3}{2}(x)^1\left(\sqrt{x^2-1}\right)^2 + \binom{3}{3}(x)^0\left(\sqrt{x^2-1}\right)^3$$

$$\left(x + \sqrt{x^2 - 1} \right)^3 = x^3 + 3x^2\sqrt{x^2 - 1} + 3x\,(x^2 - 1) + \left(\sqrt{x^2 - 1}\right)^3$$

$$\left(x - \sqrt{x^2 - 1} \right)^3 = x^3 - 3x^2\sqrt{x^2 - 1} + 3x\,(x^2 - 1) - \left(\sqrt{x^2 - 1}\right)^3$$

Adding

$$\left(x + \sqrt{x^2 - 1} \right)^3 + \left(x - \sqrt{x^2 - 1} \right)^3 = 2x^3 + 6x\,(x^2 - 1)$$

$$= 2x^3 + 6x^3 - 6x$$

$$= 8x^3 - 6x$$

Q.4 Expand the following in ascending power of x

(i) $(2 + x - x^2)^4$ **(ii)** $(1 - x + x^2)^4$

(iii) $(1 - x - x^2)^4$

Solution:

(i) $(2 + x - x^2)^4$

Let $2 + x = y$

$$(y - x^2)^4 = \binom{4}{0}(y)^4(x^2)^0 - \binom{4}{1}(y^3)(x^2)^1 + \binom{4}{2}(y^2)(x^2)^2 - \binom{4}{3}(y)(x^2)^3 + \binom{4}{4}(y)^0(x^2)^4$$

$$= y^4 - 4y^3x^2 + 6y^2x^4 - 4y\,x^6 + x^8$$

Putting value $y = 2 + x$ again

$$= (2 + x)^4 - 4\,(2 + x)^3x^2 + 6\,(2 + x)^2x^4 - 4\,(2 + x)\,x^6 + x^8$$

$$= \left[\binom{4}{0}(2)^4 + \binom{4}{1}(2)^3(x) + \binom{4}{2}(2)^2(x)^2 + \binom{4}{3}(2)^1(x)^3 + \binom{4}{4}(2)^0(x)^4 \right]$$

$$- 4\,[8 + x^3 + 6x^2 + 12x]\,x^2 + 6\,(4 + x^2 + 4x)\,x^4 - (8 + 4x)\,x^6 + x^8$$

$$= 16 + 32x - 8x^2 - 40x^3 + x^4 + 20x^5 - 2x^6 - 4x^7 + x^8$$

(ii) $\mathbf{(1 - x + x^2)^4}$

Let $1 - x = y$

$$(y + x^2)^4 = \binom{4}{0} y^4 + \binom{4}{1} y^3 x^2 + \binom{4}{2} y^2 (x^2)^2 + \binom{4}{3} y (x^2)^3 + \binom{4}{4} y^0 (x^2)^4$$

$$= y^4 + 4y^3 x^2 + 6y^2 x^4 + 4y x^6 + x^8$$

Putting value of y

$$= (1-x)^4 + 4(1-x)^3 x^2 + 6(1-x)^2 x^4 + 4(1-x) x^6 + x^8$$

$$= \left[\binom{4}{0}(1)^4 (x)^0 - \binom{4}{1}(1)^3 (x)^1 + \binom{4}{2}(1)^2 (x)^2 - \binom{4}{3}(1)^1 (x)^3 + \binom{4}{4}(1)^0 (x)^4\right]$$

$$+ 4[1 - x^3 - 3x + 3x^2] x^2 + 6(1 + x^2 - 2x) x^4 + 4(x^6 - x^7) + x^8$$

$$= 1 - 4x + 6x^2 - 4x^3 + x^4 + 4x^2 - 12x^3 + 12x^4 - 4x^5 + 6x^4 - 12x^5 + 10x^6 - 4x^7 + x^8$$

$$= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 4x^5 + 10x^6 - 4x^7 + x^8$$

(iii) $\mathbf{(1 - x - x^2)^4}$

Let $1 - x = y$

$$(y - x^2)^4 = \binom{4}{0} y^4 - \binom{4}{1} (y^3) x^2 + \binom{4}{2} (y)^2 (x^2)^2 - \binom{4}{3} y (x^2)^3 + \binom{4}{4} y^0 (x^2)^4$$

$$= y^4 - 4y^3 x^2 + 6y^2 x^4 - 4x^6 y + x^8$$

Putting value of y

$$= (1-x)^4 - 4(1-x)^3 x^2 + 6(1-x)^2 x^4 - 4x^6 (1-x) + x^8$$

$$= \left[\binom{4}{0}(1)^4 - \binom{4}{1}(1)^3 (x) + \binom{4}{2}(1)^2 (x)^2 - \binom{4}{3}(1)(x)^3 + \binom{4}{4}(1)^0 (x)^4\right]$$

$$- 4[1 - x^3 - 3x + 3x^2] x^2 + 6(1 + x^2 - 2x) x^4 - 4(x^6 - x^7) + x^8$$

$$= 1 - 4x + 6x^2 - 4x^3 + x^4 - 4x^2(1 - 3x + 3x^2 - x^3) + 6x^4 - 12x^5 + 6x^6 - 4x^6 + 4x^7 + x^8$$

$$= 1 - 4x + 2x^2 + 8x^3 - 5x^4 - 8x^5 + 2x^6 + 4x^7 + x^8$$

Q.5 Expand the following in descending power of x

(i) $\mathbf{(x^2 + x - 1)^3}$ **(ii)** $\mathbf{\left(x - 1 - \frac{1}{x}\right)^3}$

Solution:

(i) $\mathbf{(x^2 + x - 1)^3}$

Let $x - 1 = y \Rightarrow (x^2 + y)^3$

$$= \binom{3}{0}(x^2)^3 + \binom{3}{1}(x^2)^2 (y) + \binom{3}{2}(x^2)(y)^2 + \binom{3}{3}(x^2)^0 (y)^3$$

$$(x^2 + y)^3 = x^6 + 3x^4 y + 3x^2 y^2 + y^3$$

Putting value of y

$$(x^2 + x - 1)^3 = x^6 + 3x^4(x - 1) + 3x^2(x - 1)^2 + (x - 1)^3$$
$$= x^6 + 3x^5 - 3x^4 + 3x^2(x^2 + 1 - 2x) + x^3 - 1 - 3x^2 + 3x$$
$$= x^6 + 3x^5 - 3x^4 + 3x^4 + 3x^2 - 6x^3 + x^3 - 1 - 3x^2 + 3x$$
$$= x^6 + 3x^5 - 5x^3 + 3x - 1$$

(i) $\left(x - 1 - \frac{1}{x}\right)^3$

Let $x - 1 = y \Rightarrow \left(y - \frac{1}{x}\right)^3$

$$= \binom{3}{0}(y^3) - \binom{3}{1}(y^2)\left(\frac{1}{x}\right) + \binom{3}{2}(y)\left(\frac{1}{x}\right)^2 - \binom{3}{3}(y)^0\left(\frac{1}{x}\right)^3$$
$$= y^3 - \frac{3y^2}{x} + \frac{3y}{x^2} - \frac{1}{x^3}$$

Putting value of y

$$\left(x - 1 - \frac{1}{x}\right)^3 = (x - 1)^3 - \frac{3(x - 1)^2}{x} + \frac{3(x - 1)}{x^2} - \frac{1}{x^3}$$
$$= x^3 - 1 - 3x^2 + 3x - \frac{3}{x}(x^2 + 1 - 2x) + \frac{1}{x^2}(3x - 3) - \frac{1}{x^3}$$
$$= x - 3x^2 + 5 - \frac{3}{x^2} - \frac{1}{x^3}$$

GENERAL TERM OF EXPANSION $(a + x)^n$

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

Q.6 Find the term involving:

(i) x^4 **in the expansion of** $(3 - 2x)^7$ **(Lahore Board 2008)**

(ii) x^{-2} **in the expansion of** $\left(x - \frac{2}{x^2}\right)^{13}$

(iii) a^4 **in the expansion of** $\left(\frac{2}{x} - a\right)^9$ **(Lahore Board 2004)**

(iv) y^3 **in the expansion of** $(x - \sqrt{y})^{11}$

Solution:

(i) x^4 **in the expansion of** $(3 - 2x)^7$

We know that general term formula is

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

Since $n = 7, \ a = 3, \ x = (-2x)$

$$T_{r+1} = \binom{7}{r} a^{7-r} (-2x)^r$$

$$T_{r+1} = \binom{7}{r} 3^{7-r} (-2)^r x^r \qquad \ldots\ldots\ldots (1)$$

We have to find term involving x^4, so comparing the powers of x, we have $r = 4$

Putting $r = 4$ in (1)

$$T_{4+1} = \binom{7}{4} 3^{7-4} (-2)^4 x^4$$

$$= 35 \times 27 \times 16x^4$$

$$T_{4+1} = 15120\, x^4$$

(ii) **x^{-2} in the expansion of $\left(x - \dfrac{2}{x^2}\right)^{13}$**

We know that general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$a = x, \ n = 13, \ x = \left(-\frac{2}{x^2}\right)$$

$$T_{r+1} = \binom{13}{r} (x)^{13-r} \left(-\frac{2}{x^2}\right)^r$$

$$= \binom{13}{r} x^{13-r-2r} (-2)^r$$

$$= \binom{13}{r} x^{13-3r} (-2)^r \qquad \ldots\ldots\ldots (1)$$

We have to find term involving x^{-2} so comparing the power of x in (1)

$$13 - 3r = -2$$

$$13 + 2 = 3r$$

$$15 = 3r$$

$$\boxed{r = 5}$$

Put in (1)

$$T_{5+1} = \binom{13}{5} x^{13-3(5)} (-2)^5$$

$$T_6 = 1287 \times x^{-2} \times -32 = -41184\, x^{-2}$$

(iii) a^4 in the expansion of $\left(\frac{2}{x} - a\right)^9$

We know that general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$n = 9, \; a = \frac{2}{x}, \; x = (-a)$$

$$T_{r+1} = \binom{9}{r}\left(\frac{2}{x}\right)^{9-r} (-a)^r$$

$$= \binom{9}{r}\left(\frac{2}{x}\right)^{9-r} (-1)^r a^r \qquad \ldots\ldots\ldots (1)$$

We have to find term involving a^4, so comparing the powers of a, we get

$$T_{4+1} = \binom{9}{4}\left(\frac{2}{x}\right)^{9-4} (-1)^4 (a)^4$$

$$= (126) \times \frac{2^5}{x^5} \times 1 \times a^4 = 126 \times \frac{32}{x^5} a^4$$

$$T_5 = 4032 \frac{a^4}{x^5}$$

(iv) y^3 in the expansion of $(x - \sqrt{y})^{11}$

$$a = x, \; x = (-\sqrt{y}), \; n = 11$$

We know that general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{r+1} = \binom{11}{r} (x)^{11-r} (-\sqrt{y})^r$$

$$= \binom{11}{r} x^{11-r} (-1)^r y^{r/2} \qquad \ldots\ldots\ldots (1)$$

We have to find term involving y^3, so comparing the powers of y we get

$$\frac{r}{2} = 3 \quad \Rightarrow \quad r = 6 \quad \text{Put in (1)}$$

$$T_{6+1} = \binom{11}{6} x^{11-6} (-1)^6 y^{6/2}$$

$$T_7 = 462\, x^5 \times 1 \times y^3$$

$$T_7 = 462\, x^5 y^3$$

Q.7 Find the coefficient of

(i) x^5 **in the expansion of** $\left(x^2 - \frac{3}{2x}\right)^{10}$

(ii) x^n **in the expansion of** $\left(x^2 - \frac{1}{x}\right)^{2n}$

Solution:

(i) x^5 **in the expansion of** $\left(x^2 - \frac{3}{2x}\right)^{10}$ **(Lahore Board 2003-04)**

We know that general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$n = 10,\ a = x^2,\ x = \left(-\frac{3}{2x}\right)$$

$$T_{r+1} = \binom{10}{r} (x^2)^{10-r} \left(-\frac{3}{2x}\right)^r$$

$$= \binom{10}{r} (x)^{20-2r} \left(-\frac{3}{2}\right)^r \frac{1}{x^r}$$

$$= \binom{10}{r} (x)^{20-2r-r} \left(-\frac{3}{2}\right)^r$$

$$= \binom{10}{r} (x)^{20-3r} \left(-\frac{3}{2}\right)^r \quad \ldots\ldots\ldots (1)$$

we have to find the coefficient of x^5, so comparing the powers of x, we get

$$20 - 3r = 5$$

$$15 = 3r \Rightarrow \boxed{r = 5}$$

Put in (1)

$$T_{5+1} = \binom{10}{5} x^{20-15} \left(-\frac{3}{2}\right)^5$$

$$T_6 = 252 \times x^5 \times \frac{-243}{32} = \frac{-15309}{8} x^5$$

Coefficient of x^5 is $\frac{-15309}{8}$

(ii) x^n **in the expansion of** $\left(x^2 - \frac{1}{x}\right)^{2n}$

We know that general term formula is

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{r+1} = \binom{2n}{r} (x^2)^{2n-r} \left(-\frac{1}{x}\right)^r$$

$$= \binom{2n}{r} (x)^{4n-2r} \frac{(-1)^r}{x^r}$$

$$= \binom{2n}{r} x^{4n-3r} (-1)^r \quad \ldots\ldots\ldots (1)$$

we have to find the coefficient of x^n, so comparing the powers of x, we get

$$4n - 3r = n$$

$$4n - n = 3r$$

$$3n = 3r$$

$$\boxed{n = r}$$

Put in (1)

$$T_{n+1} = \binom{2n}{n} x^{4n-3n} (-1)^n = \frac{(2n)!}{n!\,(2n-n)!} x^n (-1)^n$$

$$= \frac{(2n)!}{n!\,n!} x^n (-1)^n$$

Coefficient of x^n is $\frac{(-1)^n (2n)!}{(n!)^2}$

Q.8 Find 6th term in the expansion of $\left(x^2 - \frac{3}{2x}\right)^{10}$

Solution:

$$a = x^2, \quad x = \frac{-3}{2x}, \quad n = 10, \quad r = 5$$

We know by general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{5+1} = \binom{10}{5} (x^2)^{10-5} \left(\frac{-3}{2x}\right)^5$$

$$T_6 = 252 \times x^{10} \times \frac{-243}{32\,x^5}$$

$$T_6 = \frac{-15309}{8} x^5$$

Q.9 Find the term independent of x in the following expansions.

(i) $\left(x - \frac{2}{x}\right)^{10}$ **(Gujranwala Board 2003, Lahore Board 2008)**

(ii) $\left(\sqrt{x} + \frac{1}{2x^2}\right)^{10}$

(iii) $(1 + x^2)^3 \left(1 + \frac{1}{x^2}\right)^4$

Solution:

(i) $\left(x - \frac{2}{x}\right)^{10}$

$a = x, \quad x = \frac{-2}{x} \quad n = 10, \quad r = ?$

We know that general term formula is

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$= \binom{10}{r} x^{10-r} \left(\frac{-2}{x}\right)^r$$

$$= \binom{10}{r} x^{10-r} \frac{(-2)^r}{x^r}$$

$$= \binom{10}{r} x^{10-r-r} (-2)^r$$

$$= \binom{10}{r} x^{10-2r} (-2)^r \quad \text{......... (1)}$$

We have to find the term independent of x i.e., x^0 so comparing the powers of x, we have

$10 - 2r = 0$

$10 = 2r \Rightarrow \boxed{r = 5}$ Put in (1)

$$T_{5+1} = \binom{10}{5} x^{10-2(5)} (-2)^5$$

$$T_6 = 252 \times x^{10-10} \times (-32) = -8064\, x^0 = -8064$$

Equating Index of x to 0 to get expression independent of x

(ii) $\left(\sqrt{x}+\dfrac{1}{2x^2}\right)^{10}$

$$a = \sqrt{x}, \quad x = \frac{1}{2x^2}, \quad n = 10, \quad r = ?$$

We know that general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$= \binom{10}{r} (\sqrt{x})^{10-r} \left(\frac{1}{2x^2}\right)^r$$

$$= \binom{10}{r} x^{\frac{10-r}{2}} \frac{1}{2^r x^{2r}}$$

$$= \binom{10}{r} x^{\frac{10-r}{2} - 2r} \left(\frac{1}{2^r}\right) \qquad \ldots\ldots\ldots (1)$$

We have to find independent of x i.e., x^0 so comparing the powers of 'x', we get

$$\frac{10-r}{2} - 2r = 0$$

$$10 - r - 4r = 0$$

$$10 - 5r = 0$$

$$10 = 5r$$

$$2 = r \qquad \text{Put in (1)}$$

$$T_{2+1} = \binom{10}{2} x^{\frac{10-2}{2} - 2(2)} \left(\frac{1}{2^2}\right)$$

$$= 45\, x^{4-4} \left(\frac{1}{4}\right) = \frac{45}{4} x^0 = \frac{45}{4}$$

(iii) $(1+x^2)^3\left(1+\dfrac{1}{x^2}\right)^4$

$$(1+x^2)^3\left(1+\frac{1}{x^2}\right)^4 = (1+x^2)^3 \frac{(1+x^2)^4}{x^8}$$

$$= \frac{1}{x^8}(1+x^2)^7 \qquad \ldots\ldots\ldots (1)$$

Now $(1 + x^2)^7$, we have $a = 1,\ x = x^2,\ n = 7,\ r = ?$

We know that general term formula is

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r = \binom{7}{r} (1)^{7-r} (x^2)^r$$

$$= \binom{7}{r} x^{2r} \qquad \text{equation (1) becomes}$$

$$= \frac{1}{x^8}\binom{7}{r} x^{2r}$$

$$= \binom{7}{r} x^{2r-8} \qquad \ldots\ldots\ldots (2)$$

We have to find term independent of x.

i.e. , x^0 so, comparing the powers of x.

$2r - 8 = 0$

$2r = 8 \Rightarrow r = 4$ put in (2).

$$\binom{7}{4} x^{8-8} = \frac{7!}{4! \times (7-4)!} x^0$$

$$= \frac{7!}{4! \times 3!}$$

$$= \frac{7 \times 6 \times 5 \times 4!}{4! \times 3 \times 2 \times 1} = 35$$

MIDDLE TERM

(1) If n is even then $\left(\frac{n}{2} + 1\right)^{th}$ term will be only one middle term.

(2) If n is odd then $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms will be the two middle terms.

Q.10 Determine the middle term in the following expansions.

(i) $\left(\frac{1}{x} - \frac{x^2}{2}\right)^{12}$ **(ii)** $\left(\frac{3}{2}x - \frac{1}{3x}\right)^{11}$

(iii) $\left(2x - \frac{1}{2x}\right)^{2m+1}$

Solution:

(i) $\left(\frac{1}{x} - \frac{x^2}{2}\right)^{12}$

Since $n = 12$ is even so $\left(\frac{n}{2} + 1\right)^{th}$ term i.e.,

$$\left(\frac{12}{2}+1\right)^{th} \text{ term} = 7^{th} \text{ term is the middle term}$$

Thus $r = 6$. Also $a = \frac{1}{x}$, $x = \left(\frac{-x^2}{2}\right)$, $n = 12$

We know that the general formula is

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{6+1} = \binom{12}{6} \left(\frac{1}{x}\right)^{12-6} \left(\frac{-x^2}{2}\right)^6$$

$$= 924 \frac{1}{x^6} \frac{x^{12}}{64}$$

$$T_7 = \frac{231}{16} x^6$$

(ii) $\left(\frac{3}{2}x - \frac{1}{3x}\right)^{11}$

Since $n = 11$ is odd so $\left(\frac{11+1}{2}\right)^{th}$ term and $\left(\frac{11+3}{2}\right)^{th}$ term i.e., 6^{th} & 7^{th} terms will be the two middle terms.

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{5+1} = \binom{11}{5} \left(\frac{3}{2}x\right)^{11-5} \left(-\frac{1}{3x}\right)^5$$

$$T_6 = 462 \times \left(\frac{3}{2}x\right)^6 \frac{(-1)^5}{(3x)^5}$$

$$= 462 \times \frac{(3x)^{6-5}}{64} \times -1$$

$$= \frac{-462 \times 3x}{64} = \frac{-693x}{32}$$

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{6+1} = \binom{11}{6} \left(\frac{3}{2}x\right)^{11-6} \left(-\frac{1}{3x}\right)^6$$

$$= 462 \times \frac{(3x)^5}{(2)^5} \times \frac{1}{(3x)^6}$$

$$= 462 \times \frac{1}{32} \times \frac{1}{(3x)^{6-5}}$$

$$T_6 = \frac{462}{32 \times 3x} = \frac{77}{16x}$$

Hence two middle terms are $-\frac{693x}{32}$ and $\frac{77}{16x}$

(iii) $\left(2x - \frac{1}{2x}\right)^{2m+1}$

As $2m + 1$ is odd, so there are two middle terms i.e., $\left(\frac{2m+1+1}{2}\right)$ and $\left(\frac{2m+1+3}{2}\right)$ are two middle terms.

$(m+1)^{th}$ and $(m+2)^{th}$ terms

For $(m+1)^{th}$ term

$$r = m, \quad n = 2m+1, \quad a = 2x, \quad x = \left(-\frac{1}{2x}\right)$$

$$T_{m+1} = \binom{2m+1}{m} (2x)^{2m+1-m} \left(-\frac{1}{2x}\right)^m$$

$$= \frac{(2m+1)!}{m!\,[2m+1-m]!} (2x)^{m+1-m} (-1)^m$$

$$= \frac{(2m+1)!}{m!\,(m+1)!} 2x\,(-1)^m$$

$$T_{m+1} = 2\,(-1)^m \frac{(2m+1)!}{m!\,(m+1)!} x$$

For $(m+2)^{th}$ term

$$r = m+2-1 = m+1$$

$$n = 2m+1, \quad a = 2x, \quad x = \left(-\frac{1}{2x}\right)$$

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$= \binom{2m+1}{m+1} (2x)^{2m+1-m-1} \left(-\frac{1}{2x}\right)^{m+1}$$

$$= \frac{(2m+1)!}{(m+1)!\,[(2m+1-m-1)]!} (2x)^m \frac{(-1)^{m+1}}{(2x)^{m+1}} = \frac{(2m+1)!}{(m+1)!(m)!} \frac{(-1)^{m+1}}{(2x)^{m+1-m}}$$

$$T_{m+2} = \frac{(2m+1)!\,(-1)^{m+1}}{m!\,(m+1)!\,2x}$$

T_{m+1} and T_{m+2} are two middle terms.

Q.11 Find $(2n+1)^{th}$ term from the end in the expansion of $\left(x - \frac{1}{2x}\right)^{3n}$

Solution:

To find $(2n+1)^{th}$ terms, we have $r = 2n$

And for the term from the end, we have

$$a = -\frac{1}{2x} \quad \text{and} \quad x = x$$

By general term formula

$$T_{r+1} = \binom{n}{r} a^{n-r} x^{r}$$

$$T_{2n+1} = \binom{3n}{2n} \left(-\frac{1}{2x}\right)^{3n-2n} (x)^{2n}$$

$$= \frac{3n!}{2n!\,(3n-2n)!} \left(\frac{-1}{2x}\right)^{n} x^{2n}$$

$$= \frac{3n!}{(2n)!\,n!} \frac{(-1)^n}{2^n x^n} x^{2n}$$

$$= \frac{3n!\,(-1)^n}{2n!\,n!\,2^n} x^{2n-n}$$

$$= \frac{(3n)!\,(-1)^n x^n}{2n!\,n!\,2n}$$

Q.12 Show that middle term of $(1+x)^{2n}$ is $\frac{1.3.5 \,........\, (2n-1)}{n!} 2^n . x^n$

Solution:

As $2n$ is even so $\left(\frac{2n}{2}+1\right)^{th}$ term is the middle term i.e., $(n+1)^{th}$ term $r = n$

General term formula is

$$T_{r+1} = \binom{n}{r} a^{n-r} x^{r}$$

$$T_{n+1} = \binom{2n}{n} (1)^{2n-n} x^{n}$$

$$= \frac{(2n)!}{n!\,[2n-n]!} (1)^n x^n = \frac{(2n)!}{n!\,n!} x^n$$

$$= \frac{(2n)\,(2n-1)\,(2n-2)\,(2n-3)\,(2n-4)\,........\,5 \times 4 \times 3 \times 2 \times 1}{n!\,n!} x^n$$

$$= \frac{[(2n)(2n-2)(2n-4)\ldots.4\times 2][(2n-1)(2n-3)(2n-5)\ldots.5\times 3\times 1]\,x^n}{n!\,n!}$$

$$= \frac{[2^n(n)(n-1)(n-2)\ldots(n-2)\ldots\times 2\times 1][(2n-1)(2n-3)\ldots 5\times 3\times 1]\,x^n}{n!\,n!}$$

$$= \frac{2^n\,n!\,[(2n-1)(2n-3)\ldots.5\times 3\times 1]\,x^n}{n!\,n!}$$

$$T_{n+1} = \frac{2^n\,[1\times 3\times 5\times\ldots.(2n-3)(2n-1)]\,x^n}{n!}$$

Q.13 Show that: $\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots..+\binom{n}{n-1} = 2^{n-1}$

Solution:

We know that

$$(1+x)^n = \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\ldots\ldots.+\binom{n}{n-1}x^{n-1}+\binom{n}{n}x^n \quad (1)$$

Put $x = 1$ in equation (1)

$$(1+1)^n = \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\binom{n}{4}+\ldots\ldots.+\binom{n}{n-1}+\binom{n}{n}$$

$$2^n = \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\binom{n}{4}+\ldots\ldots..+\binom{n}{n-1}+\binom{n}{n} \quad (2)$$

Next put $x = -1$ in equation (1)

$$(1-1)^n = \binom{n}{0}-\binom{n}{1}(-1)+\binom{n}{2}(-1)^2+\binom{n}{3}(-1)^3+\ldots+\binom{n}{n-1}(-1)^{n-1}+\binom{n}{n}(-1)^n$$

if n is even then

$$0 = \binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\ldots+(-1)^{n-1}\binom{n}{n-1}+(-1)^n\binom{n}{n}$$

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots\ldots..+\binom{n}{n} = \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots\ldots+\binom{n}{n-1} \ldots (3)$$

We can write (2) as.

$$2^n = \left\{\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots+\binom{n}{n}\right\}+\left\{\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right\} \quad (4)$$

Using (3) in (4)

$$2^n = \left\{\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right\}+\left\{\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right\}$$

$$2^n = 2\left\{\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}\right\}$$

$$\frac{2^n}{2} = \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots+\binom{n}{n-1}$$

$$\Rightarrow \quad \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots.+\binom{n}{n-1} = 2^{n-1}$$

Q.14 Show that $\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+\dots\dots+\frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$

Solution:

$$\text{L. H. S.} = \binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+\dots\dots+\frac{1}{n+1}\binom{n}{n}$$

$$= \frac{n!}{0!\,(n-0)!}+\frac{1}{2}\,\frac{n!}{1!\,(n-1)!}+\frac{1}{3}\,\frac{n!}{2!\,(n-2)!}+\frac{1}{4}\,\frac{n!}{3!\,(n-3)!}+\dots+\frac{1}{n+1}\,\frac{n!}{n!\,(n-n)!}$$

Taking common $n!$

$$= n!\left[\frac{1}{n!}+\frac{1}{2!\,(n-1)!}+\frac{1}{3!\,(n-2)!}+\frac{1}{4!\,(n-3)!}+\dots\dots+\frac{1}{(n+1)\,n!}\right]$$

Now multiplying and dividing by $n+1$

$$= \frac{(n+1)\,n!}{(n+1)}\left[\frac{1}{n!}+\frac{1}{2!\,(n-1)!}+\frac{1}{3!\,(n-2)!}+\frac{1}{4!\,(n-3)!}+\dots+\frac{1}{(n+1)\,n!}\right]$$

$$= \frac{(n+1)!}{n+1}\left[\frac{1}{n!}+\frac{1}{2!\,(n-1)!}+\frac{1}{3!\,(n-2)!}+\frac{1}{4!\,(n-3)!}+\dots+\frac{1}{(n+1)\,n!}\right]$$

$$= \frac{1}{n+1}\left[\frac{(n+1)!}{n!}+\frac{(n+1)!}{2!\,(n-1)!}+\frac{(n+1)!}{3!\,(n-2)!}+\frac{(n+1)!}{4!\,(n-3)!}+\dots+\frac{(n+1)!}{(n+1)\,n!}\right]$$

$$= \frac{1}{n+1}\left[\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\binom{n+1}{4}+\dots\dots+\binom{n+1}{n+1}\right]$$

Adding and subtracting $\binom{n+1}{0}$

$$= \frac{1}{n+1}\left[\binom{n+1}{0}+\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\dots\dots+\binom{n+1}{n+1}-\binom{n+1}{0}\right] \quad (1)$$

We know that

$$\binom{n+1}{0}+\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\dots\dots+\binom{n+1}{n+1} = 2^{n+1}$$

and $\binom{n+1}{0} = 1$

So (1) becomes

$$\frac{1}{n+1}\,[2^{n+1}-1] = \frac{2^{n+1}-1}{n+1} = \text{R.H.S.}$$

Hence proved.

EXERCISE 8.3

BINOMIAL SERIES **(Lahore Board 2009, 11)**

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots.$$

Here index n is negative integer or a fraction.

Q.1 Expand the following upto 4 terms, taking value of x such that the expansion in each case is valid.

(i) $(1-x)^{1/2}$

(ii) $(1+2x)^{-1}$

(iii) $(1+x)^{-1/3}$

(iv) $(4-3x)^{1/2}$

(v) $(8-2x)^{-1}$ **(Lahore Board 2008)**

(vi) $(2-3x)^{-2}$ **(Lahore Board 2010)**

(vii) $\dfrac{(1-x)^{-1}}{(1+x)^2}$

(viii) $\dfrac{\sqrt{1+2x}}{1-x}$

(ix) $\dfrac{(4+2x)^{1/2}}{(2-x)}$

(x) $(1+x-2x^2)^{1/2}$

(xi) $(1-2x+3x^2)^{-1/3}$

Solution:

(i) $(1-x)^{1/2}$

By binomial series

$$= \left(1 + \frac{1}{2}(-x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-x)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-x)^3 + \ldots\ldots\ldots\right)$$

$$= 1 - \frac{1}{2}x + \frac{1}{2}\left(-\frac{1}{2}\right) \times \frac{1}{2}x^2 + \frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right) \times \frac{1}{6}(-x^3) + \ldots\ldots.$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \ldots\ldots\ldots$$

Valid if $|x| < 1$

(ii) $(1+2x)^{-1}$

$$1 + (-1)(2x) + \frac{(-1)(-1-1)}{2!}(2x)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}(2x)^3 + \ldots.$$

$$= 1 - 2x + 4x^2 - 8x^3 + \ldots\ldots.$$

Valid if $|2x| < 1$

$2|x| < 1$

$\Rightarrow \quad |x| < \dfrac{1}{2}$

(iii) $\mathbf{(1 + x)^{-1/3}}$

$$1 + \left(-\frac{1}{3}\right)x + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}x^3 + \ldots\ldots$$

$$= 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \ldots\ldots$$

Valid if $|x| < 1$

(iv) $\mathbf{(4 - 3x)^{1/2}}$

$$(4)^{1/2}\left(1 - \frac{3x}{4}\right)^{1/2}$$

$$= 2\left[1 + \frac{1}{2}\left(-\frac{3x}{4}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{3x}{4}\right)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(-\frac{3x}{4}\right)^3 + \ldots\ldots\right]$$

$$= 2\left[1 - \frac{3x}{8} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2} \times \frac{9x^2}{16} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\left(\frac{-27x^3}{64}\right) + \ldots\ldots\right]$$

$$= 2 - \frac{3x}{4} - \frac{9}{64}x^2 - \frac{27}{512}x^3 + \ldots\ldots$$

Expansion is valid if

$$\left|\frac{3}{4}x\right| < 1$$

$$\Rightarrow \quad \frac{3}{4}|x| < 1$$

$$\Rightarrow \quad |x| < \frac{4}{3}$$

(v) $\mathbf{(8 - 2x)^{-1}}$ **(Lahore Board 2008)**

$$= 8^{-1}\left(1 - \frac{2x}{8}\right)^{-1}$$

$$= \frac{1}{8}\left[1 - \frac{x}{4}\right]^{-1}$$

$$= \frac{1}{8}\left[1 + \frac{1}{4}x + \frac{-1 \times -2}{2 \times 1}\ \frac{1}{16}x^2 + \frac{(-1) \times (-2) \times (-3)}{3 \times 2 \times 1} \times \frac{-1}{64} \times^3 + \ldots\right]$$

$$= \frac{1}{8}\left[1 + \frac{1}{4}x + \frac{1}{16}x^2 + \frac{1}{64}x^3 + \ldots\ldots\right]$$

$$= \frac{1}{8} + \frac{1}{32}x + \frac{1}{128}x^2 + \frac{1}{512}x^3 + \ldots\ldots$$

The expansion valid only if

$$\left|\frac{x}{4}\right| < 1$$

$$\Rightarrow \quad \frac{1}{4}|x| < 1$$

$$\Rightarrow \quad |x| < 4$$

(vi) $\mathbf{(2 - 3x)^{-2}}$ **(Lahore Board 2010)**

$$2^{-2}\left(1 - \frac{3x}{2}\right)^{-2}$$

$$= \frac{1}{4}\left[1 + (-2)\left(\frac{-3}{2}x\right) + \frac{(-2)(-2-1)}{2!}\left(\frac{-3x}{2}\right)^2 + \frac{(-2)(-2-1)(-2-2)}{3!}\left(\frac{-3}{2}x\right)^3 + \ldots\ldots\ldots\right]$$

$$= \frac{1}{4}\left[1 + 3x + \frac{-2x-3}{2} \times \frac{9x^2}{4} + \frac{(-2)(-3)(-4)}{6} \times \frac{-27x^3}{8} + \ldots\ldots\right]$$

$$= \frac{1}{4}\left[1 + 3x + \frac{27x^2}{4} + \frac{27x^3}{2} + \ldots\ldots.\right]$$

$$= \frac{1}{4} + \frac{3}{4}x + \frac{27x^2}{16} + \frac{27x^3}{8} + \ldots\ldots.$$

The above expansion is valid only if

$$\left|\frac{3x}{2}\right| < 1$$

$$\Rightarrow \quad \frac{3}{2}|x| < 1$$

$$\Rightarrow \quad |x| < \frac{2}{3}$$

(vii) $\mathbf{\dfrac{(1-x)^{-1}}{(1+x)^2}}$

$$= (1-x)^{-1}(1+x)^{-2}$$

$$= \left[1 + x + \frac{(-1)(-1-1)}{2!}(-x)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3 + \ldots\ldots.\right]$$

$$\left[1 - 2x + \frac{(-2)(-2-1)}{2!}x^2 + \frac{(-2)(-2-1)(-2-2)}{3!}(+x)^3 + \ldots\ldots.\right]$$

$$= \left[1 + x + \frac{(-1)(-2)}{2}(x)^2 + \frac{(-1)(-2)(-3)}{6}(-x^3) + \ldots\ldots.\right]$$

$$\left[1 - 2x + \frac{(-2)(-3)}{2}x^2 + \frac{(-2)(-3)(-4)}{6}x^3 + \ldots\ldots.\right]$$

$= [1 + x + x^2 + x^3 + \ldots\ldots] [1 - 2x + 3x^2 - 4x^3 + \ldots\ldots]$

$= 1 - 2x + 3x^2 - 4x^3 + x - 2x^2 + 3x^3 + x^2 - 2x^3 + x^3 + \ldots\ldots$

$= 1 - x + 2x^2 - 2x^3 + \ldots\ldots$

The above expansion are valid if

$|x| < 1$

(viii) $\dfrac{\sqrt{1 + 2x}}{1 - x}$

$(1 + 2x)^{1/2} (1 - x)^{-1}$

$= (1 - x)^{-1} (1 + 2x)^{1/2}$

$$= \left[1 + x + \frac{(-1)(-1-1)}{2!}(-x)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3 + \ldots\ldots\right]$$

$$\left[1 + \frac{1}{2}2x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(2x)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(2x)^3 + \ldots\ldots\right]$$

$$= \left[1 + x + \frac{-1 \times -2}{2}x^2 + \frac{-1 \times -2 \times -3}{6}(-x^3) + \ldots\ldots\right]$$

$$\left[1 + x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}4x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}8x^3 + \ldots\ldots\right]$$

$$= [1 + x + x^2 + x^3 + \ldots\ldots]\left[1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \ldots\ldots\right]$$

$$= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + x + x^2 - \frac{1}{2}x^3 + x^2 + x^3 + x^3 + \ldots\ldots$$

$$= 1 + 2x + \frac{3}{2}x^2 + 2x^3 + \ldots\ldots$$

The above expansion valid if

$|x| < \frac{1}{2}$ and $|x| < 1$

(ix) $\dfrac{(4 + 2x)^{1/2}}{(2 - x)}$

$(4 + 2x)^{1/2} (2 - x)^{-1}$

$$= 4^{1/2}\left(1 + \frac{2}{4}x\right)^{1/2} 2^{-1}\left(1 - \frac{x}{2}\right)^{-1}$$

$$= 2\left(1 + \frac{1}{2}\,\frac{x}{2} + \frac{1}{2}\;\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\,\frac{x^2}{4} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{x}{2}\right)^3 + \ldots\ldots.\right)$$

$$\frac{1}{2}\left[1 + \frac{x}{2} + \frac{(-1)(-1-1)}{2!}\left(-\frac{x}{2}\right)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}\left(\frac{-x}{2}\right)^3 + \ldots\ldots.\right]$$

$$= \left[1 + \frac{1}{4}x - \frac{1}{32}x^2 + \frac{1}{128}x^3 + \ldots\ldots.\right]\left[1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \ldots\ldots.\right]$$

$$= 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{4}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{1}{32}x^2 - \frac{1}{64}x^3 + \frac{1}{128}x^3$$

$$= 1 + \frac{3}{4}x + \frac{11}{32}x^2 + \frac{23}{128}x^3 + \ldots\ldots.$$

The expansion of $\left(1 + \frac{x}{2}\right)^{1/2}$ and $\left(1 - \frac{x}{2}\right)^{-1}$ are valid if

$$\left|\frac{x}{2}\right| < 1$$

$\Rightarrow \qquad |x| < 2$

(x) $\quad (1 + x - 2x^2)^{1/2}$

$$= 1 + \frac{1}{2}(x - 2x^2) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(x - 2x^2)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(x - 2x^2)^3 + \ldots\ldots.$$

$$= 1 + \frac{1}{2}(x - 2x^2) + \frac{\frac{1}{2}\times -\frac{1}{2}}{2}(x^2 + 4x^4 - 4x^3) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}(x^3 - 8x^6 - 6x^4 + 12x^5) + \ldots$$

$$= 1 + \frac{1}{2}x - x^2 - \frac{1}{8}x^2 - \frac{1}{2}x^4 + \frac{1}{2}x^3 + \frac{1}{16}x^3 - \frac{3}{8}x^4 + \frac{3}{4}x^5 - \frac{1}{2}x^6 + \ldots\ldots.$$

$$= 1 + \frac{1}{2}x - \frac{9}{8}x^2 + \frac{9}{16}x^3 + \ldots\ldots.$$

The above expansion is valid only if $|x - 2x^2| < 1$ that is either

$x - 2x^2 < 1 \qquad$ or $\qquad -(x - 2x^2) < 1$

$\Rightarrow \qquad -2x^2 + x - 1 < 0 \quad \ldots\ldots\ldots (1) \qquad 2x^2 - x - 1 < 0 \quad \ldots\ldots\ldots (2)$

$2x^2 - 2x + x - 1 < 0$

$(x - 1)(2x + 1) < 0$

$-\frac{1}{2} < x < 1$

(xi) $\mathbf{(1-2x+3x^2)^{-1/3}}$

$[1+(3x^2-2x)]^{-1/3}$

$$= \left[1+\left(\frac{-1}{3}\right)(3x^2-2x)+\frac{\left(-\frac{1}{3}\right)\left(\frac{-1}{3}-1\right)}{2!}(3x^2-2x)^2+\frac{\frac{-1}{3}\left(\frac{-1}{3}-1\right)\left(\frac{-1}{3}-2\right)}{3!}(3x^2-2x)^3+\ldots\right]$$

$$= 1-\frac{1}{3}(3x^2-2x)-\frac{1}{3}\times\frac{-4}{3}\times\frac{1}{2}(9x^4+4x^2-12x^3)+\frac{-1}{3}\times\frac{-4}{3}\times\frac{-7}{2}\times\frac{1}{6}$$
$$(27x^6-8x^3-54x^5+36x^4)+\ldots\ldots$$

$$= 1-x^2+\frac{2}{3}x+\frac{2}{9}(9x^4+4x^2-12x^3)-\frac{7}{27}(27x^6-8x^3-54x^5+36x^4)+\ldots\ldots$$

$$= 1-x^2+\frac{2}{3}x+2x^4+\frac{8}{9}x^2-\frac{24}{9}x^3-7x^6+\frac{56}{27}x^3+14x^5+\ldots\ldots$$

$$= 1+\frac{2}{3}x-\frac{1}{9}x^2-\frac{16}{27}x^3+\ldots\ldots$$

The above expansion is valid only if

$|3x^2-2x|<1$

$3x^2-2x<1 \qquad -(3x^2-2x)<1$

$3x^2-2x-1<1$

$3x^2-3x+x-1<1$

$(3x+1)(x-1)<1$

$\frac{-1}{3}<x<1$

Q.2 Using Binomial theorem find the value of the following to three places of decimals.

Solution:

(i) $\mathbf{\sqrt{99}}$

$$= (99)^{1/2} = (100-1)^{1/2} = (100)^{1/2}\left(1-\frac{1}{100}\right)^{1/2}$$

$$= 10\left[1+\left(\frac{1}{2}\right)\left(\frac{-1}{100}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{-1}{100}\right)^2+\ldots\ldots\right]$$

$$= 10\left[1-\frac{1}{2\times100}+\frac{1}{2}\times\frac{-1}{2}\times\frac{1}{2}\times\frac{1}{100\times100}+\ldots\ldots\right]$$

$$= 10 - \frac{1}{20} - \frac{1}{8000} + \ldots\ldots.$$

$$= 10 - 0.05 - 0.000125 + \ldots\ldots. = 9.950$$

(ii) $\mathbf{(0.98)^{1/2}}$

$$= (1 - .02)^{1/2}$$

$$= 1 + \frac{1}{2}(-.02) + \frac{1}{2}\left(\frac{1}{2} - 1\right)(-.02)^2 + \ldots\ldots.$$

$$= 1 - .01 + \frac{1}{2}\left(-\frac{1}{2}\right)(.0004) + \ldots\ldots.$$

$$= 1 - .01 - .00005 + \ldots\ldots. = .990$$

(iii) $\mathbf{(1.03)^{1/3}}$

$$= (1 + .03)^{1/3}$$

$$= \left(1 + \frac{1}{3}(.03) + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2!}(.03)^2 + \ldots\ldots.\right)$$

$$= 1 + .01 + \frac{1}{3} \times \frac{-2}{3} \times \frac{1}{2} \times .0009 + \ldots\ldots.$$

$$= 1 + .01 - .0001 + \ldots\ldots. = 1.010$$

(iv) $\mathbf{\sqrt[3]{65}}$

$$= (65)^{1/3} = (64 + 1)^{1/3}$$

$$= (64)^{1/3}\left(1 + \frac{1}{64}\right)^{1/3}$$

$$= 4\left[1 + \frac{1}{3}\left(\frac{1}{64}\right) + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2!}\left(\frac{1}{64}\right)^2 + \ldots\ldots.\right]$$

$$= 4\left[1 + \frac{1}{192} + \frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2} \times \frac{1}{4096} + \ldots\ldots.\right]$$

$$= 4 + 0.021 - 0.0001 + \ldots\ldots. = 4.021$$

(v) $\mathbf{\sqrt[4]{17}}$

$$= (17)^{1/4} = (16 + 1)^{1/4} = 16^{1/4}\left(1 + \frac{1}{16}\right)^{1/4}$$

$$= 2^{4 \times (1/4)}\left[1 + \frac{1}{4}\left(\frac{1}{16}\right) + \ldots\ldots.\right]$$

$$= 2 + \frac{1}{2 \times 16} + \ldots\ldots. = 2 + \frac{1}{32} + \ldots\ldots.$$

$$= 2 + 0.031 + \ldots\ldots. = 2.031$$

(vi) $\sqrt[5]{31}$

$$= (31)^{1/5} = (32 - 1)^{1/5}$$

$$= 32^{1/5}\left(1 - \frac{1}{32}\right)^{1/5}$$

$$= 2^{5 \times (1/5)}\left[1 + \left(\frac{1}{5}\right)\left(\frac{-1}{32}\right) + \ldots\ldots.\right]$$

$$= 2 - \frac{1}{5 \times 16} + \ldots\ldots. = 2 - 0.013 = 1.987$$

(vii) $\dfrac{1}{\sqrt[3]{998}}$

$$= \left(\frac{1}{(998)^{1/3}}\right) = (998)^{-1/3}$$

$$= (1000 - 2)^{-1/3} = (1000)^{-1/3}\left[1 - \frac{2}{1000}\right]^{-1/3}$$

$$= 10^{3 \times (-1/3)}\left[1 - \frac{1}{500}\right]^{-1/3}$$

$$= 10^{-1}\left[1 + \left(\frac{1}{3}\right)\left(\frac{1}{500}\right) + \ldots\ldots.\right]$$

$$= \frac{1}{10}\left[1 + \frac{1}{1500} + \ldots\ldots.\right] = \frac{1}{10} + \frac{1}{15000} + \ldots\ldots.$$

$$= 0.1 + 0.000067 + \ldots\ldots. = 0.1000$$

(viii) $\dfrac{1}{\sqrt[5]{252}}$

$$= \frac{1}{(252)^{1/5}} = (252)^{-1/5} = (243 + 9)^{-1/5}$$

$$= 243^{-1/5}\left[1+\frac{9}{243}\right]^{-1/5} = 3^{5\times -1/5}\left[1+\left(\frac{-1}{5}\right)\left(\frac{9}{243}\right)+\ldots\ldots\right]$$

$$= 3^{-1}\left[1-\frac{1}{5}\times\frac{1}{27}+\ldots\ldots\right] = \frac{1}{3}\left[1-\frac{1}{135}+\ldots\ldots\right]$$

$$= \frac{1}{3}[1-0.007+\ldots\ldots] = \frac{1}{3}[0.993] = 0.331$$

(ix) $\dfrac{\sqrt{7}}{\sqrt{8}}$

$$= \left(\frac{7}{8}\right)^{1/2} = \left(1-\frac{1}{8}\right)^{1/2}$$

$$= 1+\left(\frac{1}{2}\right)\left(\frac{-1}{8}\right)+\ldots\ldots$$

$$= 1-\frac{1}{16}+\ldots\ldots = 1-0.063+\ldots\ldots = 0.938$$

(x) $(.998)^{-1/3}$

$$= (1-0.002)^{-1/3}$$

$$= 1+\left(\frac{-1}{3}\right)(-0.002)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2\times 1}(-0.002)^2+\ldots\ldots$$

$$= 1+0.001+\frac{2}{9}(0)+\ldots\ldots$$

$$= 1+0.001+0+\ldots\ldots = 1.001$$

(xi) $\dfrac{1}{\sqrt[6]{486}}$

$$= \frac{1}{(486)^{1/6}} = (486)^{-1/6} = (729-243)^{-1/6}$$

$$= (729)^{-1/6}\times\left[1-\frac{243}{729}\right]^{-1/6} = (3^6)^{-1/6}\left[1-\frac{1}{3}\right]^{-1/6}$$

$$= \frac{1}{3}\left[1-\frac{1}{3}\right]^{-1/6} \Rightarrow = \frac{1}{3}\left[1+\left(\frac{-1}{6}\right)\left(\frac{-1}{3}\right)+\frac{\left(\frac{-1}{6}\right)\left(\frac{-1}{6}-1\right)}{2!}\left(\frac{-1}{3}\right)^2+\ldots\right]$$

$$= \frac{1}{3}[1+0.0555+0.0108+\ldots\ldots]$$

$$= \frac{1}{3}[1.06895] = 0.356$$

(xii) $\mathbf{(1280)^{1/4}}$

$= (1296 - 16)^{1/4} = 1296^{1/4}\left(1 - \frac{16}{1296}\right)^{1/4}$

$= 6^{4 \times (1/4)}\left[1 + \left(\frac{1}{4}\right)\left(-\frac{16}{296}\right) + \ldots\ldots.\right]$

$= 6\left[1 - \frac{1}{324} + \ldots\ldots.\right] = 6\,[1 - 0.003 + \ldots\ldots.] = 6\,[0.997] = 5.981$

Q.3 Find the coefficient of x^n in the expansion

(i) $\frac{1 + x^2}{(1 + x)^2}$ **(ii)** $\frac{(1 + x)^2}{(1 - x)^2}$

(iii) $\frac{(1 + x)^3}{(1 - x)^2}$ **(iv)** $\frac{(1 + x)^2}{(1 - x)^3}$

(v) $(1 - x + x^2 - x^3 + \ldots\ldots...)$ **(Gujranwala Board 2005)**

Solution:

(i) $\frac{1 + x^2}{(1 + x)^2}$

$= (1 + x^2)\,(1 + x)^{-2}$

$= (1 + x^2)\left[1 + (-2)\,(x) + \frac{(-2)\,(-2-1)\,x^2}{2!} + \frac{(-2)\,(-2-1)\,(-3-1)\,(x)^3}{3!} + \ldots\ldots.\right]$

$= (1 + x^2)\left[1 + (-2)\,(x) + \frac{(-2)\,(-3)\,x^2}{2!} + \frac{(-2)\,(-3)\,(-4)}{3 \times 2 \times 1}\,x^3 + \ldots\ldots.\right]$

$= (1 + x^2)\,[1 + (-1)\,2x + (-1)^2\,3x^2 + (-1)^3\,4x^3 + \ldots\ldots.$

$+ \ldots\ldots. + (-1)^{n-2}\,(n-1)^{n-2}\,x + (-1)^{n-1}\,n\,x^{n-1} + (-1)^n\,(n+1)\,x^n$

$= (-1)^{n-2}\,(n - 1)\,x^n + (-1)^n\,(n + 1)\,x^n$

$= \left[(-1)^n\,(-1)^2\,(n - 1) + (-1)^n\,(n + 1)\right]x^n = (-1)^n\,[n - 1 + n + 1]\,x^n$

$= (-1)^n\,.\,(2n)\,x^n$

Coefficient of x^n is, $(-1)^n \times (2n)$

(ii) $\frac{(1 + x)^2}{(1 - x)^2}$

$= (1 + x)^2\,(1 - x)^{-2}$

$$= (1 + 2x + x^2)\left(1 + 2x + \frac{(-2)(-2-1)(-x)^2}{2!} + \ldots\ldots\right)$$

$$= (1 + 2x + x^2)\left(1 + 2x + \frac{(-2)(-3)}{2 \times 1}x^2 + \ldots\ldots\right)$$

$$= (1 + 2x + x^2)\left[1 + 2x + 3x^2 + \ldots.. + (n-1)x^{n-2} + n\,x^{n-1} + (n+1)x^n\right]$$

Now multiplying the terms to get terms involving x^n.

$$= (n+1)x^n + 2n\,x^{n-1+1} + (n-1)x^{n-2+2}$$

$$= (n+1)x^n + 2nx^n + (n-1)x^n$$

$$= (n + 1 + 2n + n - 1)x^n$$

$$= 4\,n\,x^n$$

Hence coefficient of x^n is $4n$

(iii) $\mathbf{\dfrac{(1+x)^3}{(1-x)^2}}$

$$= (1+x)^3(1-x)^{-2}$$

$$= \left[1 + 3x + \frac{(3)(3-1)}{2!}(x)^2 + \frac{3(3-1)(3-2)}{3!}x^3\right]$$

$$\left[1 + 2x + \frac{(-2)(-2-1)(-x^2)}{2!} + \frac{(-2)(-2-1)(-2-2)}{3!}(-x)^3 + \ldots\ldots\right]$$

$$= \left[1 + 3x + \frac{3 \times 2}{2 \times 1}x^2 + \frac{3 \times 2 \times 1}{3 \times 2 \times 1}x^3\right]$$

$$\left[1 + 2x + \frac{(-2)(-3)}{2 \times 1}x^2 + \frac{(-2)(-3)(-4)}{3 \times 2 \times 1}(-x)^3 + \ldots\ldots\right]$$

$$= \left[1 + 3x + 3x^2 + x^3\right]$$

$$\left[1 + 2x + 3x^2 + 4x^3 + \ldots\ldots + (n-2)x^{n-3} + (n-1)x^{n-2} + n\,x^{n-1} + (n+1)x^n\right]$$

$$= (n+1)x^n + 3n\,x^{n-1+1} + 3(n-1)x^{n-2+2} + (n-2)x^{n-3+3}$$

$$= (n+1)x^n + 3n\,x^n + 3(n-1)x^n + (n-2)x^n$$

$$= (n + 1 + 3n + 3n - 3 + n - 2)x^n$$

$$= (8n-4).x^n = 4(2n-1)x^n$$

Hence coefficient of x^n is $4(2n-1)$.

(iv) $\dfrac{(1+x)^2}{(1-x)^3}$

$= (1+x)^2 (1-x)^{-3}$

$= \left[(1+2x+x^2)\left(1+3x+\dfrac{(-3)(-3-1)}{2!}(-x)^2+\ldots\ldots\right)\right]$

$= (1+2x+x^2)\left(1+3x+\dfrac{-3x-4}{2\times 1}x^2+\ldots\ldots\right)$

$= (1+2x+x^2)$

$\left(1+3x+\dfrac{3\times 4}{2}x^2+\dfrac{4\times 5}{2}x^3+\ldots\ldots+\dfrac{(n-1)(n)}{2}x^{n-2}+\dfrac{n(n+1)}{2}x^{n-1}+\dfrac{(n+1)(n+2)}{2}x^n\right)$

$\Rightarrow \quad = \dfrac{(n+1)(n+2)}{2}x^n+\dfrac{2(n)(n+1)}{2}x^{n-1+1}+\dfrac{(n-1)(n)}{2}x^{n-2+2}$

$= \left(\dfrac{n^2+3n+2+2n^2+2n+n^2-n}{2}\right)x^n$

$= \left(\dfrac{4n^2+4n+2}{2}\right)x^n = (2n^2+2n+1)\,x^n$

Hence coefficient of x^n is $2n^2+2n+1$.

(v) $(1 - x + x^2 - x^3 + \ldots\ldots)$

We know that,

$(1+x)^{-1} = 1-x+x^2-x^3+\ldots\ldots$

Hence given expression becomes

$[(1+x)^{-1}]^2 = (1+x)^{-2}$

$= 1+(-2)x+\dfrac{(-2)(-2-1)}{2!}(-x)^2+\dfrac{(-2)(-2-1)(-2-2)}{31}(-x)^3+\ldots\ldots$

$= 1+(-1)2x+\dfrac{(-2)(-3)}{2\times 1}x^2+\dfrac{(-2)(-3)(-4)}{3\times 2\times 1}(-x)^3+\ldots\ldots$

$= 1+(-1)2x+(-1)^2 3x^2+(-1)^3 4x^2+\ldots\ldots+(-1)^{n-2}(n-2)x^{n-2}$

$+(-1)^{n-1} n x^{n-1}+(-1)^n (n+1)x^n$

Hence coefficient of x^n is only, $(-1)^n (n+1)$

Q.4 **If x is so small that its square and higher power can be neglected, then show that**

(i) $\dfrac{1-x}{\sqrt{1-x}} \approx 1 - \dfrac{3}{2}x$ **(Lahore Board 2009)**

(ii) $\dfrac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1 + \dfrac{3}{2}x$

(iii) $\dfrac{(9+7x)^{1/2} - (16+3x)^{1/4}}{4+5x} \approx \dfrac{1}{4} - \dfrac{17}{284}x$

(iv) $\dfrac{\sqrt{4+x}}{(1-x)^3} \approx 2 + \dfrac{25}{4}x$

(v) $\dfrac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} \approx \left(1 - \dfrac{5}{6}x\right)$ **(Gujranwala Board 2006)**

(vi) $\dfrac{(1-x)^{1/2}(9-4x)^{1/2}}{(8+3x)^{1/3}} \approx \dfrac{3}{2} - \dfrac{61}{48}x$

(vii) $\dfrac{\sqrt{4-x} + (8-x)^{1/3}}{(8-x)^{1/3}} \approx 2 - \dfrac{1}{12}x$

Solution:

(i) $\dfrac{1-x}{\sqrt{1-x}} \approx 1 - \dfrac{3}{2}x$

$$\text{L.H.S.} = (1-x)(1+x)^{-1/2}$$

$$= (1-x)\left(1 - \frac{1}{2}x\right) \quad \text{(neglecting square and heigher power of } x)$$

$$= 1 - \frac{1}{2}x - x$$

$$= 1 - \frac{3}{2}x$$

$$= \text{R.H.S.}$$

(ii) $\dfrac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1 + \dfrac{3}{2}x$

$$\text{L.H.S.} = (1+2x)^{1/2}(1-x)^{-1/2}$$

$$= \left(1 + \frac{1}{2}2x\right)\left(1 + \frac{1}{2}x\right)$$

$$= (1 + x)\left(1 + \frac{1}{2}x\right)$$

$$= 1 + \frac{1}{2}x + x$$

$$= 1 + \frac{3}{2}x = \text{R.H.S.}$$

(iii) $\dfrac{(9 + 7x)^{1/2} - (16 + 3x)^{1/4}}{4 + 5x} \approx \dfrac{1}{4} - \dfrac{17}{284}x$

$$\text{L.H.S.} = \left[(9 + 7x)^{1/2} - (16 + 3x)^{1/4}\right](4 + 5x)^{-1}$$

$$= \left[9^{1/2}\left(1 + \frac{7}{9}x\right)^{1/2} - 16^{1/4}\left(1 + \frac{3x}{16}\right)^{1/4}\right].\,4^{-1}\left(1 + \frac{5x}{4}\right)^{-1}$$

$$= \left[3\left(1 + \frac{7}{8}x\right) - 2\left(1 + \frac{3}{64}x\right)\right]\frac{1}{4}\left(1 - \frac{5x}{4}\right)$$

$$= \frac{1}{4}\left[3 + \frac{7}{6}x - 2 - \frac{3}{32}x\right]\left(1 - \frac{5}{4}x\right)$$

$$= \frac{1}{4}\left[\left(1 + \frac{103}{96}x\right)\left(1 - \frac{5}{4}x\right)\right]$$

$$= \frac{1}{4}\left(1 - \frac{5}{4}x + \frac{103}{96}x\right) \qquad [\because \text{ neglecting heigher power of } x]$$

$$= \frac{1}{4}\left(1 - \frac{17}{96}x\right)$$

$$= \frac{1}{4} - \frac{17}{384}x = \text{R.H.S.}$$

(iv) $\dfrac{\sqrt{4 + x}}{(1 - x)^3} \approx 2 + \dfrac{25}{4}x$

$$\text{L.H.S.} = (4 + x)^{1/2}(1 - x)^{-3}$$

$$= 4^{1/2}\left(1 + \frac{x}{4}\right)^{1/2}(1 - x)^{-3}$$

$$= 2\left(1 + \frac{1}{8}x\right)(1 + 3x)$$

$$= 2\left(1 + 3x + \frac{1}{8}x\right)$$

$$= 2\left(1 + \frac{25}{8}x\right)$$

$$= 2 + \frac{25}{4}x = \text{R.H.S.}$$

(v) $\dfrac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} \approx \left(1-\dfrac{5}{6}x\right)$

$$\text{L.H.S.} = (1+x)^{1/2}(4-3x)^{3/2}(8+5x)^{-1/3}$$

$$= (1+x)^{1/2}\,4^{3/2}\left(1-\frac{3x}{4}\right)^{3/2}(8)^{-1/3}\left(1+\frac{5x}{8}\right)^{-1/3}$$

$$= \left(1+\frac{1}{2}x\right)2^{3}\left(1-\frac{9}{8}x\right)2^{-1}\left(1-\frac{5}{24}x\right)$$

$$= 2^{3}\,2^{-1}\left(1+\frac{1}{2}x\right)\left(1-\frac{9}{8}x\right)\left(1-\frac{5}{24}x\right)$$

$$= 2^{2}\left(1+\frac{1}{2}x\right)\left(1-\frac{5}{24}x-\frac{9}{8}x\right)$$

$$= 4\left(1-\frac{5}{24}x-\frac{9}{8}x+\frac{1}{2}x\right)$$

$$= 4\left(1-\frac{5}{6}x\right)$$

$$= \text{R.H.S.}$$

(vi) $\dfrac{(1-x)^{1/2}(9-4x)^{1/2}}{(8+3x)^{1/3}} \approx \dfrac{3}{2}-\dfrac{61}{48}x$

$$\text{L.H.S.} = (1-x)^{1/2}(9-4x)^{1/2}(8+3x)^{-1/3}$$

$$= (1-x)^{1/2}\,9^{1/2}\left(1-\frac{4x}{9}\right)^{1/2}8^{-1/3}\left(1+\frac{3x}{8}\right)^{-1/3}$$

$$= \left(1-\frac{1}{2}x\right)3\left(1-\frac{4}{18}x\right)2^{-1}\left(1-\frac{3x}{24}\right)$$

$$= 3^{1}\,2^{-1}\left(1-\frac{1}{2}x\right)\left(1-\frac{2}{9}x\right)\left(1-\frac{1}{8}x\right)$$

$$= \frac{3}{2}\left(1-\frac{1}{2}x\right)\left(1-\frac{1}{8}x-\frac{2}{9}x\right)$$

$$= \frac{3}{2}\left(1-\frac{1}{8}x-\frac{2}{9}x-\frac{1}{2}x\right)$$

$$= \frac{3}{2}\left(1-\frac{61}{72}x\right)$$

$$= \frac{3}{2} - \frac{3}{2} \times \frac{61}{72}x$$

$$= \frac{3}{2} - \frac{61}{48}$$

$$= \text{R.H.S.}$$

(vii) $\dfrac{\sqrt{4-x} + (8-x)^{1/3}}{(8-x)^{1/3}} \approx 2 - \dfrac{1}{12}x$

$$\text{L.H.S.} = \left[(4-x)^{1/2} + (8-x)^{1/3}\right](8-x)^{-1/3}$$

$$= \left[4^{1/2}\left(1-\frac{x}{4}\right)^{1/2} + 8^{1/3}\left(1-\frac{x}{8}\right)^{1/3}\right](8)^{-1/3}\left(1-\frac{x}{8}\right)^{-1/3}$$

$$= \left[2\left(1-\frac{1}{8}x\right) + 2\left(1-\frac{x}{24}\right)\right]2^{-1}\left(1+\frac{1}{24}x\right)$$

$$= \left[2-\frac{1}{4}x + 2 - \frac{x}{12}\right]\frac{1}{2}\left(1+\frac{1}{24}x\right)$$

$$= \frac{1}{2}\left(4-\frac{1}{3}x\right)\left(1+\frac{1}{24}x\right)$$

$$= \frac{1}{2}\left(4+\frac{1}{6}x-\frac{1}{3}x\right)$$

$$= \frac{1}{2}\left(4-\frac{1}{6}x\right)$$

$$= 2-\frac{1}{12}x$$

$$= \text{R.H.S.}$$

Q.5 If x is so small that its cube and heigher power can be neglected, show that

(i) $\sqrt{1-x-2x^2} \approx 1 - \dfrac{1}{2}x - \dfrac{9}{8}x^2$

(ii) $\sqrt{\dfrac{1+x}{1-x}} \approx 1 + x + \dfrac{1}{2}x$

Solution:

(i) $\sqrt{1-x-2x^2} \approx 1 - \dfrac{1}{2}x - \dfrac{9}{8}x^2$

$$\text{L.H.S.} = (1-x-2x^2)^{1/2}$$

$$= \left[1-(x+2x^2)\right]^{1/2}$$

$$= 1 - \frac{1}{2}(x + 2x^2) + \frac{1}{2}\left(\frac{1}{2} - 1\right)[-(x + 2x^2)^2]$$

$$= 1 - \frac{1}{2}(x + 2x^2) + \frac{1}{2}\left(-\frac{1}{2}\right) \times \frac{1}{2}(x^2 + 4x^4 + 4x^3)$$

$$= 1 - \frac{1}{2}(x + 2x^2) - \frac{1}{8}(x^2 + 4x^4 + 4x^3)$$

$$= 1 - \frac{1}{2}x - x^2 - \frac{1}{8}x^2 \qquad \text{(neglecting cube and heigher power of } x\text{)}$$

$$= 1 - \frac{1}{2}x - \frac{9}{8}x^2$$

$$= \text{R.H.S.}$$

(ii) $\sqrt{\dfrac{1 + x}{1 - x}} \approx 1 + x + \dfrac{1}{2}x$

$$\text{L.H.S.} = (1 + x)^{1/2}(1 - x)^{-1/2}$$

$$= \left[1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}x^2\right]\left[1 + \frac{1}{2}x + \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2} - 1\right)}{2!}(-x)^2\right]$$

$$= \left[1 + \frac{1}{2}x - \frac{1}{8}x^2\right]\left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right)$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{3}{16}x^3 - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{3}{64}x^4$$

$$= 1 + \frac{1}{2}x + \frac{1}{2}x + \frac{3}{8}x^2 - \frac{1}{8}x^2 + \frac{1}{4}x^2$$

$$= 1 + x + \frac{1}{2}x^2$$

$$= \text{R.H.S.}$$

Q.6 If x is very nearly equal to 1, then prove that

$Px^p - qx^q \approx (p - q)\, x^{p+q}$

(Gujranwala Board 2005, 2003), (Lahore Board 2003, 2009, 2011)

Solution:

Since $x \approx 1$

Let $x = 1 + h$ where h is so small that its square and heigher powers can be neglected.

$$\text{L.H.S.} = P\,x^p - q\,x^q$$

$$\approx P(1 + h)^p - q(1 + h)^q$$

$$\approx P(1+ph)-q(1+qh)$$

$$\approx P+p^2h-q-q^2h$$

$$\approx (p-q)+(p^2-q^2)h$$

$$\approx (p-q)+(p-q)(p+q)h$$

$$\approx (p-q)[1+(p+q)h] \quad \ldots\ldots\ldots (1)$$

$$\text{R.H.S.} = (P-q)x^{p+q}$$

$$\approx (P-q)(1+h)^{p+q}$$

$$\approx (P-q)\left[1+(p+q)h\right] \quad \ldots\ldots\ldots (2)$$

From (1) and (2) we have

L.H.S. = R.H.S.

Hence proved.

Q.7 If $p-q$ is small, when compared with p or q show that

$$\frac{(2n+1)p+(2n-1)q}{(2n-1)p+(2n+1)q} = \left[\frac{p+q}{2q}\right]^{1/n}$$

Solution:

$$\text{L.H.S.} = \frac{(2n+1)p+(2n-1)q}{(2n-1)p+(2n+1)q}$$

Let $p-q = h$

$p = q+h$, where 'h' is a small that it square and higher powers can be neglected.

$$= \frac{(2n+1)(q+h)+(2n-1)q}{(2n-1)(q+h)+(2n+1)q}$$

$$= \frac{2nq+2nh+q+h+2nq-q}{2nq+2nh-q-h+2nq+q}$$

$$= \frac{4nq+2nh+h}{4nq+2nh-h} = \frac{4nq+(2n+1)h}{4nq+(2n-1)h}$$

$$= \frac{4nq\left[1+\left(\frac{2n+1}{4nq}\right)h\right]}{4nq\left[1+\left(\frac{2n-1}{4nq}\right)h\right]}$$

$$= \left[1+\left(\frac{2n+1}{4nq}\right)h\right]\left[1+\left(\frac{2n-1}{4nq}\right)h\right]^{-1}$$

$$= \left[1+\left(\frac{2n+1}{4nq}\right)h\right]\left[1-\left(\frac{2n-1}{4nq}\right)h\right]$$

$$\approx P(1+ph) - q(1+qh)$$

$$\approx P + p^2 h - q - q^2 h$$

$$\approx (p-q) + (p^2 - q^2) h$$

$$\approx (p-q) + (p-q)(p+q) h$$

$$\approx (p-q)[1+(p+q)h] \quad \text{.........} (1)$$

$$\text{R.H.S.} = (P-q) x^{p+q}$$

$$\approx (P-q)(1+h)^{p+q}$$

$$\approx (P-q)\left[1+(p+q)h\right] \quad \text{.........} (2)$$

From (1) and (2) we have

L.H.S. = R.H.S.

Hence proved.

Q.7 If $p-q$ is small, when compared with p or q show that

$$\frac{(2n+1)p+(2n-1)q}{(2n-1)p+(2n+1)q} = \left[\frac{p+q}{2q}\right]^{1/n}$$

Solution:

$$\text{L.H.S.} = \frac{(2n+1)p+(2n-1)q}{(2n-1)p+(2n+1)q}$$

Let $p - q = h$

$p = q + h$, where 'h' is a small that it square and higher powers can be neglected.

$$= \frac{(2n+1)(q+h)+(2n-1)q}{(2n-1)(q+h)+(2n+1)q}$$

$$= \frac{2nq+2nh+q+h+2nq-q}{2nq+2nh-q-h+2nq+q}$$

$$= \frac{4nq+2nh+h}{4nq+2nh-h} = \frac{4nq+(2n+1)h}{4nq+(2n-1)h}$$

$$= \frac{4nq\left[1+\left(\frac{2n+1}{4nq}\right)h\right]}{4nq\left[1+\left(\frac{2n-1}{4nq}\right)h\right]}$$

$$= \left[1+\left(\frac{2n+1}{4nq}\right)h\right]\left[1+\left(\frac{2n-1}{4nq}\right)h\right]^{-1}$$

$$= \left[1+\left(\frac{2n+1}{4nq}\right)h\right]\left[1-\left(\frac{2n-1}{4nq}\right)h\right]$$

$$= 1 + \left(\frac{2n+1}{4nq}\right)h - \left(\frac{2n-1}{4nq}\right)h$$

$$= 1 + \frac{2nh + h - 2nh + h}{4nq}$$

$$= 1 + \frac{2h}{4nq} = 1 + \frac{h}{2nq} \qquad \text{.........} \ (1)$$

$$\text{R.H.S.} = \left[\frac{p+q}{2q}\right]^{1/n}$$

$$= \left[\frac{q+h+q}{2q}\right]^{1/n}$$

$$= \left[\frac{2q+h}{2q}\right]^{1/n} = \left[\frac{2q}{2q} + \frac{h}{2q}\right]^{1/n}$$

$$= \left[1 + \frac{h}{2q}\right]^{1/n} = 1 + \frac{h}{2nq} \qquad \text{.........} \ (2)$$

By (1) and (2)

L.H.S. = R.H.S.

Q.8 Show that $\left[\frac{n}{2(n+N)}\right]^{1/2} = \frac{8n}{9n-N} - \frac{n+N}{4n}$ where n and N are nearly equal.

Solution:

Since, $N \approx n$

$\Rightarrow$ $N = n + h$, where 'h' is so small that its square and higher powers can be neglected.

$$\text{L.H.S.} = \left[\frac{n}{2(n+N)}\right]^{1/2}$$

$$= \left[\frac{n}{2(n+n+h)}\right]^{1/2} = \left[\frac{n}{2(2n+h)}\right]^{1/2} = \left[\frac{n}{4n+2h}\right]^{1/2}$$

$$= \left[\frac{n}{4n\left(1+\frac{2h}{4n}\right)}\right]^{1/2} = \left[\frac{1}{4^{1/2}\left(1+\frac{2h}{4n}\right)^{1/2}}\right]$$

$$= \frac{1}{\sqrt{4}}\left[1+\frac{2h}{4n}\right]^{-1/2} = \frac{1}{2}\left[1-\frac{2h}{8n}\right]$$

$$= \frac{1}{2}\left[1-\frac{h}{4n}\right] \qquad \text{.........} \ (1)$$

$$\text{R.H.S.} = \frac{8n}{9n-n} - \frac{n+N}{4n}$$

$$= \frac{8n}{9n-(n+h)} - \frac{(n+n+h)}{4n}$$

$$= \frac{8n}{9n-n-h} - \frac{2n+h}{4n}$$

$$= \frac{8n}{8n-h} - \frac{2n+h}{4n}$$

$$= \frac{8n}{8n\left(1-\frac{h}{8n}\right)} - \frac{2n+h}{4n}$$

$$= \left(1-\frac{h}{8n}\right)^{-1} - \frac{2n+h}{4n}$$

$$\approx 1 + \frac{h}{8n} - \left(\frac{2n+h}{4n}\right)$$

$$\approx \frac{8n+h-4n-2h}{8n}$$

$$\approx \frac{4n-h}{8n} \approx \frac{4n}{8n} - \frac{h}{8n} \approx \frac{1}{2} - \frac{h}{8n}$$

$$\approx \frac{1}{2}\left[1-\frac{h}{4n}\right] \qquad \text{..........} (2)$$

From (1) and (2), we have

L.H.S. = R.H.S.

Q.9 Identify the following series as binomial expansion and find the sum in ease.

(i) $1 - \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1.3}{2!.4}\left(\frac{1}{4}\right)^2 - \frac{1.3.5}{3!.8}\left(\frac{1}{4}\right)^3 + \ldots\ldots$

(ii) $1 - \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1.3}{2.4}\left(\frac{1}{2}\right)^2 - \frac{1.3.5}{2.4.6}\left(\frac{1}{2}\right)^3 + \ldots\ldots$

(iii) $1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \ldots\ldots$

(iv) $1 - \frac{1}{2}\cdot\frac{1}{3} + \frac{1.3}{2.4}\left(\frac{1}{3}\right)^2 - \frac{1.3.5}{2.4.6}\left(\frac{1}{3}\right)^3 + \ldots\ldots$

Solution:

(i) $1 - \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1 \,.\, 3}{2! \,.\, 4}\left(\frac{1}{4}\right)^2 - \frac{1 \,.\, 3 \,.\, 5}{3! \,.\, 8}\left(\frac{1}{4}\right)^3 + \ldots\ldots$

Let $(1 + x)^n = 1 - \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1.3}{2!\,4}\left(\frac{1}{4}\right)^2 - \ldots\ldots$

Also, $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots$

Now, comparing term by term of the above two equations, we have

$$nx = \frac{-1}{8} \qquad \ldots\ldots\ldots (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{3}{128} \qquad \ldots\ldots\ldots (2)$$

$$\therefore \qquad \frac{1.3}{2!\,4}\left(\frac{1}{4}\right)^2 = \frac{1.3}{2\,.\,1\,.\,4}\,\frac{1}{16} = \frac{3}{128}$$

By (1), we have

$$x = \frac{-1}{8n} \qquad \ldots\ldots\ldots (3)$$

Putting the value of x in (2)

$$\frac{n(n-1)}{2!}\cdot\frac{1}{64\,n^2} = \frac{3}{128}$$

$$\frac{n-1}{128n} = \frac{3}{128}$$

Multiplying both sides by 128

$$\frac{n-1}{n} = 3 \quad \Rightarrow \quad n - 1 = 3n$$

$$\Rightarrow \qquad -1 = 2n$$

$$\Rightarrow \qquad \boxed{n = \frac{-1}{2}}$$

Putting value of n in (3)

$$x = -\frac{1}{48\left(\frac{-1}{2}\right)}$$

$$\Rightarrow \qquad \boxed{x = \frac{1}{4}}$$

Now, putting the values of x and n in,

$$(1+x)^n = \left(1+\frac{1}{4}\right)^{-1/2} = \left(\frac{5}{4}\right)^{-1/2}$$

$$\text{Required sum} = \left(\frac{4}{5}\right)^{1/2} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

(ii) $$\mathbf{1-\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1.3}{2.4}\left(\frac{1}{2}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{2}\right)^3+\ldots\ldots}$$

$$\text{Let } (1+x)^n = 1-\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)+\frac{1.3}{2.4}\left(\frac{1}{2}\right)^2-\ldots\ldots.$$

$$\text{Also, } (1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\ldots\ldots.$$

Now comparing term by term of the above two equations, we have

$$nx = -\frac{1}{4} \qquad \ldots\ldots\ldots (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{3}{32} \qquad \ldots\ldots\ldots (2)$$

$$\therefore \quad \frac{1.3}{2.4}\,\frac{1}{4} = \frac{3}{32}$$

By (1), we have

$$x = -\frac{1}{4n} \qquad \ldots\ldots\ldots (3)$$

[Putting the value of x in (2)]

$$\frac{n(n-1)}{2!}\,\frac{1}{16n^2} = \frac{3}{32}$$

$$\frac{n-1}{32n} = \frac{3}{32}$$

$$\Rightarrow \quad \frac{n-1}{n} = 3 \qquad (\because \text{ since multiply both sides by } 32)$$

$$\Rightarrow \quad n-1 = 3n$$

$$\Rightarrow \quad -1 = 3n-n$$

$$-1 = 2n$$

$$\Rightarrow \quad \boxed{n = \frac{-1}{2}}$$

Putting the value of n in (3)

$$x = -\frac{1}{2 + \left(-\frac{1}{2}\right)}$$

$$\boxed{x = \frac{1}{2}}$$

Now, putting the values of x and n in

$$(1 + x)^n = \left(1 + \frac{1}{2}\right)^{-1/2} = \left(\frac{3}{2}\right)^{-1/2} = \left(\frac{2}{3}\right)^{1/2} = \sqrt{\frac{2}{3}}, \text{ required sum}$$

(iii) $\mathbf{1 + \frac{3}{4} + \frac{3\,.\,5}{4\,.\,8} + \frac{3\,.\,5\,.\,7}{4\,.\,8\,.\,12} + \ldots\ldots}$

Let $(1 + x)^n = 1 + \frac{3}{4} + \frac{3\,.\,5}{4\,.\,8} + \ldots\ldots$

Also, $(1 + x)^n = 1 + nx + \frac{n\,(n-1)}{2!}x^2 + \ldots\ldots$

Now comparing term by term above two equations, we have

$$nx = \frac{3}{4} \qquad \ldots\ldots\ldots (1)$$

$$\frac{n\,(n-1)}{2!}x^2 = \frac{15}{32} \qquad \ldots\ldots\ldots (2)$$

By (1), we have

$$x = \frac{3}{4n} \qquad \ldots\ldots\ldots (3)$$

Putting the value of x in (2)

$$\frac{n\,(n-1)}{2!}\,\frac{9}{16n^2} = \frac{15}{32}$$

$$\frac{(n-1)\,9}{32n} = \frac{15}{32}$$

$$\frac{(n-1)\,9}{n} = 15 \qquad (\because \text{ multiplying both sides by } 32)$$

$$9n - 9 = 15n$$

$$-9 = 15n - 9n$$

$$-9 = 6n \Rightarrow n = \frac{-9}{6} \Rightarrow \boxed{n = \frac{-3}{2}}$$

Putting the value of n in (3).

$$x = \frac{3}{4\left(\frac{-3}{2}\right)}$$

$$\Rightarrow \quad \boxed{x = \frac{-1}{2}}$$

Now putting the values of x and n in,

$$(1+x)^n = \left(1 - \frac{1}{2}\right)^{-3/2} = \left(\frac{1}{2}\right)^{-3/2} = (2)^{3/2} = 2 \,.\, \sqrt{2}$$

(iv) $\mathbf{1 - \frac{1}{2} \cdot \frac{1}{3} + \frac{1\,.\,3}{2\,.\,4}\left(\frac{1}{3}\right)^2 - \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\left(\frac{1}{3}\right)^3 + \ldots\ldots}$

Let $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots.$

Now comparing term by term of the above two equations, we have

$$nx = \frac{-1}{6} \qquad \ldots\ldots\ldots (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{3}{8}\,\frac{1}{9}$$

$$\frac{n(n-1)}{2!}x^2 = \frac{1}{24} \qquad \ldots\ldots\ldots (2)$$

From (1) $\quad x = \frac{-1}{6n} \quad$ Put in

$$\frac{n^2 - n}{2}\left(\frac{1}{36\,n^2}\right) = \frac{1}{24}$$

$$\frac{n(n-1)}{2} \times \frac{1}{36\,n^2} = \frac{1}{24}$$

$$\frac{n-1}{72n} = \frac{1}{24}$$

$$24n - 24 = 72n$$

$$-24 = 48n$$

$$\boxed{n = \frac{-1}{2}} \quad \text{Put in (1)}$$

$$x = \frac{-1}{6} \times \frac{-2}{1}$$

$$x = \frac{1}{3}$$

Hence $\left(1 + \frac{1}{3}\right)^{-1/2} = \left(\frac{4}{3}\right)^{-1/2} = \left(\frac{3}{4}\right)^{1/2} = \frac{\sqrt{3}}{2}$

Q.10 Use binomial theorem to show that, $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \ldots\ldots = \sqrt{2}$

Solution:

L.H.S.

Let $(1 + x)^n = 1 + \frac{1}{4} + \frac{1\,.\,3}{4\,.\,8} + \ldots\ldots$

Also, $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots$

Now comparing term by term the above two equations, we have

$$nx = \frac{1}{4} \quad \ldots\ldots\ldots (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{3}{32} \quad \ldots\ldots\ldots (2)$$

By (1), we have

$$x = \frac{1}{4n} \quad \ldots\ldots\ldots (3)$$

Putting the value of x in (2)

$$\frac{n(n-1)}{2!}\,\frac{1}{16n^2} = \frac{3}{32}$$

$$\frac{n-1}{32n} = \frac{3}{32}$$

$$\frac{n-1}{n} = 3 \quad (\because \text{ multiplying both sides by 32})$$

$$n - 1 = 3n$$

$$-1 = 3n - n$$

$$-1 = 2n \quad \Rightarrow \quad \boxed{n = \frac{-1}{2}}$$

Putting the values of n and x in (3)

$$(1+x)^n = \left(1 - \frac{1}{2}\right)^{-1/2} = \left(\frac{1}{2}\right)^{-1/2} = (2)^{1/2} = \sqrt{2} \text{ R.H.S.}$$

Hence proved.

Q.11 If $y = \frac{1}{3} + \frac{1.3}{2!}\left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3!}\left(\frac{1}{3}\right)^2 + ...$ then prove that $y^2 + 2y - 2 = 0$

Solution:

By adding '1' on both sides,

$$1 + y = 1 + \frac{1}{3} + \frac{1.3}{2!}\left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3!}\left(\frac{1}{3}\right)^2 + ...$$

$$\text{Let } (1+x)^n = 1 + \frac{1}{3} + \frac{1.3}{2!}\left(\frac{1}{3}\right)^2 + ...$$

Also, $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...$

Now comparing term by term above two equations, we have

$$nx = \frac{1}{3} \qquad .... (1)$$

$$\frac{n(n-1)x^2}{2!} = \frac{1.3}{2!}\left(\frac{1}{3}\right)^2 = \frac{3}{2}\left(\frac{1}{9}\right) = \frac{3}{18} \qquad .... (2)$$

By (1) we have,

$$x = \frac{1}{3n} \qquad .... (3)$$

Putting the value of x in (2)

$$\frac{n(n-1)}{2}\left(\frac{1}{3n}\right)^2 = \frac{3}{18}$$

$$\frac{n(n-1)}{2} \times \frac{1}{9n^2} = \frac{3}{18}$$

$$\frac{n-1}{18n} = \frac{3}{18}$$

$$\frac{n-1}{n} = 3 \qquad (\because \text{ multiplying both sides by } 18)$$

$$n - 1 = 3n$$

$$\Rightarrow \quad \boxed{n = \frac{-1}{2}}$$

Putting the value of n in (3)

$$x = \frac{1}{3\left(\frac{-1}{2}\right)} = \frac{1}{\frac{-3}{2}} = \frac{-2}{3}$$

$$\Rightarrow \quad \boxed{x = \frac{-2}{3}}$$

Now, putting the values of x and n in,

$$(1 + x)^n = \left(1 + \left(\frac{-2}{3}\right)\right)^{-1/2} = \left(1 - \frac{2}{3}\right)^{-1/2}$$

$$(1 + y) = \left(\frac{1}{3}\right)^{-1/2} = \sqrt{3}$$

Taking square on both sides,

$$(1 + y)^2 = (\sqrt{3})^2$$

$$1 + y^2 + 2y = 3$$

$$y^2 + 2y + 1 = 3$$

$$\Rightarrow \quad y^2 + 2y + 1 - 3 = 0$$

$$\Rightarrow \quad y^2 + 2y - 2 = 0$$

Hence proved.

Q.12 If $2y = \frac{1}{2^2} + \frac{1.3}{2!} \cdot \frac{1}{2^4} + \frac{1.3.5}{3!}\left(\frac{1}{2^6}\right) + \ldots\ldots$ then prove that $4y^2 + 4y - 1 = 0$.

(Lahore Board 2006)

Solution:

By adding '1' on both sides

$$1 + 2y = 1 + \frac{1}{2^2} + \frac{1.3}{2!}\,\frac{1}{2^4} + \frac{1.3.5}{3!}\,\frac{1}{2^6} + \ldots\ldots$$

$$\text{Let } (1 + x)^n = 1 + \frac{1}{2^2} + \frac{1.3}{2!}\,\frac{1}{2^4} + \ldots\ldots$$

$$\text{Also, } (1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots$$

Now, comparing term by term of the above two equations, we have

$$nx = \frac{1}{2^2} = \frac{1}{4} \qquad \text{.........} (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{1.3}{2!}\frac{1}{2^4} = \frac{3}{2}\times\frac{1}{16} = \frac{3}{32} \qquad \text{.........} (2)$$

By (1) we have,

$$x = \frac{1}{4n} \qquad \text{.........} (3)$$

Putting the value of x in (2)

$$\frac{n(n-1)}{2!}\left(\frac{1}{4n}\right)^2 = \frac{3}{32}$$

$$\frac{n(n-1)}{2!}\times\frac{1}{16n^2} = \frac{3}{32}$$

$$\frac{n-1}{2}\frac{1}{16n} = \frac{3}{32}$$

$$\frac{n-1}{32n} = \frac{3}{32}$$

$$\frac{n-1}{n} = 3 \qquad (\because \text{ multiplying both sides by } 32)$$

$$\frac{n-1}{n} = 3$$

$$\Rightarrow \quad n-1 = 3n$$

$$\Rightarrow \quad \boxed{n = -\frac{1}{2}}$$

Putting the value of n in (3)

$$x = \frac{1}{2^{4(-1/2)}}$$

$$\Rightarrow \quad \boxed{x = \frac{-1}{2}}$$

Now putting the values of x and n in

$$(1+x)^n = \left(1+\left(\frac{-1}{2}\right)\right)^{-1/2}$$

$$(1 + 2y) = \left(1 - \frac{1}{2}\right)^{-1/2} = \left(\frac{1}{2}\right)^{-1/2}$$

$$(1 + 2y) = \sqrt{2}$$

Taking square on both sides

$$(1 + 2y)^2 = \left(\sqrt{2}\right)^2$$

$$1 + 4y^2 + 4y = 2$$

$$\Rightarrow \quad 4y^2 + 4y + 1 - 2 = 0$$

$$\Rightarrow \quad 4y^2 + 4y - 1 = 0$$

Hence proved.

Q.13 If $y = \frac{2}{5} + \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3 + \ldots\ldots\ldots$ then prove that $y^2 + 2y - 4 = 0$.

(Gujranwala Board 2003)

Solution:

By adding '1' on both sides,

$$1 + y = 1 + \frac{2}{5} + \frac{1\,.\,3}{2!}\left(\frac{2}{5}\right)^2 + \frac{1\,.\,3\,.\,5}{3!}\left(\frac{2}{5}\right)^3 + \ldots\ldots$$

$$\text{Let } (1 + x)^n = 1 + \frac{2}{5} + \frac{1\,.\,3}{2!}\left(\frac{2}{5}\right)^2 + \ldots\ldots$$

$$\text{Also, } (1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots\ldots$$

Now, comparing term by term of the above two equations we have

$$nx = \frac{2}{5} \qquad \ldots\ldots\ldots (1)$$

$$\frac{n(n-1)}{2!}x^2 = \frac{1\,.\,3}{2!}\left(\frac{2}{5}\right)^5 = \frac{3}{2} \times \frac{4}{25} = \frac{6}{25} \qquad \ldots\ldots\ldots (2)$$

By (1) we have,

$$x = \frac{2}{5n} \qquad \ldots\ldots\ldots (3)$$

Putting the value of x in (2)

$$\frac{n(n-1)}{2} \times \left(\frac{2}{5n}\right)^2 = \frac{6}{25}$$

$$\frac{n(n-1)}{2} \times \frac{4}{25n^2} = \frac{6}{25}$$

$$\frac{n-1}{2} \times \frac{4}{25} = \frac{6}{25}$$

$$\frac{2(n-1)}{25n} = \frac{6}{25}$$

$$\frac{2(n-1)}{n} = 6 \qquad (\because \text{ multiplying both sides by } 25)$$

$$2(n-1) = 6n$$

$$n - 1 = 3n \quad \Rightarrow \quad \boxed{n = \frac{-1}{2}}$$

Putting the value of n in (3)

$$x = \frac{2}{5\left(\frac{-1}{2}\right)} = \frac{2}{\frac{-5}{2}}$$

$$\boxed{x = \frac{-4}{5}}$$

Now, putting the values of x and n in,

$$(1+x)^n = \left(1 - \frac{4}{5}\right)^{-1/2}$$

$$(1+y) = \left(\frac{1}{5}\right)^{-1/2}$$

$$(1+y) = \sqrt{5}$$

Taking square on both sides,

$$(1+y)^2 = \left(\sqrt{5}\right)^2$$

$$1 + y^2 + 2y = 5$$

$$\Rightarrow \qquad y^2 + 2y - 4 = 0$$

Hence proved.