

**Distance formula:**

let  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  be two points. If  $d$  denotes the distance between them then

$$d = |PQ| = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

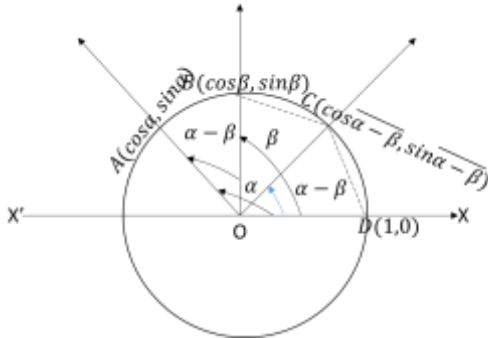
**Fundamental law of trigonometry:**

let  $\alpha$  and  $\beta$  be any two angles (real numbers) then

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Which is called the fundamental law of trigonometry.

Proof:



Consider a unit circle at O.

where  $\angle AOD = \alpha$ ,  $\angle BOD = \beta$

$$\angle AOB = \angle COB = \alpha - \beta$$

Now  $\triangle AOB$  and  $\triangle COB$  are congruent.

Then  $|AB| = |CD|$

$$\Rightarrow |AB|^2 = |CD|^2$$

Using distance formula we have

$$\begin{aligned} (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 \\ = (\cos\alpha - \cos\beta - 1)^2 + (\sin\alpha - \sin\beta - 0)^2 \end{aligned}$$

$$\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta$$

$$\cos^2\alpha - \cos^2\beta + 1 - 2\cos\alpha - \cos\beta + \sin^2\alpha - \sin^2\beta$$

$$\cos^2\alpha + \sin^2\alpha + \cos^2\beta - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

$$= \cos^2\alpha - \cos^2\beta + \sin^2\alpha - \sin^2\beta + 1 - 2\cos(\alpha - \beta)$$

$$\Rightarrow 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 2 - 2\cos(\alpha - \beta)$$

Subtract 2 from both sides

$$\Rightarrow -2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = -2\cos(\alpha - \beta)$$

$$\Rightarrow (\cos\alpha\cos\beta + \sin\alpha\sin\beta) = \cos(\alpha - \beta) \div -2$$

Or

$$\cos(\alpha - \beta) = (\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

**Note:**

We have proved this law for  $\alpha > \beta > 0$ , it is true for all values of  $\alpha$  and  $\beta$

**Deduction from fundamental law:**

1) we know that

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\text{put } \alpha = \frac{\pi}{2} \text{ we get}$$

$$\cos\left(\frac{\pi}{2} - \beta\right) = \cos\frac{\pi}{2}\cos\beta + \sin\frac{\pi}{2}\sin\beta$$

$$= (0)\cos\beta + (1)\sin\beta$$

$$\Rightarrow \cos\left(\frac{\pi}{2} - \beta\right) = \sin\beta$$

$$(\because \cos\frac{\pi}{2} = 0 \quad \sin\frac{\pi}{2} = 1)$$

2) we know that

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\text{put } \beta = -\frac{\pi}{2} \text{ we get}$$

$$\cos\left(\alpha - \left(-\frac{\pi}{2}\right)\right) = \cos\alpha\cos\left(-\frac{\pi}{2}\right) + \sin\alpha\sin\left(-\frac{\pi}{2}\right)$$

$$\cos\left(\alpha + \frac{\pi}{2}\right) = \cos\alpha\cos\frac{\pi}{2} - \sin\alpha\sin\frac{\pi}{2}$$

$$= \cos\alpha(0) - \sin\alpha(1)$$

$$\Rightarrow \cos\left(\alpha + \frac{\pi}{2}\right) = -\sin\alpha$$

$$(\because \cos\left(-\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0)$$

$$\because \sin\left(-\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$$

3) we know that

$$\cos\left(\frac{\pi}{2} - \beta\right) = \sin\beta$$

Put  $\beta = \frac{\pi}{2} + \alpha$  we get

$$\Rightarrow \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} + \alpha\right)\right) = \sin\left(\frac{\pi}{2} + \alpha\right)$$

$$\Rightarrow \cos\left(\frac{\pi}{2} - \frac{\pi}{2} - \alpha\right) = \sin\left(\frac{\pi}{2} + \alpha\right)$$

$$\Rightarrow \cos(-\alpha) = \sin\left(\frac{\pi}{2} + \alpha\right)$$

$$\Rightarrow \cos\alpha = \sin\left(\frac{\pi}{2} + \alpha\right)$$

$$\Rightarrow \sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha$$

4) we know that

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

replacing  $\beta$  by  $-\beta$  we get

$$\cos[\alpha - (-\beta)] = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$$

$$\Rightarrow \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$(\because \cos(-\beta) = \cos\beta, \quad \sin(-\beta) = -\sin\beta)$$

5) we know that

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Replacing  $\alpha$  by  $\frac{\pi}{2} + \alpha$

$$\cos\left(\frac{\pi}{2} + \alpha + \beta\right) = \cos\frac{\pi}{2}\cos\beta - \sin\frac{\pi}{2}\sin\beta$$

$$\cos\left(\left(\frac{\pi}{2} + \alpha\right) + \beta\right) = -\sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$-\sin(\alpha + \beta) = -(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$(\because \sin\left(\frac{\pi}{2} + \alpha\right) = -\cos\alpha, \cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha)$$

6) we know that

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Replacing  $\beta$  by  $(-\beta)$  we get

$$\sin(\alpha + (-\beta)) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\because \cos(-\beta) = \cos\beta, \quad \sin(-\beta) = -\sin\beta$$

7) we know that

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\text{let } \alpha = 2\pi \text{ and } \beta = \theta$$

$$\cos(2\pi - \theta) = \cos 2\pi\cos\theta + \sin 2\pi\sin\theta$$

$$\cos(2\pi - \theta) = (1)\cos\theta + (0)\sin\theta$$

$$\Rightarrow \cos(2\pi - \theta) = \cos\theta \quad \because \cos 2\pi = 1 \text{ and } \sin 2\pi = 0$$

8) we know that

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\text{let } \alpha = 2\pi \text{ and } \beta = \theta$$

$$\sin(2\pi - \theta) = \sin 2\pi\cos\theta - \cos 2\pi\sin\theta$$

$$\sin(2\pi - \theta) = (0)\cos\theta - (1)\sin\theta$$

$$\sin(2\pi - \theta) = -\sin\theta$$

$$9) \because \tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$$

$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

During up and down by  $\cos\alpha\cos\beta$

$$\begin{aligned} & \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta} \\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta} \end{aligned}$$

Thus,

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$10) \because \tan(\alpha - \beta) = \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)}$$

$$= \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

During up and down by  $\cos\alpha\cos\beta$

$$\begin{aligned} & \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} - \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta} \\ &= \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta} \end{aligned}$$

Thus,

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Trigonometric Ratio of allied Angle:

Allied angles:

The angle associated with basic angle of measure  $\theta$  to a right angle or multiple are called allied angles.

Examples:

$$90^\circ \pm \theta, \quad 180^\circ \pm \theta, \quad 270^\circ \pm \theta, \quad 360^\circ \pm \theta \text{ etc.}$$

Remember some basic results of allied angles:

1) If  $\theta$  is add to or subtracted from odd multiple of right angle, trigonometric ratio change into co-ratio and versa. i.e

$$\sin \rightleftharpoons \cos, \quad \tan \rightleftharpoons \cot, \quad \sec \rightleftharpoons \csc$$

**sin → cos**

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \quad \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$$

$$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta$$

**cos → sin**

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \quad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

$$\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$$

$$\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$$

**tan → cot**

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta, \quad \tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$$

$$\tan\left(\frac{3\pi}{2} - \theta\right) = \cot\theta$$

$$\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta$$

2) if  $\theta$  is add or subtracted from an even multiple of  $\frac{\pi}{2}$ , the trigonometric ratio shall remain the same.

3) so far as the sign of result is concerned, it is determined by the quadrant in which the terminal arm of the angle lies.

**sin → sin**

$$\sin(\pi - \theta) = \sin\theta, \quad \sin(\pi + \theta) = -\sin\theta$$

$$\sin(2\pi - \theta) = -\sin\theta, \quad \sin(2\pi + \theta) = \sin\theta$$

**COS → COS**

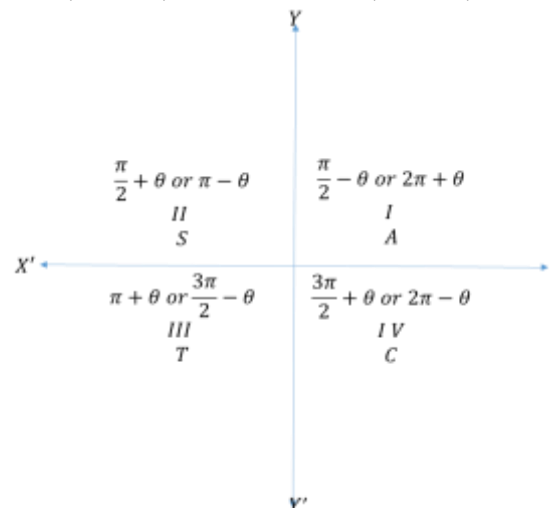
$$\cos(\pi - \theta) = -\cos\theta, \quad \cos(\pi + \theta) = -\cos\theta$$

$$\cos(2\pi - \theta) = \cos\theta, \quad \cos(2\pi + \theta) = \cos\theta$$

**tan → tan**

$$\tan(\pi - \theta) = -\tan\theta, \quad \tan(\pi + \theta) = \tan\theta$$

$$\tan(2\pi - \theta) = -\tan\theta, \quad \tan(2\pi + \theta) = \tan\theta$$



## Exercise 10.1

**Question # 1**

Without using the tables, find the value of :

**(i).  $\sin(-780^\circ)$** 

Solution.

$$\sin(-780^\circ) = -\sin(780^\circ)$$

$$\sin(-780^\circ) = -\sin(8(90^\circ) + 60^\circ)$$

$$\sin(-780^\circ) = -\sin(60^\circ)$$

$$\sin(-780^\circ) = -\frac{\sqrt{3}}{2}$$

Answer.

**(ii).  $\cot(-885^\circ)$** 

Solution.

$$\cot(-885^\circ) = -\cot(885^\circ)$$

$$\cot(-885^\circ) = -\cot(9(90^\circ) + 45^\circ)$$

$$\cot(-885^\circ) = -\tan(45^\circ)$$

$$\cot(-885^\circ) = -1$$

Answer.

**(iii).  $\csc(2040^\circ)$** 

Solution.

$$\csc(2040^\circ) = \csc(22(90^\circ) + 60^\circ)$$

$$\csc(2040^\circ) = -\csc(60^\circ)$$

$$\csc(2040^\circ) = -\frac{2}{\sqrt{3}}$$

Answer.

**(iv).  $\sec(-960^\circ)$** 

Solution.

$$\sec(960^\circ) = \sec(10(90^\circ) + 60^\circ)$$

$$\sec(960^\circ) = -\sec(60^\circ)$$

$$\sec(-960^\circ) = -2.$$

Answer.

**(v).  $\tan(1110^\circ)$** 

Solution.

$$\tan(1110^\circ) = \tan(12(90^\circ) + 30^\circ)$$

$$\tan(1110^\circ) = \tan(30^\circ)$$

$$\tan(1110^\circ) = \frac{1}{\sqrt{3}}$$

Answer.

**(vi).  $\sin(-300^\circ)$** 

Solution.

$$\because \sin(-\theta) = -\sin\theta$$

$$= -\sin 300^\circ$$

$$= -\sin(360^\circ - 60^\circ)$$

$$= -\sin\left(4\frac{\pi}{2} - 60^\circ\right)$$

$$= -(-\sin 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\because \sin\left(\frac{4\pi}{2} - \theta\right) = -\sin\theta$$

**Question # 2**

Express each of the following as a trigonometric function of an angle of positive degree measure of less than  $45^\circ$ .

**(i).  $\sin 196^\circ$** 

Solution:

$$\sin(196^\circ) = \sin(180^\circ + 16^\circ)$$

$$\sin(196^\circ) = \sin 180^\circ \cos 16^\circ + \cos 180^\circ \sin 16^\circ$$

$$\sin(196^\circ) = (0) \cos 16^\circ + (-1) \sin 16^\circ$$

$$\sin(196^\circ) = -\sin 16^\circ.$$

Answer.

**(ii).  $\cos 147^\circ$** 

Solution.

$$\cos(147^\circ) = \cos(180^\circ - 33^\circ)$$

$$\cos(147^\circ) = \cos 180^\circ \cos 33^\circ + \sin 180^\circ \sin 33^\circ$$

$$\cos(147^\circ) = (-1) \cos 33^\circ + (0) \sin 33^\circ$$

$$\cos(147^\circ) = -\cos 33^\circ.$$

Answer.

**(iii).  $\sin 319^\circ$** 

Solution.

$$\sin(319^\circ) = \sin(360^\circ - 41^\circ)$$

$$\sin(319^\circ) = \sin 360^\circ \cos 41^\circ - \cos 360^\circ \sin 41^\circ$$

$$\sin(319^\circ) = (0) \cos 41^\circ - (1) \sin 41^\circ$$

$$\sin(319^\circ) = -\sin 41^\circ.$$

Answer.

**(iv).  $\cos 254^\circ$** 

Solution.

$$\cos(254^\circ) = \cos(270^\circ - 16^\circ)$$

$$\cos(254^\circ) = \cos 270^\circ \cos 16^\circ + \sin 270^\circ \sin 16^\circ$$

$$\cos(254^\circ) = (0) \cos 16^\circ + (-1) \sin 16^\circ$$

$$\cos(254^\circ) = -\sin 16^\circ.$$

Answer.

**(v).  $\tan 294^\circ$** 

Solution.

$$\tan 294^\circ = \frac{\sin 294^\circ}{\cos 294^\circ}$$

$$\tan 294^\circ = \frac{\sin(270^\circ + 24^\circ)}{\cos(270^\circ + 24^\circ)}$$

$$\tan 294^\circ = \frac{\sin 270^\circ \cos 24^\circ + \cos 270^\circ \sin 24^\circ}{(-1) \cos 24^\circ + (0) \sin 24^\circ}$$

$$\tan 294^\circ = \frac{\cos 24^\circ}{(-1) \cos 24^\circ + (0) \sin 24^\circ}$$

$$\tan 294^\circ = \frac{\cos 24^\circ}{(-1) \cos 24^\circ - (-1) \sin 24^\circ}$$

$$\tan 294^\circ = \frac{\cos 24^\circ}{\cos 24^\circ - \sin 24^\circ}$$

$$\tan 294^\circ = -\frac{\cos 24^\circ}{\sin 24^\circ}$$

$$\tan 294^\circ = -\cot 24^\circ.$$

$$\tan 294^\circ = -\cot 24^\circ.$$

Answer.

**(vi).  $\cos(728^\circ)$** 

Solution.

$$\cos(728^\circ) = \cos(720^\circ + 8^\circ)$$

$$\cos(728^\circ) = \cos 720^\circ \cos 8^\circ - \sin 720^\circ \sin 8^\circ$$

$$\cos(728^\circ) = (1) \cos 8^\circ + (0) \sin 8^\circ$$

$$\cos(728^\circ) = \cos 8^\circ.$$

Answer.

**(vii).  $\sin(-625^\circ)$**

Solution.

$$\sin(-625^\circ) = -\sin(625^\circ)$$

$$\sin(-625^\circ) = -\sin(630^\circ - 5^\circ)$$

$$\sin(-625^\circ) = -(\sin 630^\circ \cos 5^\circ - \cos 630^\circ \sin 5^\circ)$$

$$\sin(-625^\circ) = -((-1)\cos 5^\circ - (0)\sin 5^\circ)$$

$$\sin(-625^\circ) = \cos 5^\circ.$$

Answer.

(viii).  $\cos(-435^\circ)$

Solution.

$$\cos(-435^\circ) = \cos(435^\circ)$$

$$\cos(-435^\circ) = \cos(450^\circ - 15^\circ)$$

$$\cos(-435^\circ) = \cos 450^\circ \cos 15^\circ + \sin 450^\circ \sin 15^\circ$$

$$\cos(-435^\circ) = (0)\cos 15^\circ + (1)\sin 15^\circ$$

$$\cos(-435^\circ) = \sin 15^\circ.$$

Answer.

(ix)  $\sin 150^\circ$

$$= \sin(180^\circ - 30^\circ)$$

$$= \sin(\pi - 30^\circ)$$

$$= \sin 30^\circ \quad \because \begin{pmatrix} \sin(\pi - \theta) \\ = \sin \theta \end{pmatrix}$$

Question # 3

Prove the following:

(i).  $\sin(180^\circ + \alpha)\sin(90^\circ - \alpha) = -\sin\alpha\cos\alpha$

Solution.

$$L.H.S = \sin(180^\circ + \alpha)\sin(90^\circ - \alpha)$$

$$L.H.S$$

$$= (\sin 180^\circ \cos \alpha + \cos 180^\circ \sin \alpha)(\sin 90^\circ \cos \alpha - \cos 90^\circ \sin \alpha)$$

$$L.H.S$$

$$= ((0)\cos \alpha + (-1)\sin \alpha)((1)\cos \alpha - (0)\sin \alpha)$$

$$L.H.S = (-\sin \alpha)(\cos \alpha)$$

$$L.H.S = -\sin \alpha \cos \alpha$$

$$L.H.S = R.H.S$$

Hence Proved.

(ii).  $\sin 780^\circ \sin 480^\circ + \cos 120^\circ \sin 30^\circ = \frac{1}{2}$

Solution.

$$L.H.S = \sin 780^\circ \sin 480^\circ + \cos 120^\circ \sin 30^\circ$$

$$L.H.S = \sin(720^\circ + 60^\circ)\sin(450^\circ + 30^\circ)$$

$$+ \cos 120^\circ \sin 30^\circ$$

$$L.H.S = \sin(60^\circ)\sin(30^\circ) + \cos 120^\circ \sin 30^\circ$$

$$L.H.S = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$L.H.S = \frac{3}{4} - \frac{1}{4}$$

$$L.H.S = \frac{2}{4}$$

$$L.H.S = \frac{1}{2}$$

$$L.H.S = R.H.S$$

Hence Proved.

(iii).  $\sin 306^\circ + \cos 234^\circ + \cos 162^\circ + \cos 18^\circ = 0$

Solution.

$$L.H.S = \sin 306^\circ + \cos 234^\circ + \cos 162^\circ + \cos 18^\circ$$

$$L.H.S = \sin(3(90^\circ) + 36^\circ) + \cos(3(90^\circ) - 36^\circ)$$

$$+ \cos(180^\circ - 18^\circ) + \cos 18^\circ$$

$$L.H.S = \cos(36^\circ) - \sin(36^\circ) - \cos(18^\circ) + \cos 18^\circ$$

$$L.H.S = 0$$

$$L.H.S = R.H.S$$

Hence Proved.

(iv).  $\cos 330^\circ \sin 600^\circ + \cos 120^\circ \sin 150^\circ = -1$

Solution.

$$L.H.S = \cos 330^\circ \sin 600^\circ + \cos 120^\circ \sin 150^\circ$$

$$L.H.S = \cos(360^\circ - 30^\circ)\sin(3(180^\circ) + 60^\circ) + \cos(90^\circ + 30^\circ)\sin(90^\circ + 60^\circ)$$

$$L.H.S = \cos(30^\circ)(-\sin(60^\circ)) + (-\sin(30^\circ))\cos(60^\circ)$$

$$L.H.S = \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$L.H.S = -\frac{3}{4} - \frac{1}{4}$$

$$L.H.S = -1$$

$$L.H.S = R.H.S$$

Hence Proved.

Question # 4

Prove that;

$$(i). \frac{\sin^2(\pi + \theta)\tan\left(\frac{3\pi}{2} + \theta\right)}{\cot^2\left(\frac{3\pi}{2} - \theta\right)\cos^2(\pi - \theta)\operatorname{cosec}(2\pi - \theta)} = \cos\theta$$

Solution.

$$L.H.S = \frac{\sin^2(\pi + \theta)\tan\left(\frac{3\pi}{2} + \theta\right)}{\cot^2\left(\frac{3\pi}{2} - \theta\right)\cos^2(\pi - \theta)\operatorname{cosec}(2\pi - \theta)}$$

$$L.H.S = \frac{(-\sin\theta)^2(-\cot\theta)}{(\tan\theta)^2(-\cos\theta)^2(-\operatorname{cosec}\theta)}$$

$$L.H.S = \frac{\sin^2\theta \frac{\cos\theta}{\sin\theta}}{\frac{\sin^2\theta}{\cos^2\theta} \cos^2\theta \frac{1}{\sin\theta}}$$

$$L.H.S = \frac{\sin\theta \cos\theta}{\sin\theta}$$

$$L.H.S = \cos\theta$$

$$L.H.S = R.H.S$$

Hence Proved.

$$(ii). \frac{\cos(90^\circ + \theta)\sec(-\theta)\tan(180^\circ - \theta)}{\sec(360^\circ - \theta)\sin(180^\circ + \theta)\cot(90^\circ - \theta)} = -1$$

Solution.

$$L.H.S = \frac{\cos(90^\circ + \theta)\sec(-\theta)\tan(180^\circ - \theta)}{\sec(360^\circ - \theta)\sin(180^\circ + \theta)\cot(90^\circ - \theta)}$$

$$L.H.S = \frac{-\sin(\theta)\sec(\theta)(-\tan(\theta))}{\sec(\theta)(-\sin(\theta))\tan\theta}$$

$$L.H.S = \frac{-\sin(\theta)(-1)}{(-\sin(\theta))}$$

$$L.H.S = -1$$

$$L.H.S = R.H.S$$

Hence Proved.

## Question # 5

If  $\alpha, \beta, \gamma$  are the angles of a triangle  $ABC$ , then prove that;

(i).  $\sin(\alpha + \beta) = \sin\gamma$

Solution.

Since  $\alpha, \beta, \gamma$  are the angles of a triangle  $ABC$ , then

$$\alpha + \beta + \gamma = 180^\circ$$

$$\Rightarrow \alpha + \beta = 180^\circ - \gamma$$

$$\sin(\alpha + \beta) = \sin(180^\circ - \gamma)$$

$$\sin(\alpha + \beta) = \sin 180^\circ \cos\gamma - \cos 180^\circ \sin\gamma$$

$$\sin(\alpha + \beta) = (0)\cos\gamma - (-1)\sin\gamma$$

$$\sin(\alpha + \beta) = \sin\gamma$$

Hence Proved.

(ii).  $\cos\left(\frac{\alpha+\beta}{2}\right) = \sin\frac{\gamma}{2}$

Solution.

Since  $\alpha, \beta, \gamma$  are the angles of a triangle  $ABC$ , then

$$\alpha + \beta + \gamma = 180^\circ$$

$$\Rightarrow \alpha + \beta = 180^\circ - \gamma$$

$$\frac{\alpha + \beta}{2} = \frac{180^\circ - \gamma}{2}$$

$$\cos\left(\frac{\alpha + \beta}{2}\right) = \cos\left(\frac{180^\circ - \gamma}{2}\right)$$

$$\cos\left(\frac{\alpha + \beta}{2}\right) = \cos\left(90^\circ - \frac{\gamma}{2}\right)$$

$$\cos\left(\frac{\alpha + \beta}{2}\right) = \sin\left(90^\circ - \frac{\gamma}{2}\right)$$

$$\cos\left(\frac{\alpha + \beta}{2}\right) = \sin\left(\frac{\gamma}{2}\right)$$

Hence Proved.

(iii).  $\cos(\alpha + \beta) = -\cos\gamma$

Solution.

Since  $\alpha, \beta, \gamma$  are the angles of a triangle  $ABC$ , then

$$\alpha + \beta + \gamma = 180^\circ$$

$$\Rightarrow \alpha + \beta = 180^\circ - \gamma$$

$$\cos(\alpha + \beta) = \cos(180^\circ - \gamma)$$

$$\cos(\alpha + \beta) = \cos 180^\circ \cos\gamma + \sin 180^\circ \sin\gamma$$

$$\cos(\alpha + \beta) = (-1)\cos\gamma + (0)\sin\gamma$$

$$\cos(\alpha + \beta) = -\cos\gamma$$

Hence Proved.

(iv).  $\tan(\alpha + \beta) = \tan\gamma$

Solution.

Since  $\alpha, \beta, \gamma$  are the angles of a triangle  $ABC$ , then

$$\alpha + \beta + \gamma = 180^\circ$$

$$\Rightarrow \alpha + \beta = 180^\circ - \gamma$$

$$\tan(\alpha + \beta) = \tan(180^\circ - \gamma)$$

$$\tan(\alpha + \beta) = -\tan\gamma$$

$$\tan(\alpha + \beta) + \tan\gamma = 0$$

Hence Proved.

## Further Application of Basic identities

## Exercise 10.2

## Question#1 Prove that:

(i)  $\sin(180^\circ + \theta) = -\sin\theta$

L.H.S =  $\sin(180^\circ + \theta)$

$$= \sin 180^\circ \cos\theta + \cos 180^\circ \sin\theta$$

$$\therefore \sin 180^\circ = 0$$

$$= (0)\cos\theta + (-1)\sin\theta$$

$$\therefore \cos 180^\circ = -1$$

$$= -\sin\theta$$

(ii)  $\cos(180^\circ + \theta) = -\cos\theta$

L.H.S =  $\cos(180^\circ + \theta)$

$$= \cos 180^\circ \cos\theta - \sin 180^\circ \sin\theta$$

$$\therefore \sin 180^\circ = 0$$

$$= (-1)\cos\theta - (0)\sin\theta$$

$$\therefore \cos 180^\circ = -1$$

$$= -\cos\theta$$

(iii)  $\tan(270^\circ - \theta) = \cot\theta$

Solution:

$$L.H.S = \tan(270^\circ - \theta)$$

$$= \frac{\sin(270^\circ - \theta)}{\cos(270^\circ - \theta)}$$

$$= \frac{\sin 270^\circ \cos\theta - \cos 270^\circ \sin\theta}{\cos 270^\circ \cos\theta + \sin 270^\circ \sin\theta}$$

$$= \frac{(-1)\cos\theta - (0)\sin\theta}{(0)\cos\theta + (-1)\sin\theta} = \frac{-\cos\theta}{-\sin\theta}$$

$$= \cot\theta = R.H.S$$

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$$= \cot\theta = R.H.S$$

(iv)  $\cos(\theta - 180^\circ) = -\cos\theta$

Solution:

$$L.H.S = \cos(\theta - 180^\circ)$$

$$= \cos\theta \cos 180^\circ - \sin\theta \sin 180^\circ$$

$$= \cos(-1) - \sin\theta(0)$$

$$= -\cos\theta = R.H.S$$

$$= -\cos\theta = R.H.S$$

$$= -\cos\theta = R.H.S$$

$$= -\cos\theta = R.H.S$$

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$$= -\cos\theta = R.H.S$$

$$= -\cos\theta = R.H.S$$

$$= -\cos\theta = R.H.S$$

(vii)  $\tan(180^\circ + \theta) = \tan\theta$

Solution:

$$L.H.S = \tan(180^\circ + \theta)$$

$$= \frac{\sin(180^\circ + \theta)}{\cos(180^\circ + \theta)}$$

$$= \frac{\sin 180^\circ \cos\theta + \cos 180^\circ \sin\theta}{\cos 180^\circ \cos\theta - \sin 180^\circ \sin\theta}$$

$$= \frac{(0)\cos\theta + (-1)\sin\theta}{(-1)\cos\theta - (0)\sin\theta} = \frac{-\sin\theta}{-\cos\theta} = \tan\theta = R.H.S$$

$$= \frac{-\sin\theta}{-\cos\theta} = \tan\theta = R.H.S$$

$$= \frac{-\sin\theta}{-\cos\theta} = \tan\theta = R.H.S$$

$$= \frac{-\sin\theta}{-\cos\theta} = \tan\theta = R.H.S$$

$$= \frac{-\sin\theta}{-\cos\theta} = \tan\theta = R.H.S$$

$$\begin{aligned}
 &= \frac{\sin 180^\circ \cos \theta + \cos 180^\circ \sin \theta}{\cos 180^\circ \cos \theta - \sin 180^\circ \sin \theta} \\
 &= \frac{(0)\cos \theta + (-1)\sin \theta}{(-1)\cos \theta - (0)\sin \theta} \\
 &= \frac{-\sin \theta}{-\cos \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta = R.H.S
 \end{aligned}$$

Hence proved.

(viii) **Cos(360 - θ) = cos θ**

$$\begin{aligned}
 \text{L.H.S} &= \text{Cos}(360-\theta) \\
 &= \cos 360 \cos \theta + \sin 360 \sin \theta && \because \cos 360=1 \\
 &= (1)\cos \theta + (0)\sin \theta && \because \sin 360=0 \\
 &= \cos \theta = \text{R.H.S}
 \end{aligned}$$

Question#2 Find the values of the following

Note

We use  $15^\circ = 60^\circ - 45^\circ$  and  $105^\circ = 60^\circ + 45^\circ$

(i) Sin 15

$$\begin{aligned}
 &\Rightarrow \sin(60-45) \\
 &= \sin 60 \cos 45 - \cos 60 \sin 45 \\
 &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) \\
 &= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \\
 &= \frac{\sqrt{3}-1}{2\sqrt{2}}
 \end{aligned}$$

(iv) Sin 105

$$\begin{aligned}
 &\Rightarrow \sin(60+45) \\
 &= \sin 60 \cos 45 + \cos 60 \sin 45 \\
 &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) \\
 &= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \\
 &= \frac{\sqrt{3}+1}{2\sqrt{2}}
 \end{aligned}$$

(ii) Cos 15

$$\begin{aligned}
 &\Rightarrow \cos(60-45) \\
 &= \cos 60 \cos 45 - \sin 60 \sin 45 \\
 &= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) \\
 &= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}
 \end{aligned}$$

$$\frac{1 - \sqrt{3}}{2\sqrt{2}}$$

(v) Cos 105

$$\begin{aligned}
 &\Rightarrow \cos(60-45) \\
 &= \cos 60 \cos 45 + \sin 60 \sin 45 \\
 &= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) \\
 &= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}
 \end{aligned}$$

$$\frac{1 + \sqrt{3}}{2\sqrt{2}}$$

(iii) tan 15°

Solution:

$$\begin{aligned}
 \tan 15^\circ &= \tan(45^\circ - 30^\circ) \\
 &= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1 - \frac{1}{\sqrt{3}}}{1 + (1)\left(\frac{1}{\sqrt{3}}\right)} \\
 &= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}
 \end{aligned}$$

(vi) tan 105°

Solution:

$$\begin{aligned}
 \tan 105^\circ &= \tan(60^\circ + 45^\circ) \\
 &= \frac{\tan 60^\circ + \tan 45^\circ}{1 - \tan 60^\circ \tan 45^\circ} \\
 &= \frac{\sqrt{3} + 1}{1 - \sqrt{3}(1)} = \frac{\sqrt{3} + 1}{1 - \sqrt{3}}
 \end{aligned}$$

Question#3 Prove the following

i.  $\sin(45+\alpha) = \frac{1}{\sqrt{2}}(\sin \alpha + \cos \alpha)$

$$\begin{aligned}
 \text{L.H.S} &= \sin(45+\alpha) \\
 &= \sin 45 \cos \alpha + \cos 45 \sin \alpha \\
 &= \left(\frac{1}{\sqrt{2}}\right) \cos \alpha + \left(\frac{1}{\sqrt{2}}\right) \sin \alpha \\
 &= \frac{1}{\sqrt{2}}(\cos \alpha + \sin \alpha) = \text{R.H.S}
 \end{aligned}$$

ii.  $\cos(\alpha+45) = \frac{1}{\sqrt{2}}(\sin \alpha - \cos \alpha)$

$$\begin{aligned}
 &= \cos \alpha \cos 45 - \sin \alpha \sin 45 \\
 &= \left(\frac{1}{\sqrt{2}}\right) \cos \alpha - \left(\frac{1}{\sqrt{2}}\right) \sin \alpha \\
 &= \frac{1}{\sqrt{2}}(\cos \alpha - \sin \alpha) = \text{R.H.S}
 \end{aligned}$$

Question#4 Prove that

(i)  $\tan(45+A) \tan(45-A) = 1$

$$\begin{aligned}
 \text{solution:} & \text{L.H.S} = \tan(45+A) \tan(45-A) \\
 &= \left(\frac{\tan 45 + \tan A}{1 - \tan 45 \tan A}\right) \left(\frac{\tan 45 - \tan A}{1 + \tan 45 \tan A}\right) \\
 &= \left(\frac{1 + \tan A}{1 - (1)\tan A}\right) \left(\frac{1 - \tan A}{1 + (1)\tan A}\right) \because \tan 45 = 1 \\
 &= \left(\frac{1 + \tan A}{1 - \tan A}\right) \left(\frac{1 - \tan A}{1 + \tan A}\right) \\
 &= 1 \text{ (after cancellation)} \\
 &= \text{R.H.S}
 \end{aligned}$$

ii:-  $\tan\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{3\pi}{4} + \theta\right) = 0$

$$\begin{aligned}
 \text{solution:} & \text{L.H.S} = \tan\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{3\pi}{4} + \theta\right) \\
 &= \tan(45-\theta) + \tan(135+\theta) \\
 &= \left(\frac{\tan 45 - \tan \theta}{1 + \tan 45 \tan \theta}\right) + \left(\frac{\tan 135 + \tan \theta}{1 - \tan 135 \tan \theta}\right) \\
 &= \left(\frac{(1) - \tan \theta}{1 + (1)\tan \theta}\right) + \left(\frac{(-1) + \tan \theta}{1 - (-1)\tan \theta}\right) \\
 &= \left(\frac{1 - \tan \theta}{1 + \tan \theta}\right) + \left(\frac{-1 + \tan \theta}{1 + \tan \theta}\right) \\
 &= \left(\frac{1 - \tan \theta}{1 + \tan \theta}\right) - \left(\frac{1 - \tan \theta}{1 - \tan \theta}\right) \text{ (cancellation)} \\
 &= 0 \text{ R.H.S}
 \end{aligned}$$



$$\text{iii. } \sin\left(\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$$

$$\text{L.H.S.:- } \sin\left(\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{\pi}{3}\right)$$

$$\begin{aligned} & \left(\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6}\right) + \left(\cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3}\right) \\ & \left(\sin\theta\left(\frac{\sqrt{3}}{2}\right) + \cos\theta\left(\frac{1}{2}\right)\right) + \left(\cos\theta\left(\frac{1}{2}\right) - \sin\theta\left(\frac{\sqrt{3}}{2}\right)\right) \\ & \left(\frac{\sqrt{3}}{2}\right)\sin\theta + \left(\frac{1}{2}\right)\cos\theta + \left(\frac{1}{2}\right)\cos\theta - \left(\frac{\sqrt{3}}{2}\right)\sin\theta \text{ (cancellation)} \end{aligned}$$

$$\left(\frac{1}{2}\right)\cos\theta + \left(\frac{1}{2}\right)\cos\theta = \cos\theta = \text{R.H.S}$$

$$\text{iv:- } \frac{\sin\theta - \cos\theta\tan\frac{\theta}{2}}{\cos\theta + \sin\theta\tan\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\text{L.H.S} = \frac{\sin\theta - \cos\theta\tan\frac{\theta}{2}}{\cos\theta + \sin\theta\tan\frac{\theta}{2}}$$

$$\frac{\sin\theta - \cos\theta\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{\cos\theta + \sin\theta\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}} = \frac{\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}}{\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}}$$

$$= \frac{\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}}{\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}} = \frac{\sin\left(\theta - \frac{\theta}{2}\right)}{\cos\left(\theta - \frac{\theta}{2}\right)}$$

$$= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\text{(v): } \frac{1 - \tan\theta\tan\phi}{1 + \tan\theta\tan\phi} = \frac{\cos(\theta + \phi)}{\cos(\theta - \phi)}$$

$$\begin{aligned} & \frac{1 - \frac{\sin\theta}{\cos\theta}\frac{\sin\phi}{\cos\phi}}{1 + \frac{\sin\theta}{\cos\theta}\frac{\sin\phi}{\cos\phi}} \\ & \frac{\frac{\cos\theta\cos\phi - \sin\theta\sin\phi}{\cos\theta\cos\phi}}{\frac{\cos\theta\cos\phi + \sin\theta\sin\phi}{\cos\theta\cos\phi}} \\ & \frac{\cos\theta\cos\phi - \sin\theta\sin\phi}{\cos\theta\cos\phi + \sin\theta\sin\phi} \\ & \frac{\cos(\theta + \phi)}{\cos(\theta - \phi)} = \text{R.H.S} \end{aligned}$$

$$\text{Question #5 prove that } \cos(\alpha + \beta)\cos(\alpha - \beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha$$

$$\text{L.H.S} = \cos(\alpha + \beta)\cos(\alpha - \beta)$$

$$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$

[Difference between two square]

$$= \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta$$

$$= \cos^2\alpha(1 - \sin^2\beta) - (1 - \cos^2\alpha)\sin^2\beta$$

$$= \cos^2\alpha - \cos^2\alpha\sin^2\beta - \sin^2\beta + \cos^2\alpha\sin^2\beta$$

(cancellation)

$$= \cos^2\alpha - \sin^2\beta = \text{Middle side}$$

$$= (1 - \sin^2\alpha) - (1 - \cos^2\beta)$$

$$= 1 - \sin^2\alpha - 1 + \cos^2\beta$$

$$= \cos^2\beta - \sin^2\alpha = \text{R.H.S}$$

$$\text{Question #6 show that } \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{\cos(\alpha + \beta) + \cos(\alpha - \beta)} = \tan\alpha$$

$$\text{Solution:- L.H.S} = \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{\cos(\alpha + \beta) + \cos(\alpha - \beta)}$$

$$\begin{aligned} & \frac{(\sin\alpha\cos\beta + \cos\alpha\sin\beta) + (\sin\alpha\cos\beta - \cos\alpha\sin\beta)}{(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + (\cos\alpha\cos\beta + \sin\alpha\sin\beta)} \\ & \frac{2\sin\alpha\cos\beta}{2\cos\alpha\cos\beta} = \frac{\sin\alpha}{\cos\alpha} \\ & = \tan\alpha = \text{R.H.S} \end{aligned}$$

**Question #7 show that**

$$\text{i. } \cot(\alpha + \beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$$

$$\text{Solution:- R.H.S} = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$$

$$\frac{\frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} - 1}{\frac{\cos\alpha}{\sin\alpha} + \frac{\cos\beta}{\sin\beta}} = \frac{\frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\frac{\cos\alpha\sin\beta + \sin\alpha\cos\beta}{\sin\alpha\sin\beta}} \text{ (cancellation)}$$

$$= \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\sin\beta + \sin\alpha\cos\beta}$$

$$= \frac{\cos(\alpha + \beta)}{\sin(\alpha + \beta)}$$

$$= \cot(\alpha + \beta) = \text{R.H.S}$$

$$\text{ii. } \cot(\alpha - \beta) = \frac{\cot\alpha\cot\beta + 1}{\cot\alpha - \cot\beta}$$

$$\text{Solution:- R.H.S} = \frac{\cot\alpha\cot\beta + 1}{\cot\alpha - \cot\beta}$$

$$\frac{\frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} + 1}{\frac{\cos\alpha}{\sin\alpha} - \frac{\cos\beta}{\sin\beta}} = \frac{\frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\frac{\cos\alpha\sin\beta - \sin\alpha\cos\beta}{\sin\alpha\sin\beta}} \text{ (cancellation)}$$

$$= \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{\cos\alpha\sin\beta - \sin\alpha\cos\beta}$$

$$= \frac{\cos(\alpha - \beta)}{\sin(\alpha - \beta)}$$

$$= \cot(\alpha - \beta) = \text{R.H.S}$$

$$\text{iii. } \frac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta} = \frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)}$$

$$\text{Solution:- } \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{\frac{\sin\alpha}{\cos\alpha} - \frac{\sin\beta}{\cos\beta}}$$

$$= \frac{\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta}} \text{ (cancellation)}$$

$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta}$$

$$= \frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \text{R.H.S}$$

$$\text{Question #8 if } \sin\alpha = \frac{4}{5} \text{ and } \cos\beta = \frac{40}{41}, \text{ where } 0 < \alpha < \frac{\pi}{2} \text{ and } 0 < \beta < \frac{\pi}{2}, \text{ show that } \sin(\alpha - \beta) = \frac{133}{205}$$

**Solution:-** As  $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$  so we first need to find  $\cos\alpha$  and  $\sin\beta$

$$\Rightarrow \cos^2\alpha = 1 - \sin^2\alpha$$

$$= \cos^2\alpha = 1 - \left(\frac{4}{5}\right)^2$$

$$= \cos^2\alpha = 1 - \frac{16}{25}$$

$$= \cos^2\alpha = \frac{25 - 16}{25}$$

$$= \cos^2\alpha = \frac{9}{25} \text{ taking square root}$$

$$= \cos\alpha = \pm\frac{3}{5} \text{ (only take positive because of first quadrant)}$$

$$\text{So } \cos\alpha = \frac{3}{5}$$

Also

$$\Rightarrow \sin^2\beta = 1 - \cos^2\beta$$

$$= \sin^2 \beta = 1 - \left(\frac{40}{41}\right)^2$$

$$= \sin^2 \beta = 1 - \frac{1600}{1681}$$

$$= \sin^2 \beta = \frac{81}{1681}$$

$$= \sin^2 \beta = \frac{81}{1681} \text{ taking square root}$$

$$= \sin \beta = \pm \frac{9}{41} \text{ (only take positive because of first quadrant)}$$

$$\text{So } \sin \beta = + \frac{9}{41}$$

$$\Rightarrow \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$= \sin(\alpha - \beta) = \left(\frac{4}{5}\right) \left(\frac{40}{41}\right) - \left(\frac{3}{5}\right) \left(\frac{9}{41}\right)$$

$$= \frac{160 - 27}{205} = \frac{133}{205}$$

$$= \sin(\alpha - \beta) = \frac{133}{205} \text{ (proved)}$$

**Question #9** if  $\sin \alpha = \frac{4}{5}$  and  $\sin \beta = \frac{12}{13}$  where  $\frac{\pi}{2} < \alpha < \pi$  and  $\frac{\pi}{2} < \beta < \pi$ . Find

**Note:-** first of all we find  $\cos \alpha$  and  $\cos \beta$  and use all these four values to find all values given below and quadrant is 2<sup>nd</sup> so only  $\sin$  (cosec) is positive

$$\Rightarrow \cos^2 \alpha = 1 - \sin^2 \alpha$$

$$= \cos^2 \alpha = 1 - \left(\frac{4}{5}\right)^2$$

$$= \cos^2 \alpha = 1 - \frac{16}{25}$$

$$= \cos^2 \alpha = \frac{25 - 16}{25}$$

$$= \cos^2 \alpha = \frac{9}{25} \text{ taking square root}$$

$$= \cos \alpha = \pm \frac{3}{5} \text{ (only } \sin \text{ is positive because of 2<sup>nd</sup> quadrant)}$$

$$\text{So } \cos \alpha = - \frac{3}{5}$$

$$\Rightarrow \cos^2 \beta = 1 - \sin^2 \beta$$

$$= \cos^2 \beta = 1 - \left(\frac{12}{13}\right)^2$$

$$= \cos^2 \beta = 1 - \frac{144}{169}$$

$$= \cos^2 \beta = \frac{169 - 144}{169}$$

$$= \cos^2 \beta = \frac{25}{169} \text{ taking square root}$$

$$= \cos \beta = \pm \frac{5}{13} \text{ (only } \sin \text{ is positive because of 2<sup>nd</sup> quadrant)}$$

$$\text{So } \cos \beta = - \frac{5}{13}$$

From above calculations we have  $\sin \alpha = \frac{4}{5}$ ,  $\sin \beta = \frac{12}{13}$ ,

$$\cos \alpha = - \frac{3}{5} \text{ and } \cos \beta = - \frac{5}{13}$$

**i.  $\sin(\alpha + \beta)$**

solution:- As  $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$$\sin(\alpha + \beta) = \left(\frac{4}{5}\right) \left(-\frac{5}{13}\right) + \left(-\frac{3}{5}\right) \left(\frac{12}{13}\right)$$

$$= \left(-\frac{20}{65}\right) + \left(-\frac{36}{65}\right)$$

$$= -\frac{20}{65} - \frac{36}{65}$$

$$= \frac{-20 - 36}{65} = -\frac{56}{65}$$

**ii.  $\cos(\alpha + \beta)$**

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \left(-\frac{3}{5}\right) \left(-\frac{5}{13}\right) - \left(\frac{4}{5}\right) \left(\frac{12}{13}\right)$$

$$= \left(\frac{15}{65}\right) - \left(\frac{48}{65}\right)$$

$$= \frac{15}{65} - \frac{48}{65}$$

$$= \frac{15 - 48}{65}$$

$$= -\frac{33}{65}$$

**iii.  $\tan(\alpha + \beta)$**

solution:- To find the value of  $\tan(\alpha + \beta)$  we use

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$$

$$\tan(\alpha + \beta) = \frac{-\frac{56}{65}}{-\frac{33}{65}} = \frac{56}{33}$$

**iv.  $\sin(\alpha - \beta)$**

solution:- As  $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

$$\sin(\alpha - \beta) = \left(\frac{4}{5}\right) \left(-\frac{5}{13}\right) - \left(-\frac{3}{5}\right) \left(\frac{12}{13}\right)$$

$$= \left(-\frac{20}{65}\right) - \left(-\frac{36}{65}\right)$$

$$= -\frac{20}{65} + \frac{36}{65}$$

$$= \frac{-20 + 36}{65} = \frac{16}{65}$$

**v.  $\cos(\alpha - \beta)$**

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \left(-\frac{3}{5}\right) \left(-\frac{5}{13}\right) + \left(\frac{4}{5}\right) \left(\frac{12}{13}\right)$$

$$= \left(\frac{15}{65}\right) + \left(\frac{48}{65}\right)$$

$$= \frac{15}{65} + \frac{48}{65}$$

$$= \frac{15 + 48}{65}$$

$$= \frac{63}{65}$$

**vi.  $\tan(\alpha - \beta)$**

solution:- To find the value of  $\tan(\alpha - \beta)$  we use

$$\tan(\alpha - \beta) = \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)}$$

$$\tan(\alpha - \beta) = \frac{\frac{16}{65}}{\frac{63}{65}} = \frac{16}{63}$$

**Question #10** find  $\sin(\alpha + \beta)$  and  $\cos(\alpha + \beta)$ , given that

(i)  $\tan \alpha = \frac{3}{4}$ ,  $\cos \beta = \frac{5}{13}$  and neither the terminal side of the angle of measure  $\alpha$  nor that of  $\beta$  is the I quadrant.

**Solution:**

$$\tan \alpha = \frac{3}{4} \text{ (}\alpha \text{ is not in I quadrant)}$$

so  $\alpha$  lies in III quadrant.

$$\cos \beta = \frac{5}{13} \text{ (}\beta \text{ not in I quadrant so } \beta \text{ lies in IV quadr.)}$$

$$\therefore \sec^2 \alpha = 1 + \tan^2 \alpha = 1 + \left(\frac{3}{4}\right)^2$$

$$\sec^2 \alpha = 1 + \tan^2 \alpha = 1 + \frac{9}{16} = \frac{16 + 9}{16} = \frac{25}{16}$$



$$\Rightarrow \sec \alpha = -\frac{5}{4} (\because \alpha \text{ is in IV quadrant})$$

$$\text{or } \cos \alpha = -\frac{4}{5}$$

$$\text{also } \sin \alpha = \pm \sqrt{1 - \cos^2 \alpha}$$

$$\sin \alpha = \pm \sqrt{1 - \left(-\frac{4}{5}\right)^2}$$

$$\sin \alpha = \pm \sqrt{1 - \frac{16}{25}} = \pm \sqrt{\frac{25 - 16}{25}}$$

$$\sin \alpha = \pm \sqrt{\frac{9}{25}} = \pm \frac{3}{5}$$

$$\Rightarrow \sin \alpha = -\frac{3}{5} (\because \alpha \text{ is in III quadrant})$$

$$\because \sin \beta = \pm \sqrt{1 - \cos^2 \beta}$$

$$= \pm \sqrt{1 - \left(\frac{5}{13}\right)^2}$$

$$= \pm \sqrt{1 - \frac{25}{169}}$$

$$= \pm \sqrt{\frac{169 - 25}{169}} = \pm \sqrt{\frac{144}{169}}$$

$$= \pm \frac{12}{13}$$

$$\Rightarrow \sin \beta = -\frac{12}{13} (\because \beta \text{ is in IV quad})$$

Now

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$= \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(-\frac{3}{5}\right)\left(-\frac{12}{13}\right)$$

$$= -\frac{15}{65} + \frac{36}{65} = \frac{-15 + 36}{65} = \frac{21}{65}$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$= \left(-\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(-\frac{12}{13}\right)$$

$$= -\frac{20}{65} - \frac{36}{65} = \frac{-20 - 36}{65} = -\frac{56}{65}$$

$$(ii) \tan \alpha = -\frac{15}{8} \text{ and } \sin \beta = -\frac{7}{25} \text{ and neither the}$$

terminal arm of the angle of measure nor of  $\beta$  is in forth quadrant.

**Solution:-**

**NOTE:-** from above signs of  $\alpha$  and  $\beta$  we have  $\alpha$  is in 2<sup>nd</sup> quadrant and  $\beta$  is in 3<sup>rd</sup> quadrant.

We first find  $\sin \alpha$  and  $\cos \alpha$  by using  $\tan \alpha$  and we also find  $\cos \beta$  using by  $\sin \beta$

$$\Rightarrow \sec^2 \alpha = 1 + \tan^2 \alpha$$

$$= \sec^2 \alpha = 1 + \left(\frac{15}{8}\right)^2$$

$$= \sec^2 \alpha = 1 + \frac{225}{64}$$

$$= \sec^2 \alpha = \frac{64 + 225}{64} = \frac{289}{64} \text{ (taking square root)}$$

$$= \sec \alpha = -\frac{17}{8} \text{ (-ve sign is due to 2<sup>nd</sup> quadrant)}$$

$$\text{Also } \cos \alpha = -\frac{8}{17}$$

$$\sin^2 \alpha = 1 - \cos^2 \alpha$$

$$\sin^2 \alpha = 1 - \left(-\frac{8}{17}\right)^2$$

$$\sin^2 \alpha = 1 - \frac{64}{289} = \frac{289 - 64}{169}$$

$$\sin^2 \alpha = \frac{225}{169} \text{ (square root both sides)}$$

$$\sin \alpha = +\frac{15}{17} \text{ (2<sup>nd</sup> quadrant)}$$

$$\Rightarrow \cos^2 \beta = 1 - \sin^2 \beta$$

$$= \cos^2 \beta = 1 - \left(-\frac{7}{25}\right)^2$$

$$= \cos^2 \beta = 1 - \frac{49}{625}$$

$$= \cos^2 \beta = \frac{625 - 49}{625}$$

$$= \cos^2 \beta = \frac{576}{625} \text{ (taking square root)}$$

$$= \cos \beta = -\frac{24}{25} \text{ (Because of 3<sup>rd</sup> quadrant)}$$

$$\text{So } \cos \beta = -\frac{24}{25}$$

$$\text{Now } \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \left(\frac{15}{17}\right)\left(-\frac{24}{25}\right) + \left(-\frac{8}{17}\right)\left(-\frac{7}{25}\right)$$

$$= \left(-\frac{360}{425}\right) + \left(\frac{56}{425}\right)$$

$$= \frac{-360 + 56}{425} = -\frac{304}{425}$$

$$\sin(\alpha + \beta) = -\frac{304}{425}$$

Now we find  $\cos(\alpha + \beta)$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \left(-\frac{8}{17}\right)\left(-\frac{24}{25}\right) - \left(\frac{15}{17}\right)\left(-\frac{7}{25}\right)$$

$$= \frac{192}{425} + \frac{105}{425}$$

$$= \frac{192 + 105}{425} = \frac{297}{425}$$

$$\cos(\alpha + \beta) = \frac{297}{425}$$

**Question#11:-** Prove that:  $\frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ} = \tan 37^\circ$

**Solution:-** R.H.S =  $\tan 37^\circ$

$$\text{We put } 37^\circ = 45^\circ - 8^\circ$$

$$\Rightarrow \tan(45^\circ - 8^\circ)$$

$$= \frac{\tan 45^\circ - \tan 8^\circ}{1 + \tan 45^\circ \tan 8^\circ}$$

$$= \frac{(1) - \tan 8^\circ}{1 + (1)\tan 8^\circ}$$

$$= \frac{1 - \tan 8^\circ}{1 + \tan 8^\circ} = \frac{1 - \frac{\sin 8^\circ}{\cos 8^\circ}}{1 + \frac{\sin 8^\circ}{\cos 8^\circ}}$$

$$= \frac{\frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ}}{\frac{\cos 8^\circ + \sin 8^\circ}{\cos 8^\circ}} \text{ (cancellation)}$$

$$= \frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ} = \text{R.H.S}$$

**Question#12:-** If  $\alpha$ , and  $\gamma$  are the angles of a triangle ABC, show that  $\cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} = \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2}$

**Solution:-** if  $\alpha, \beta$  and  $\gamma$  are the angles of a triangle

$$\Rightarrow \alpha + \beta + \gamma = 180$$

$$= \alpha + \beta = 180 - \gamma$$

Dividing B.S by 2

$$\frac{\alpha + \beta}{2} = \frac{180 - \gamma}{2}$$

$$\frac{\alpha}{2} + \frac{\beta}{2} = 90 - \frac{\gamma}{2}$$

Apply tan on both side

$$\tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \tan\left(90 - \frac{\gamma}{2}\right)$$

$$= \frac{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = \cot\frac{\gamma}{2} \quad (\text{put } \tan = \frac{1}{\cot})$$

$$= \frac{\frac{1}{\cot\frac{\alpha}{2}} + \frac{1}{\cot\frac{\beta}{2}}}{1 - \frac{1}{\cot\frac{\alpha}{2}} \frac{1}{\cot\frac{\beta}{2}}} = \cot\frac{\gamma}{2}$$

$$= \frac{\cot\frac{\beta}{2} + \cot\frac{\alpha}{2}}{\cot\frac{\alpha}{2}\cot\frac{\beta}{2} - 1} = \cot\frac{\gamma}{2} \quad (\text{cancellation})$$

$$= \frac{\cot\frac{\beta}{2} + \cot\frac{\alpha}{2}}{\cot\frac{\alpha}{2}\cot\frac{\beta}{2} - 1} = \cot\frac{\gamma}{2} \quad (\text{by cross multiplication})$$

$$= \cot\frac{\beta}{2} + \cot\frac{\alpha}{2} = \cot\frac{\gamma}{2} \left(\cot\frac{\alpha}{2}\cot\frac{\beta}{2} - 1\right)$$

$$= \cot\frac{\beta}{2} + \cot\frac{\alpha}{2} = \cot\frac{\gamma}{2} \cot\frac{\alpha}{2} \cot\frac{\beta}{2} - \cot\frac{\gamma}{2}$$

$$= \cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2} \quad (\text{Proved})$$

**Question#13** If  $\alpha + \beta + \gamma = 180$ ,

show that  $\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = 1$

**Solution:-**

$\alpha + \beta = 180 - \gamma$  apply tan on both sides

$$\tan(\alpha + \beta) = \tan(180 - \gamma)$$

Apply formula of "tan"

$$\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{\tan 180 - \tan\gamma}{1 + \tan 180\tan\gamma} \quad \therefore \tan 180 = 0$$

$$\Rightarrow \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = -\tan\gamma \quad (\text{put } \tan = \frac{1}{\cot})$$

$$= \frac{\frac{1}{\cot\alpha} - \frac{1}{\cot\beta}}{1 - \frac{1}{\cot\alpha} \frac{1}{\cot\beta}} = -\frac{1}{\cot\gamma} \quad (\text{L.C.M})$$

$$= \frac{\cot\beta + \cot\alpha}{\cot\alpha\cot\beta - 1} = -\frac{1}{\cot\gamma} \quad (\text{cancellation})$$

$$= \frac{\cot\beta + \cot\alpha}{\cot\alpha\cot\beta - 1} = -\frac{1}{\cot\gamma} \quad (\text{cross multiplication})$$

$$= \cot\gamma(\cot\beta + \cot\alpha) = -(\cot\alpha\cot\beta - 1)$$

$$= \cot\gamma\cot\beta + \cot\gamma\cot\alpha = -\cot\alpha\cot\beta + 1$$

$$= \cot\gamma\cot\beta + \cot\gamma\cot\alpha + \cot\alpha\cot\beta = 1$$

OR

$$\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = 1 \quad (\text{Proved})$$

**Question#14:-** Express the following in the form of  $r \sin(\theta + \phi)$  where terminal arm of the angle lies in first quadrant

i.  $12\sin\theta + 5\cos\theta$

**Solution:-**

$$\text{As } r \sin(\theta + \phi) = r[\sin\theta\cos\phi + \cos\theta\sin\phi] \rightarrow (1)$$

By comparing we have  $\cos\phi = 12$  and  $\sin\phi = 5$

$$\text{As } r = \sqrt{12^2 + 5^2}$$

$$\Rightarrow r = \sqrt{169} = +13 \quad (\text{because of 1<sup>st</sup> quadrant}) \dots \dots \dots (2)$$

Now  $\tan\phi = \frac{\sin\phi}{\cos\phi}$

$$= \tan\phi = \frac{5}{12}$$

$$\Rightarrow \phi = \tan^{-1} \frac{5}{12} \dots \dots \dots (3)$$

**Put (2) and (3) in (1)**

$$\Rightarrow r \sin(\theta + \phi) = 13[\sin\theta\cos\phi + \cos\theta\sin\phi]$$

$$\text{Where } \phi = \tan^{-1} \frac{5}{12}$$

ii.  $3\sin\theta - 4\cos\theta$

**Solution:-**

$$\text{As } r \sin(\theta + \phi) = r[\sin\theta\cos\phi + \cos\theta\sin\phi] \rightarrow (1)$$

By comparing we have  $\cos\phi = 3$  and  $\sin\phi = -4$

$$\text{As } r = \sqrt{3^2 + (-4)^2}$$

$$\Rightarrow r = \sqrt{25} = +5 \quad (\text{because of 1<sup>st</sup> quadrant}) \dots \dots \dots (2)$$

Now  $\tan\phi = \frac{\sin\phi}{\cos\phi}$

$$= \tan\phi = \frac{-4}{3}$$

$$\Rightarrow \phi = \tan^{-1} \frac{-4}{3} \dots \dots \dots (3)$$

**Put (2) and (3) in (1)**

$$\Rightarrow r \sin(\theta + \phi) = 5[\sin\theta\cos\phi + \cos\theta\sin\phi]$$

$$\text{Where } \phi = \tan^{-1} \frac{5}{12}$$

(iii)  $\sin\theta - \cos\theta$

**solution:**

$$\text{let } 1 = r\cos\phi \rightarrow (i)$$

$$-1 = r\sin\phi \rightarrow (ii)$$

$$\text{By } (i)^2 + (ii)^2$$

$$\Rightarrow (1)^2 + (-1)^2 = r^2\cos^2\phi + r^2\sin^2\phi$$

$$\Rightarrow 1 + 1 = r^2(\cos^2\phi + \sin^2\phi)$$

$$\Rightarrow r^2 = 2 \Rightarrow r = \sqrt{2}$$

$$\text{by } \frac{ii}{i} \Rightarrow \frac{r\sin\phi}{r\cos\phi} = \frac{-1}{1} \Rightarrow \tan\phi = -1$$

$$\Rightarrow \phi = \tan^{-1}(-1)$$

Now

$$\begin{aligned} \sin\theta - \cos\theta &= r\cos\phi\sin\theta + r\sin\phi\cos\theta \\ &= r(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &= r\sin(\theta + \phi) \text{ where } r = \sqrt{2} \end{aligned}$$

$$\text{and } \phi = \tan^{-1}(-1)$$

$$\text{(iv) } 5\sin\theta - 4\cos\theta$$

**Solution:**

$$\text{let } 5 = r\cos\phi \rightarrow \text{(i)}$$

$$-4 = r\sin\phi \rightarrow \text{(ii)}$$

$$\text{by (i)}^2 + \text{(ii)}^2$$

$$\Rightarrow (5)^2 + (-4)^2 = r^2\cos^2\phi + r^2\sin^2\phi$$

$$\Rightarrow 25 + 16 = r^2(\cos^2\phi + \sin^2\phi)r^2(1)$$

$$\Rightarrow r^2 = 41 \Rightarrow r = \sqrt{41}$$

$$\text{by } \frac{\text{(ii)}}{\text{(i)}} \Rightarrow -\frac{4}{5} = \frac{r\sin\phi}{r\cos\phi} \Rightarrow \tan\phi = -\frac{4}{5}$$

$$\text{or } \phi = \tan^{-1}\left(-\frac{4}{5}\right)$$

$$\begin{aligned} 5\sin\theta - 4\cos\theta &= r\cos\phi\sin\theta + r\sin\phi\cos\theta \\ &= r(\sin\theta\cos\phi + \cos\theta\sin\phi) \end{aligned}$$

$$= r\sin(\theta + \phi), \quad \text{where } r = \sqrt{41}$$

$$\text{And } \phi = \tan^{-1}\left(-\frac{4}{5}\right)$$

$$\text{(v) } \sin\theta + \cos\theta$$

**Solution:**

$$\text{let } 1 = r\cos\phi \rightarrow \text{(i)}$$

$$1 = r\sin\phi \rightarrow \text{(ii)}$$

$$\text{by (i)}^2 + \text{(ii)}^2$$

$$\Rightarrow (1)^2 + (1)^2 = r^2\cos^2\phi + r^2\sin^2\phi$$

$$\Rightarrow 1 + 1 = r^2(\cos^2\phi + \sin^2\phi)r^2(1)$$

$$\Rightarrow r^2 = 2 \Rightarrow r = \sqrt{2}$$

$$\text{by } \frac{\text{(ii)}}{\text{(i)}} \Rightarrow \frac{1}{1} = \frac{r\sin\phi}{r\cos\phi} \Rightarrow \tan\phi = 1$$

$$\text{or } \phi = \tan^{-1}(1)$$

Now

$$\begin{aligned} \sin\theta + \cos\theta &= r\cos\phi\sin\theta + r\sin\phi\cos\theta \\ &= r(\sin\theta\cos\phi + \cos\theta\sin\phi) \end{aligned}$$

$$= r\sin(\theta + \phi), \quad \text{where } r = \sqrt{2}$$

$$\text{And } \phi = \tan^{-1}(1)$$

$$\text{(vi) } 3\sin\theta - 5\cos\theta$$

**Solution:**

$$\text{let } 3 = r\cos\phi \rightarrow \text{(i)}$$

$$-5 = r\sin\phi \rightarrow \text{(ii)}$$

$$\text{by (i)}^2 + \text{(ii)}^2$$

$$\Rightarrow (3)^2 + (-5)^2 = r^2\cos^2\phi + r^2\sin^2\phi$$

$$\Rightarrow 9 + 25 = r^2(\cos^2\phi + \sin^2\phi)r^2(1)$$

$$\Rightarrow r^2 = 34 \Rightarrow r = \sqrt{34}$$

$$\text{by } \frac{\text{(ii)}}{\text{(i)}} \Rightarrow \frac{-5}{3} = \frac{r\sin\phi}{r\cos\phi} \Rightarrow \tan\phi = -\frac{5}{3}$$

$$\text{or } \phi = \tan^{-1}\left(-\frac{5}{3}\right)$$

Now

$$3\sin\theta - 5\cos\theta = r\cos\phi\sin\theta + r\sin\phi\cos\theta$$

$$= r(\sin\theta\cos\phi + \cos\theta\sin\phi)$$

$$= r\sin(\theta + \phi), \quad \text{where } r = \sqrt{34}$$

$$\text{And } \phi = \tan^{-1}\left(-\frac{5}{3}\right)$$

## Double angle identities

$$\text{i) } \sin 2\alpha = 2\sin\alpha\cos\alpha$$

we know that

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

put  $\beta = \alpha$  we get

$$\sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$$

$$\Rightarrow \sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\text{ii) } \cos 2\alpha = 2\sin\alpha\cos\alpha$$

we know that

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

put  $\beta = \alpha$  we get

$$\cos(\alpha + \alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$$

$$\Rightarrow \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\Rightarrow \cos 2\alpha = \cos^2\alpha - (1 - \cos^2\alpha)$$

$$= \cos^2\alpha - 1 + \cos^2\alpha$$

$$\Rightarrow \cos 2\alpha = 2\cos^2\alpha - 1$$

$$\Rightarrow \cos 2\alpha = 2(1 - \sin^2\alpha) - 1$$

$$= 2 - 2\sin^2\alpha - 1$$

$$\Rightarrow \cos 2\alpha = 1 - 2\sin^2\alpha$$

$$\text{(iii) } \tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

we know that

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Put  $\beta = \alpha$ , we get

$$\Rightarrow \tan(\alpha + \beta) = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}$$

$$\Rightarrow \tan(\alpha + \beta) = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

## Half angle identities

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos\alpha}{2}}$$

we know that

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

Similarly

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 \text{ in form half angle}$$

$$\Rightarrow 2\cos^2\frac{\alpha}{2} = 1 + \cos\alpha$$

$$\Rightarrow \cos^2\frac{\alpha}{2} = \frac{1 + \cos\alpha}{2}$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1+\cos\alpha}{2}}$$

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1-\cos\alpha}{2}}$$

We know that

$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

Similarly

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} \text{ in form half angle}$$

$$\Rightarrow 2\sin^2\frac{\alpha}{2} = 1 - \cos\alpha$$

$$\Rightarrow \sin^2\frac{\alpha}{2} = \frac{1 - \cos\alpha}{2}$$

$$\sin\frac{\alpha}{2} = \pm \sqrt{\frac{1-\cos\alpha}{2}}$$

$$\text{iii } \tan \frac{\alpha}{2} = \pm \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

We know that

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

Similarly

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \text{ (inform of half angle)}$$

$$\Rightarrow \tan\frac{\alpha}{2} = \pm \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

$$\Rightarrow \tan\frac{\alpha}{2} = \pm \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

Triple angle Identities

$$\text{i } \sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

$$\text{L.H.S} = \sin 3\alpha$$

$$= \sin(2\alpha + \alpha)$$

$$= 2\sin\alpha\cos\alpha + \cos 2\alpha\sin\alpha$$

$$= 2\sin\alpha\cos\alpha\cos\alpha + (1 - 2\sin^2\alpha)\sin\alpha$$

$$= 2\sin\alpha\cos^2\alpha + \sin\alpha - 2\sin^3\alpha$$

$$= 2\sin\alpha(1 - \sin^2\alpha) + \sin\alpha - 2\sin^3\alpha$$

$$= 2\sin\alpha - 2\sin^3\alpha + \sin\alpha - 2\sin^3\alpha$$

$$= 3\sin\alpha - 4\sin^3\alpha = \text{R.H.S}$$

$$\text{Hence } \sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

$$\text{ii } \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$\text{L.H.S} = \cos 3\alpha$$

$$= \cos(2\alpha + \alpha)$$

$$= \cos 2\alpha\cos\alpha - \sin 2\alpha\sin\alpha$$

$$(2\cos^2\alpha - 1)\cos\alpha - 2\sin\alpha\cos\alpha\sin\alpha$$

$$= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha$$

$$= 2\cos^3\alpha - \cos\alpha - 2(1 - \cos^2\alpha)\cos\alpha$$

$$= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha$$

$$= 4\cos^3\alpha - 3\cos\alpha = \text{R.H.S}$$

$$\text{iii } \tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$

$$\text{L.H.S} = \tan 3\alpha$$

$$= \tan(2\alpha + \alpha)$$

$$= \frac{\tan 2\alpha + \tan\alpha}{1 - \tan 2\alpha\tan\alpha}$$

$$= \frac{\frac{2\tan\alpha}{1 - \tan^2\alpha} + \tan\alpha}{1 - \frac{2\tan\alpha}{1 - \tan^2\alpha} \cdot \tan\alpha}$$

$$= \frac{2\tan\alpha + \tan\alpha - \tan^3\alpha}{1 - \tan^2\alpha}$$

$$= \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha} = \text{R.H.S}$$

$$\text{Hence}$$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$

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$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$

### Exercise 10.3

1. Find the values of  $\sin 2\alpha$ ,  $\cos 2\alpha$  and  $\tan 2\alpha$ , when

where  $0 < \alpha < \frac{\pi}{2}$

$$\text{i. } \sin\alpha = \frac{12}{13}$$

Solution : - To find all the above values we first calculate

$\cos\alpha$  by using  $\sin\alpha$

$$\Rightarrow \cos^2\alpha = 1 - \sin^2\alpha$$

$$= \cos^2\alpha = 1 - \left(\frac{12}{13}\right)^2$$

$$= \cos^2\alpha = 1 - \frac{144}{169}$$

$$= \cos^2\alpha = \frac{169-144}{169}$$

$$= \cos^2\alpha = \frac{25}{169} \text{ (taking square root)}$$

$$\Rightarrow \cos\alpha = \frac{5}{13} \text{ (+is because of first quadrant)}$$

$$\text{Now } \sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(\frac{12}{13}\right)\left(\frac{5}{13}\right)$$

$$\sin 2\alpha = \frac{120}{169}$$

$$\Rightarrow \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\cos 2\alpha = \left(\frac{5}{13}\right)^2 - \left(\frac{12}{13}\right)^2$$

$$\cos 2\alpha = \frac{25}{169} - \frac{144}{169}$$

$$\cos 2\alpha = \frac{25-144}{169}$$

$$\cos 2\alpha = \frac{-119}{169}$$

$$\Rightarrow \tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{\frac{120}{169}}{\frac{-119}{169}}$$

$$\tan 2\alpha = -\frac{120}{119}$$

$$\text{ii. } \cos\alpha = \frac{3}{5}$$

(Same as above)

Prove the following identities

$$2. \quad \cot\alpha - \tan\alpha = 2\cot 2\alpha$$

$$\text{L.H.S} = \cot\alpha - \tan\alpha$$

$$\begin{aligned} & \frac{\cos\alpha}{\sin\alpha} - \frac{\sin\alpha}{\cos\alpha} \\ &= \frac{\cos^2\alpha - \sin^2\alpha}{\sin\alpha\cos\alpha} \\ &= \frac{\cos 2\alpha}{\sin 2\alpha} \quad (\times \text{ and } \div \text{ by } 2) \\ &= \frac{\cos 2\alpha}{\sin 2\alpha} = 2\cot 2\alpha = \text{R.H.S} \end{aligned}$$

$$3. \quad \frac{\sin 2\alpha}{1 + \cos 2\alpha} = \tan\alpha$$

$$\begin{aligned} \text{L.H.S} &= \frac{\sin 2\alpha}{1 + \cos 2\alpha} \\ &= \frac{2\sin\alpha\cos\alpha}{1 + 2\cos^2\alpha - 1} \\ &= \frac{2\sin\alpha\cos\alpha}{2\cos^2\alpha} \\ &= \frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \text{R.H.S} \end{aligned}$$

$$4. \quad \frac{1 - \cos\alpha}{\sin\alpha} = \tan\frac{\alpha}{2}$$

$$\begin{aligned} \text{L.H.S} &= \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} \quad (\text{cancellation}) \\ &= \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \\ &= \tan\frac{\alpha}{2} = \text{R.H.S} \end{aligned}$$

$$5. \quad \frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} = \sec 2\alpha - \tan 2\alpha$$

$$\begin{aligned} \text{L.H.S} &= \frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} \\ &= \frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} \times \frac{\cos\alpha - \sin\alpha}{\cos\alpha - \sin\alpha} \\ &= \frac{(\cos\alpha - \sin\alpha)^2}{\cos^2\alpha - \sin^2\alpha} \\ &= \frac{\cos^2\alpha + \sin^2\alpha + 2\sin\alpha\cos\alpha}{\cos 2\alpha} \\ &= \frac{1 + \sin 2\alpha}{\cos 2\alpha} = \frac{1}{\cos 2\alpha} + \frac{\sin 2\alpha}{\cos 2\alpha} \\ &= \sec 2\alpha - \tan 2\alpha = \text{R.H.S} \end{aligned}$$

$$6. \quad \sqrt{\frac{1 + \sin\alpha}{1 - \sin\alpha}} = \frac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}}$$

$$\text{L.H.S} = \sqrt{\frac{1 + \sin\alpha}{1 - \sin\alpha}}$$

$$= \sqrt{\frac{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}} \quad \therefore \cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} = 1$$

$$\begin{aligned} &= \sqrt{\frac{(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2})^2}{(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2})^2}} \\ &= \frac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}} = \text{R.H.S} \end{aligned}$$

$$\begin{aligned} 7. \quad \frac{\operatorname{cosec}\theta + \operatorname{cosec} 2\theta}{\sec\theta} &= \cot\frac{\theta}{2} \\ \text{L.H.S} &= \frac{\operatorname{cosec}\theta + \operatorname{cosec} 2\theta}{\sec\theta} \\ &= \frac{\frac{1}{\sin\theta} + \frac{2}{\sin 2\theta}}{\frac{1}{\cos\theta}} \\ &= \left(\frac{1}{\sin\theta} + \frac{2}{\sin 2\theta}\right)\cos\theta \\ &= \left(\frac{1}{\sin\theta} + \frac{2}{2\sin\theta\cos\theta}\right)\cos\theta \\ &= \left(\frac{1}{\sin\theta} + \frac{1}{\sin\theta\cos\theta}\right)\cos\theta \\ &= \left(\frac{\cos\theta + 1}{\sin\theta\cos\theta}\right)\cos\theta \\ &= \frac{\cos\theta + 1}{\sin\theta} \\ &= \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \\ &= \cot\frac{\theta}{2} = \text{R.H.S} \end{aligned}$$

$$8. \quad 1 + \tan\alpha \tan 2\alpha = \sec 2\alpha$$

$$\text{L.H.S} = 1 + \tan\alpha \tan 2\alpha$$

$$\begin{aligned} &= 1 + \frac{\sin\alpha}{\cos\alpha} \frac{\sin 2\alpha}{\cos 2\alpha} \\ &= \frac{\cos 2\alpha \cos\alpha + \sin 2\alpha \sin\alpha}{\cos\alpha \cos 2\alpha} \\ &= \frac{\cos(2\alpha - \alpha)}{\cos\alpha \cos 2\alpha} \\ &= \frac{1}{\cos 2\alpha} = \sec 2\alpha = \text{R.H.S} \end{aligned}$$

$$9. \quad \frac{2\sin\theta\sin 2\theta}{\cos\theta + \cos 3\theta} = \tan 2\theta \tan\theta$$

$$\begin{aligned} \text{L.H.S} &= \frac{2\sin\theta\sin 2\theta}{\cos\theta + \cos 3\theta} \\ &= \frac{2\sin\theta\sin 2\theta}{\cos 3\theta + \cos\theta} \quad (\text{using formula from 10.4}) \\ &= \frac{2\sin\theta\sin 2\theta}{2\cos\left(\frac{3\theta + \theta}{2}\right)\cos\left(\frac{3\theta - \theta}{2}\right)} \\ &= \frac{2\sin\theta\sin 2\theta}{2\cos\left(\frac{4\theta}{2}\right)\cos\left(\frac{2\theta}{2}\right)} \\ &= \frac{2\sin\theta\sin 2\theta}{2\cos 2\theta \cos\theta} = \frac{\sin 2\theta}{\cos 2\theta} \frac{\sin\theta}{\cos\theta} \\ &= \tan 2\theta \tan\theta = \text{R.H.S} \end{aligned}$$

$$10. \quad \frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2$$

$$\begin{aligned} \text{L.H.S} &= \frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} \\ &= \frac{\sin 3\theta \cos\theta - \cos 3\theta \sin\theta}{\sin\theta \cos\theta} \\ &= \frac{\sin(3\theta - \theta)}{\sin\theta \cos\theta} = \frac{\sin 2\theta}{\sin\theta \cos\theta} \\ &= \frac{2\sin\theta \cos\theta}{\sin\theta \cos\theta} \quad (\text{cancellation}) \\ &= 2 = \text{R.H.S} \end{aligned}$$

$$11. \quad \frac{\cos 3\theta}{\cos\theta} + \frac{\sin 3\theta}{\sin\theta} = 4\cos 2\theta$$

$$\begin{aligned} \text{L.H.S} &= \frac{\cos 3\theta}{\cos\theta} + \frac{\sin 3\theta}{\sin\theta} \\ &= \frac{\cos 3\theta \sin\theta + \sin 3\theta \cos\theta}{\cos\theta \sin\theta} \\ &= \frac{\sin(3\theta + \theta)}{\cos\theta \sin\theta} = \frac{\sin 4\theta}{\cos\theta \sin\theta} \quad (\times \text{ and } \div \text{ by } 2) \end{aligned}$$

$$\begin{aligned}
 &= \frac{2\sin 4\theta}{2\cos\theta\sin\theta} \\
 &= \frac{2\sin 4\theta}{\sin 2\theta} \quad \therefore \sin 4\theta = 2\sin 2\theta\cos 2\theta \\
 &= \frac{2 \cdot 2\sin 2\theta\cos 2\theta}{\sin 2\theta} \quad (\text{cancellation}) \\
 &= 4 \cos 2\theta \text{ R.H.S}
 \end{aligned}$$

$$12. \frac{\tan \frac{\theta}{2} + \cot \frac{\theta}{2}}{\cot \frac{\theta}{2} - \tan \frac{\theta}{2}} = \sec \theta$$

$$\begin{aligned}
 L.H.S &= \frac{\tan \frac{\theta}{2} + \cot \frac{\theta}{2}}{\cot \frac{\theta}{2} - \tan \frac{\theta}{2}} \\
 &= \frac{\frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} + \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}}}{\frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} - \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}} = \frac{\frac{\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}}{\cos \frac{\theta}{2} \sin \frac{\theta}{2}}}{\frac{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}}{\cos \frac{\theta}{2} \sin \frac{\theta}{2}}} \\
 &= \frac{\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}} \quad \therefore \sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} = 1
 \end{aligned}$$

$$13. \frac{\sin 3\theta}{\cos \theta} + \frac{\cos 3\theta}{\sin \theta} = 2\cot 2\theta$$

$$\begin{aligned}
 L.H.S &= \frac{\sin 3\theta}{\cos \theta} + \frac{\cos 3\theta}{\sin \theta} \\
 &= \frac{\sin 3\theta \sin \theta + \cos 3\theta \cos \theta}{\cos \theta \sin \theta} \\
 &= \frac{\cos(3\theta - \theta)}{\cos \theta \sin \theta} = \frac{\cos 2\theta}{\cos \theta \sin \theta} \quad (\times \text{ and } \div \text{ by } 2) \\
 &= \frac{2\cos \theta \sin \theta}{2\cos \theta \sin \theta} \\
 &= \frac{\sin 2\theta}{\sin 2\theta} \quad (\text{cancellation}) \\
 &= 2 \cot 2\theta \text{ R.H.S}
 \end{aligned}$$

14. Reduce  $\sin^4 \theta$  to an expression involving function of  $\theta$  raised to the first power.

$$\begin{aligned}
 \text{Solution:- } &(\sin^2 \theta)^2 \\
 &= \left(\frac{1 - \cos 2\theta}{2}\right)^2 \\
 &= \frac{1 - 2\cos 2\theta + \cos^2 2\theta}{4} \\
 &= \frac{1}{4}(1 - 2\cos 2\theta + \cos^2 2\theta) \\
 &= \frac{1}{4}\left(1 - 2\cos 2\theta + \frac{1 + \cos 4\theta}{2}\right) \\
 &= \frac{1}{4}\left(\frac{2 - 4\cos 2\theta + 1 + \cos 4\theta}{2}\right) \\
 &= \frac{1}{8}(2 - 4\cos 2\theta + 1 + \cos 4\theta) \\
 &= \frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)
 \end{aligned}$$

#### Question No.15

Find the value of  $\sin \theta$  and  $\cos \theta$  without using table or calculator where  $\theta$

(i)  $18^\circ$  (ii)  $36^\circ$  (iii)  $54^\circ$  (iv)  $72^\circ$

Hence prove that

$$\cos 36^\circ \cos 72^\circ \cos 108^\circ \cos 144^\circ = \frac{1}{16}$$

Hint

$$\begin{aligned}
 \text{let } \theta &= 18^\circ \\
 \Rightarrow 5\theta &= 90^\circ \\
 3\theta + 2\theta &= 90^\circ \\
 3\theta + 2\theta &= 90^\circ \\
 3\theta &= 90^\circ - 2\theta \\
 \text{let } \theta &= 36^\circ \\
 (5\theta = 180^\circ \Rightarrow 3\theta + 2\theta = 180^\circ) \\
 (\sin 3\theta = \sin(180^\circ - 2\theta) \text{ e.t.c.})
 \end{aligned}$$

Solution:

$$\begin{aligned}
 (i) &18^\circ \\
 \text{let } \theta &= 18^\circ \\
 \Rightarrow 5\theta &= 90^\circ \\
 \Rightarrow 2\theta + 3\theta &= 90^\circ \\
 \Rightarrow 2\theta &= 90^\circ - 3\theta \\
 \Rightarrow \sin 2\theta &= \sin(90^\circ - 3\theta) \\
 2\sin \theta \cos \theta &= 4\cos^3 \theta - 3\cos \theta \\
 2\sin \theta \cos \theta &= \cos \theta(4\cos^2 \theta - 3) \\
 2\sin \theta &= 4(1 - \sin^2 \theta) \\
 &= 4 - 4\sin^2 \theta - 3 \\
 2\sin \theta &= 1 - 4\sin^2 \theta \\
 \Rightarrow 4\sin^2 \theta + 2\sin \theta - 1 &= 0
 \end{aligned}$$

Here  $a = 4$ ,  $b = 2$ ,  $c = -1$

$$\begin{aligned}
 \sin \theta &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
 \sin \theta &= \frac{-2 \pm \sqrt{(2)^2 - 4(4)(-1)}}{2(4)} \\
 \sin \theta &= \frac{-2 \pm \sqrt{4 + 16}}{8} \\
 \sin \theta &= \frac{-2 \pm \sqrt{20}}{8} = \frac{-2 \pm 2\sqrt{5}}{8} \\
 \sin \theta &= \frac{-1 \pm \sqrt{5}}{4}
 \end{aligned}$$

put  $\theta = 18^\circ$  so

$$\begin{aligned}
 \sin \theta &= \frac{-1 + \sqrt{5}}{4} \quad (\because 18^\circ \text{ lies in I quad.}) \\
 \therefore \cos^2 \theta &= (1 - \sin^2 \theta) \\
 \cos^2 \theta &= 1 - \left(\frac{-1 + \sqrt{5}}{4}\right)^2 = 1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2 \\
 \cos^2 \theta &= 1 - \left(\frac{5 + 1 - 2\sqrt{5}}{16}\right) = \frac{16 - 6 + 2\sqrt{5}}{16} \\
 \cos^2 \theta &= \frac{10 + 2\sqrt{5}}{16} \\
 \cos \theta &= \pm \sqrt{\frac{10 + 2\sqrt{5}}{16}} \\
 \Rightarrow \cos \theta &= \pm \frac{\sqrt{10 + 2\sqrt{5}}}{4} \\
 \Rightarrow \cos 18^\circ &= \pm \frac{\sqrt{10 + 2\sqrt{5}}}{4} \quad (\because 18^\circ \text{ lies in I quad})
 \end{aligned}$$

(ii)



$36^\circ$ Let  $\theta = 36^\circ$ 

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\text{Put } \theta = 18^\circ \Rightarrow \cos 2(18^\circ) = 2\cos^2 18^\circ - 1$$

$$\Rightarrow \cos 36^\circ = 2(\cos 18^\circ)^2 - 1$$

$$\Rightarrow \cos 36^\circ = 2(\cos 18^\circ)^2 - 1$$

$$= 2\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2 - 1$$

$$\cos 36^\circ = 2\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2 - 1$$

$$\cos 36^\circ = 2\left(\frac{10+2\sqrt{5}}{16}\right) - 1$$

$$\cos 36^\circ = \frac{10+2\sqrt{5}}{8} - 1 = \frac{10+2\sqrt{5}-8}{8}$$

$$\cos 36^\circ = \frac{2+2\sqrt{5}}{8} = \frac{2(1+\sqrt{5})}{8}$$

$$\Rightarrow \cos 36^\circ = \frac{1+\sqrt{5}}{4}$$

$$\because \sin^2\theta = 1 - \cos^2\theta$$

$$\Rightarrow \sin^2 36^\circ = 1 - \cos^2 36^\circ$$

$$= 1 - (\cos 36^\circ)^2$$

$$= 1 - \left(\frac{1+\sqrt{5}}{4}\right)^2$$

$$= 1 - \left(\frac{1+\sqrt{5}}{4}\right)^2$$

$$= 1 - \frac{1+5+2\sqrt{5}}{16}$$

$$\sin^2 36^\circ = \frac{16-6-2\sqrt{5}}{16}$$

$$\sin^2 36^\circ = \frac{10-2\sqrt{5}}{16}$$

$$\Rightarrow \sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

(iii)  $54^\circ$ Let  $\theta = 54^\circ$ let  $\cos 54^\circ$ 

$$\because \cos 54^\circ = \cos(90^\circ - 54^\circ)$$

$$\cos 54^\circ = \sin 36^\circ = \frac{10-2\sqrt{5}}{4}$$

$$\cos 54^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

$$\because \sin 54^\circ = \sin(90^\circ - 54^\circ)$$

$$= \cos 36^\circ = \frac{1+\sqrt{5}}{4}$$

$$\Rightarrow \sin 54^\circ = 1 + \frac{\sqrt{1+\sqrt{5}}}{4}$$

(iv)  $72^\circ$ let  $\theta = 72^\circ$ 

$$\because \sin 72^\circ = \sin(90^\circ - 18^\circ)$$

$$= \cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$\Rightarrow \sin 72^\circ = \frac{10+2\sqrt{5}}{4}$$

$$\cos 72^\circ = \cos(90^\circ - 18^\circ)$$

$$= \sin 18^\circ = \frac{-1+\sqrt{5}}{4}$$

$$\Rightarrow \cos 72^\circ = \frac{-1+\sqrt{5}}{4}$$

Now

$$L.H.S = \cos 36^\circ \cos 72^\circ \cos 108^\circ \cos 144^\circ$$

$$= \cos 36^\circ \cos 72^\circ \cos(180^\circ - 72^\circ) \cos(180^\circ - 36^\circ)$$

$$= \cos 36^\circ \cos 72^\circ (-\cos 72^\circ) (-\cos 36^\circ)$$

$$= \cos^2 36^\circ \cos^2 72^\circ$$

$$= \left(\frac{1+\sqrt{5}}{4}\right)^2 \left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= \left(\frac{1+5+2\sqrt{5}}{16}\right) \left(\frac{5+1-2\sqrt{5}}{16}\right)$$

$$= \left(\frac{6+2\sqrt{5}}{16}\right) \left(\frac{6-2\sqrt{5}}{16}\right)$$

$$= \frac{(6)^2 - (2\sqrt{5})^2}{(16)^2}$$

$$= \frac{36 - 4(5)}{16 \times 16} = \frac{36 - 20}{16 \times 16}$$

$$= \frac{16}{16 \times 16} = \frac{1}{16} = R.H.S$$

### Sum, difference and products of sine and cosines

We know that

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \rightarrow (i)$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \rightarrow (ii)$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \rightarrow (iii)$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \rightarrow (iv)$$

$$eq(i) + eq(ii) \Rightarrow$$

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$$

$$eq(i) - eq(ii) \Rightarrow$$

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\cos\beta$$

$$by(iii) + (iv) \Rightarrow$$

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$$

$$by(iii) - (iv) \Rightarrow$$

$$\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin\alpha\sin\beta$$

So we get four identities as

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$$

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\cos\beta$$

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$$

$$\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin\alpha\sin\beta$$

$$\text{Now put } \alpha + \beta = p \rightarrow (1)$$

$$\text{And } \alpha - \beta = Q \rightarrow (2)$$

$$\text{By (1)+(2)} \Rightarrow 2\alpha = P + Q$$

$$\Rightarrow \alpha = \frac{P + Q}{2}$$

$$\text{by (1) - (2)} \Rightarrow 2\beta = P - Q \Rightarrow \beta = \frac{P - Q}{2}$$

Note:

$$\sin P + \sin Q = 2 \sin \frac{P + Q}{2} \cos \frac{P - Q}{2}$$

$$\sin P - \sin Q = 2 \cos \frac{P + Q}{2} \sin \frac{P - Q}{2}$$

$$\cos P + \cos Q = 2 \cos \frac{P + Q}{2} \cos \frac{P - Q}{2}$$

$$\cos P - \cos Q = -2 \sin \frac{P + Q}{2} \sin \frac{P - Q}{2}$$

### Exercise 10.4

1. Express the following product as sum or differences:

i.  $2 \sin 3\theta \cos \theta$

$$\text{Solution:- } 2 \sin 3\theta \cos \theta$$

$\therefore$  using formula

$$= \sin(3\theta + \theta) + \sin(3\theta - \theta)$$

$$= \sin 4\theta + \sin 2\theta$$

ii.  $2 \cos 5\theta \sin 3\theta$

$$\text{Solution:- } 2 \cos 5\theta \sin 3\theta$$

$\therefore$  using formula

$$= \sin(5\theta + 3\theta) - \sin(5\theta - 3\theta)$$

$$= \sin 8\theta - \sin 2\theta$$

iii.  $\sin 5\theta \cos 2\theta$

$$\text{Solution:- } \sin 5\theta \cos 2\theta$$

$$= \frac{1}{2} [2 \sin 5\theta \cos 2\theta]$$

$\therefore$  using formula

$$= \frac{1}{2} [\sin(5\theta + 2\theta) + \sin(5\theta - 2\theta)]$$

$$= \frac{1}{2} [\sin 7\theta + \sin 3\theta]$$

iv.  $2 \sin 7\theta \sin 2\theta$

$$\text{Solution:- } 2 \sin 7\theta \sin 2\theta$$

$$= -[-2 \sin 7\theta \sin 2\theta] \quad \therefore \text{using formula}$$

$$= -[\cos(7\theta + 2\theta) - \cos(7\theta - 2\theta)]$$

$$= -[\cos 9\theta - \cos 5\theta]$$

v.  $\cos(x+y) \sin(x-y)$

$$\text{Solution:- } \frac{1}{2} [2 \cos(x+y) \sin(x-y)]$$

$\therefore$  using formula

$$= \frac{1}{2} [\sin(x+y+x-y) - \sin(x+y-x-y)]$$

$$= \frac{1}{2} [\sin(x+x) - \sin(y+y)]$$

$$= \frac{1}{2} [\sin(2x) - \sin(2y)]$$

vi.  $\cos(2x+30) \cos(2x-30)$

$$\text{Solution:- } \frac{1}{2} [2 \cos(2x+30) \cos(2x-30)]$$

using formula

$$= \frac{1}{2} [\cos(2x+30+2x-30) + \cos(2x+30-2x+30)]$$

$$= \frac{1}{2} [\cos(4x) + \cos(60)]$$

$$= \frac{1}{2} [\cos(4x) + \cos(60)]$$

vii.  $\sin 12 \sin 46$

$$\text{Solution:- } -\frac{1}{2} [-2 \sin(12) \sin(46)]$$

$\therefore$  using formula

$$= -\frac{1}{2} [\cos(12+46) - \cos(12-46)]$$

$$= -\frac{1}{2} [\cos(58) - \cos(-34)]$$

$\therefore \cos(-\alpha) = \cos \alpha$

$$= -\frac{1}{2} [\cos(58) - \cos(34)]$$

viii.  $\sin(x+45) \sin(x-45)$

$$\text{Solution:- } -\frac{1}{2} [-2 \sin(x+45) \sin(x-45)]$$

$\therefore$  using formula

$$= -\frac{1}{2} [\cos(x+45+x-45) - \cos(x+45-x-45)]$$

$$= -\frac{1}{2} [\cos(2x) - \cos(90)]$$

$$= -\frac{1}{2} [\cos(2x) - \cos(90)]$$

2. Express the following product as sum or differences:

i.  $\sin 5\theta + \sin 3\theta$

$$\text{Solution:- } \sin 5\theta + \sin 3\theta$$

$$= 2 \sin\left(\frac{5\theta+3\theta}{2}\right) \cos\left(\frac{5\theta-3\theta}{2}\right)$$

$$= 2 \sin\left(\frac{8\theta}{2}\right) \cos\left(\frac{2\theta}{2}\right)$$

$$= 2 \sin(4\theta) \cos(\theta)$$

ii.  $\sin 8\theta - \sin 4\theta$

$$\text{Solution:- } \sin 8\theta - \sin 4\theta$$

$$= 2 \cos\left(\frac{8\theta+4\theta}{2}\right) \sin\left(\frac{8\theta-4\theta}{2}\right)$$

$$= 2 \cos\left(\frac{12\theta}{2}\right) \sin\left(\frac{4\theta}{2}\right)$$

$$= 2 \cos(6\theta) \sin(2\theta)$$

iii.  $\cos 6\theta + \cos 3\theta$

$$\text{Solution:- } \cos 6\theta + \cos 3\theta$$

$$= 2 \cos\left(\frac{6\theta+3\theta}{2}\right) \cos\left(\frac{6\theta-3\theta}{2}\right)$$

$$= 2 \cos\left(\frac{9\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right)$$

iv.  $\cos 7\theta - \cos \theta$

Solution:-  $\cos 7\theta - \cos \theta$

$$= -2\sin\left(\frac{7\theta - \theta}{2}\right) \sin\left(\frac{7\theta + \theta}{2}\right)$$

$$= -2\sin\left(\frac{6\theta}{2}\right) \sin\left(\frac{8\theta}{2}\right)$$

$$= -2\sin(3\theta) \sin(4\theta)$$

v.  $\cos 12 + \cos 48$

Solution:-  $\cos 12 + \cos 48$

$$= 2\cos\left(\frac{12+48}{2}\right) \cos\left(\frac{12-48}{2}\right)$$

$$= 2\cos\left(\frac{60}{2}\right) \cos\left(\frac{-36}{2}\right)$$

$$= 2\cos(30) \cos(-18) \quad \therefore \cos(-\alpha) = \cos \alpha$$

$$= 2\cos(30) \cos(18)$$

vi.  $\sin(x+30) + \sin(x-30)$

Solution:-  $\sin(x+30) + \sin(x-30)$

$$= 2\sin\left(\frac{x+30+x-30}{2}\right) \cos\left(\frac{x+30-x-30}{2}\right)$$

$$= 2\sin\left(\frac{x+x}{2}\right) \cos\left(\frac{30+30}{2}\right)$$

$$= 2\sin\left(\frac{x+x}{2}\right) \cos\left(\frac{30+30}{2}\right)$$

$$= 2\sin\left(\frac{2x}{2}\right) \cos\left(\frac{60}{2}\right)$$

$$= 2\sin(x) \cos(30)^\circ$$

3. Prove the following identities.

i.  $\frac{\sin 3x - \sin x}{\cos x - \cos 3x} = \cot 2x$

$$\text{L.H.S.} = \frac{\sin 3x - \sin x}{\cos x - \cos 3x}$$

$$= \frac{2\cos\left(\frac{3x+x}{2}\right) \sin\left(\frac{3x-x}{2}\right)}{-2\sin\left(\frac{x+3x}{2}\right) \sin\left(\frac{x-3x}{2}\right)}$$

$$= \frac{2\cos\left(\frac{4x}{2}\right) \sin\left(\frac{2x}{2}\right)}{-2\sin\left(\frac{4x}{2}\right) \sin\left(\frac{-2x}{2}\right)}$$

$$= \frac{2\cos(2x) \sin(x)}{-2\sin(2x) \sin(-x)} \quad \therefore \sin(-\alpha) = -\sin \alpha$$

$$= \frac{2\cos(2x) \sin(x)}{2\sin(2x) \sin(x)}$$

$$= \frac{\cos 2x}{\sin 2x}$$

$$= \cot 2x \quad \text{R.H.S}$$

ii.  $\frac{\sin 8x + \sin 2x}{\cos 8x + \cos 2x} = \tan 5x$

$$\text{L.H.S} = \frac{2\sin\left(\frac{8x+2x}{2}\right) \cos\left(\frac{8x-2x}{2}\right)}{2\cos\left(\frac{8x+2x}{2}\right) \cos\left(\frac{8x-2x}{2}\right)}$$

$$= \frac{2\sin\left(\frac{10x}{2}\right) \cos\left(\frac{6x}{2}\right)}{2\cos\left(\frac{10x}{2}\right) \cos\left(\frac{6x}{2}\right)}$$

$$= \frac{2\sin(5x) \cos(3x)}{2\cos(5x) \cos(3x)} \quad (\text{cancellation})$$

$$= \frac{\sin 5x}{\cos 5x}$$

$$= \tan 5x \quad \text{R.H.S}$$

iii.  $\frac{\sin \alpha - \sin \beta}{\sin \alpha + \sin \beta} = \tan \frac{\alpha - \beta}{2} \cot \frac{\alpha + \beta}{2}$

Solution:-  $\frac{\sin \alpha - \sin \beta}{\sin \alpha + \sin \beta}$

$$= \frac{2\cos\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right) \cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$= \frac{\cos\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right) \cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$= \frac{\cos\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right) \cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$= \tan \frac{\alpha - \beta}{2} \cot \frac{\alpha + \beta}{2} = \text{R.H.S}$$

4. Prove that:

i.  $\cos 20 + \cos 100 + \cos 140 = 0$

L.H.S =  $\cos 20 + \cos 100 + \cos 140$  (re-arranging)

$$= \cos 140 + \cos 100 + \cos 20$$

$$= \cos 140 + 2\cos\left(\frac{100+20}{2}\right) \cos\left(\frac{100-20}{2}\right)$$

$$= \cos 140 + 2\cos\left(\frac{120}{2}\right) \cos\left(\frac{80}{2}\right)$$

$$= \cos 140 + 2\cos(60) \cos(40)$$

$$= \cos 140 + 2\left(\frac{1}{2}\right) \cos(40)$$

$$= \cos 140 + \cos 40$$

$$= 2\cos\left(\frac{140+40}{2}\right) \cos\left(\frac{140-40}{2}\right)$$

$$= 2\cos\left(\frac{180}{2}\right) \cos\left(\frac{120}{2}\right)$$

$$= 2\cos(90) \cos(60)$$

$$= 2(0) \cos 60$$

$$= 0 = \text{R.H.S}$$

ii.  $\sin\left(\frac{\pi}{4} - \theta\right) \sin\left(\frac{\pi}{4} + \theta\right) = \frac{1}{2} \cos 2\theta$

Solution:-  $\sin\left(\frac{\pi}{4} - \theta\right) \sin\left(\frac{\pi}{4} + \theta\right)$

$$= \left(\sin \frac{\pi}{4} \cos \theta - \cos \frac{\pi}{4} \sin \theta\right) \left(\sin \frac{\pi}{4} \cos \theta + \cos \frac{\pi}{4} \sin \theta\right)$$

$$= \left(\left(\frac{1}{\sqrt{2}}\right) \cos \theta - \left(\frac{1}{\sqrt{2}}\right) \sin \theta\right) \left(\left(\frac{1}{\sqrt{2}}\right) \cos \theta + \left(\frac{1}{\sqrt{2}}\right) \sin \theta\right)$$

$$= \left(\frac{1}{\sqrt{2}} \cos \theta\right)^2 - \left(\frac{1}{\sqrt{2}} \sin \theta\right)^2$$

$$= \left(\frac{1}{\sqrt{2}} \cos \theta\right)^2 - \left(\frac{1}{\sqrt{2}} \sin \theta\right)^2$$

$$= \frac{1}{2} (\cos^2 \theta - \sin^2 \theta)$$

$$= \frac{1}{2} \cos 2\theta$$

iii.  $\frac{\sin \theta + \sin 3\theta + \cos 5\theta + \sin 7\theta}{\cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta} = \tan 4\theta$

$$\text{L.H.S} = \frac{\sin \theta + \sin 3\theta + \cos 5\theta + \sin 7\theta}{\cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta}$$

$$= \frac{\left[2\sin \frac{\theta + 3\theta}{2} \cos \frac{\theta - 3\theta}{2}\right] + \left[2\sin \frac{5\theta + 7\theta}{2} \cos \frac{5\theta - 7\theta}{2}\right]}{\left[2\cos \frac{\theta + 3\theta}{2} \cos \frac{\theta - 3\theta}{2}\right] + \left[2\cos \frac{5\theta + 7\theta}{2} \cos \frac{5\theta - 7\theta}{2}\right]}$$

$$= \frac{\left[2\sin \frac{\theta + 3\theta}{2} \cos \frac{\theta - 3\theta}{2}\right] + \left[2\sin \frac{5\theta + 7\theta}{2} \cos \frac{5\theta - 7\theta}{2}\right]}{\left[2\cos \frac{\theta + 3\theta}{2} \cos \frac{\theta - 3\theta}{2}\right] + \left[2\cos \frac{5\theta + 7\theta}{2} \cos \frac{5\theta - 7\theta}{2}\right]}$$

$$\begin{aligned}
&= \frac{\left[2\sin\frac{4\theta}{2}\cos\left(-\frac{2\theta}{2}\right)\right] + \left[2\sin\frac{12\theta}{2}\cos\left(-\frac{2\theta}{2}\right)\right]}{\left[2\cos\frac{4\theta}{2}\cos\left(-\frac{2\theta}{2}\right)\right] + \left[2\cos\frac{12\theta}{2}\cos\left(-\frac{2\theta}{2}\right)\right]} \\
&= \frac{2\sin 2\theta \cos(-\theta) + 2\sin 6\theta \cos(-\theta)}{2\cos 2\theta \cos(-\theta) + 2\cos 6\theta \cos(-\theta)} \\
&= \frac{2\sin 2\theta \cos\theta + 2\sin 6\theta \cos(-\theta)}{2\cos 2\theta \cos(-\theta) + 2\cos 6\theta \cos(-\theta)} \\
&= \frac{2\cos(\sin 2\theta + \sin\theta)}{2\cos\theta(\cos 2\theta + \cos 6\theta)} \\
&= \frac{\sin 2\theta + \sin 6\theta}{\cos 2\theta + \cos 6\theta} \\
&= \frac{2\sin\frac{2\theta + 6\theta}{2}\cos\frac{2\theta - 6\theta}{2}}{2\cos\frac{2\theta + 6\theta}{2}\cos\frac{2\theta - 6\theta}{2}} \\
&= \frac{2\sin\left(\frac{8\theta}{2}\right)\cos\left(-\frac{4\theta}{2}\right)}{2\cos\left(\frac{8\theta}{2}\right)\cos\left(-\frac{4\theta}{2}\right)} \\
&= \frac{\sin 4\theta}{\cos 4\theta} = \tan 4\theta = R.H.S
\end{aligned}$$

Hence proved.

5. Prove that:

i.  $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}$

Solution:

$$\begin{aligned}
L.H.S &= \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ \\
&= \cos 20^\circ \cos 40^\circ \left(\frac{1}{2}\right) \cos 80^\circ \\
&= \frac{1}{2}(\cos 20^\circ \cos 40^\circ) \cos 80^\circ \\
&= \frac{1}{4}(2\cos 20^\circ \cos 40^\circ) \cos 80^\circ \\
&= \frac{1}{4}[\cos(20^\circ + 40^\circ) + \cos(20^\circ - 40^\circ)] \cos 80^\circ \\
&= \frac{1}{4}[\cos 60^\circ + \cos(-20^\circ)] \cos 80^\circ \\
&= \frac{1}{4}\left[\frac{1}{2} + \cos 20^\circ\right] \cos 80^\circ \\
&= \frac{1}{8} \cos 80^\circ + \frac{1}{4} \cos 20^\circ \cos 80^\circ \\
&= \frac{1}{8} \cos 80^\circ + \frac{1}{8}(2\cos 20^\circ \cos 80^\circ) \\
&= \frac{1}{8} \cos 80^\circ + \frac{1}{8}[\cos(20^\circ + 80^\circ) + \cos(20^\circ - 80^\circ)] \\
&= \frac{1}{8} \cos 80^\circ + \frac{1}{8}[\cos 100^\circ + \cos 60^\circ] \\
&= \frac{1}{8} \cos 80^\circ + \frac{1}{8} \cos 100^\circ + \frac{1}{8}\left(\frac{1}{2}\right) \\
&= \frac{1}{8} \cos 80^\circ + \frac{1}{8} \cos(180^\circ - 80^\circ) + \frac{1}{16} \\
&= \frac{1}{8} \cos 80^\circ - \frac{1}{8} \cos 80^\circ + \frac{1}{16} \\
&= \frac{1}{16} R.H.S \because \cos(180^\circ - \theta) = -\cos\theta
\end{aligned}$$

Hence proved.

ii.  $\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9} = \frac{3}{16}$

solution:

$$\begin{aligned}
L.H.S &= \sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9} \\
&= \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ \\
&= \sin 20^\circ \sin 40^\circ \left(\frac{\sqrt{3}}{2}\right) \sin 80^\circ \\
&= \frac{\sqrt{3}}{2}(\sin 20^\circ \sin 40^\circ) \sin 80^\circ \\
&= -\frac{\sqrt{3}}{2}[\cos(20^\circ + 40^\circ) - \cos(20^\circ - 40^\circ)] \sin 80^\circ \\
&= -\frac{\sqrt{3}}{4}(\cos 60^\circ - \cos(-20^\circ)) \sin 80^\circ \\
&= -\frac{\sqrt{3}}{4}\left(\frac{1}{2} - \cos 20^\circ\right) \sin 80^\circ \\
&= \left(-\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{4} \cos 20^\circ\right) \sin 80^\circ \\
&= -\frac{\sqrt{3}}{8} \sin 80^\circ + \frac{\sqrt{3}}{4} \cos 20^\circ \sin 80^\circ \\
&= -\frac{\sqrt{3}}{8} \sin 80^\circ + \frac{\sqrt{3}}{8}(2\cos 20^\circ \sin 80^\circ) \\
&= -\frac{\sqrt{3}}{8} \sin 80^\circ + \frac{\sqrt{3}}{8}(\sin(20^\circ + 80^\circ) - \sin(20^\circ - 80^\circ)) \\
&= -\frac{\sqrt{3}}{8} \sin 80^\circ + \frac{\sqrt{3}}{8}(\sin 100^\circ - \sin(-60^\circ)) \\
&= -\frac{\sqrt{3}}{8} \sin 180^\circ + \frac{\sqrt{3}}{8}(\sin 100^\circ + \sin 60^\circ) \\
&= -\frac{\sqrt{3}}{8} \sin 80^\circ + \frac{\sqrt{3}}{8} \sin 100^\circ + \frac{\sqrt{3}}{8} \sin 60^\circ \\
&= -\frac{\sqrt{3}}{8} \sin 80^\circ + \frac{\sqrt{3}}{8} \sin(180^\circ - 80^\circ) + \frac{\sqrt{3}}{8}\left(\frac{\sqrt{3}}{2}\right) \\
&= -\frac{\sqrt{3}}{8} \sin 80^\circ + \frac{\sqrt{3}}{8} \sin 80^\circ + \frac{3}{16} \\
&= \frac{3}{16} = R.H.S
\end{aligned}$$

iii.  $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{16}$

solution:

$$\begin{aligned}
L.H.S &= \sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ \\
&= \sin 10^\circ \left(\frac{1}{2}\right) \sin 50^\circ \sin 70^\circ \\
&= \frac{1}{2}(\sin 10^\circ \sin 50^\circ) \sin 70^\circ \\
&= -\frac{1}{4}(-2\sin 10^\circ \sin 50^\circ) \sin 70^\circ \\
&= -\frac{1}{4}(\cos(10^\circ + 50^\circ) - \cos(10^\circ - 50^\circ)) \sin 70^\circ \\
&= -\frac{1}{4}(\cos 60^\circ - \cos(-40^\circ)) \sin 70^\circ \\
&= -\frac{1}{4}\left(\frac{1}{2} - \cos 40^\circ\right) \sin 70^\circ \\
&= \left(-\frac{1}{8} + \frac{1}{4} \cos 40^\circ\right) \sin 70^\circ \\
&= -\frac{1}{8} \sin 70^\circ + \frac{1}{4} \cos 40^\circ \sin 70^\circ \\
&= -\frac{1}{8} \sin 70^\circ + \frac{1}{8}(2\cos 40^\circ \sin 70^\circ) \\
&= -\frac{1}{8} \sin 70^\circ + \frac{1}{8}(\sin(40^\circ + 70^\circ) - \sin(40^\circ - 70^\circ))
\end{aligned}$$

$$\begin{aligned} &= -\frac{1}{8}\sin 70^\circ + \frac{1}{8}(\sin 110^\circ - \sin(-30^\circ)) \\ &= -\frac{1}{8}\sin 70^\circ + \frac{1}{8}\sin 110^\circ + \frac{1}{8}\sin 30^\circ \\ &= -\frac{1}{8}\sin 70^\circ + \frac{1}{8}\sin(180^\circ - 70^\circ) + \frac{1}{8}\sin 30^\circ \\ &= -\frac{1}{8}\sin 70^\circ + \frac{1}{8}\sin 70^\circ + \frac{1}{8}\left(\frac{1}{2}\right) \\ &= \frac{1}{16} = R.H.S \end{aligned}$$

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